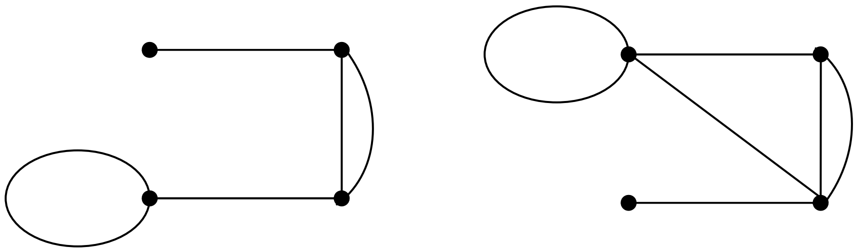
1.7. Isomorphic Graphs

Example:

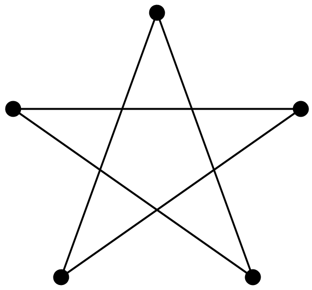
Consider the following graphs, are they the isomorphic, i.e. the “same”?

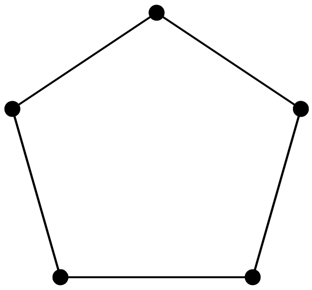
◆



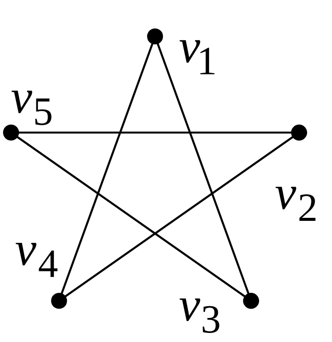
No. The left-hand graph has 5 edges; the right hand graph has 6 edges.

◆

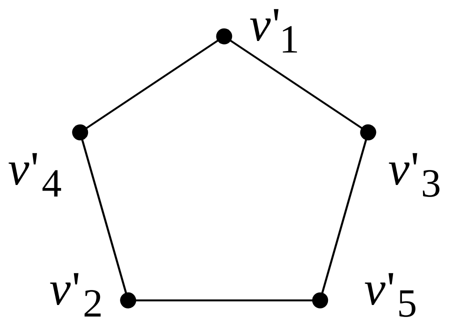




Firstly, label the graphs. It "looks" true, so check all the things we know:



G



G′

Number of vertices: both 5.

Number of edges: both 5.

Degrees of corresponding vertices: all degree 2.

Connectedness: Each is fully connected.

Number of connected components: Both 1.

Pairs of connected vertices: All correspond.

Number of loops: 0.

Number of parallel edges: 0.

Everything is equal and so the graphs are isomorphic.

More formally:

$G = \{V, E\}$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and

$$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1)\}$$
$$= \{e_1, e_2, e_3, e_4, e_5\}$$

$G' = \{V', E'\}$ where $V' = \{v_1', v_2', v_3', v_4', v_5'\}$ and

$$E' = \{(v_1', v_2'), (v_2', v_3'), (v_3', v_4'), (v_4', v_5'), (v_5', v_1')\}$$
$$= \{e_1', e_2', e_3', e_4', e_5'\}.$$

Construct 2 functions: $f : V \to V'$ and $g : E \to E'$

$f : V \to V'$		$g : E \to E'$	
V	V'	E	E'
v_1	v_1'	e_1	e_1'
v_2	v_2'	e_2	e_2'
v_3	v_3'	e_3	e_3'
v_4	v_4'	e_4	e_4'
v_5	v_5'	e_5	e_5'

1.7.1. Definition

Let $G = \{V, E\}$ and $G' = \{V', E'\}$ be graphs. G and G' are said to be isomorphic if there exist a pair of functions $f : V \to V'$ and $g : E \to E'$ such that f associates each element in V with exactly one element in V' and vice versa; g associates each element in E with exactly one element in E' and vice versa, and for each $v \in V$, and each $e \in E$, if v is an endpoint of the edge e, then $f(v)$ is an endpoint of the edge $g(e)$.

Notes:

* To prove two graphs are isomorphic you must give a formula (picture) for the functions f and g.

* If two graphs are isomorphic, they must have:

- the same number of vertices
- the same number of edges
- the same degrees for corresponding vertices
- the same number of connected components
- the same number of loops

- the same number of parallel edges.

* Further,

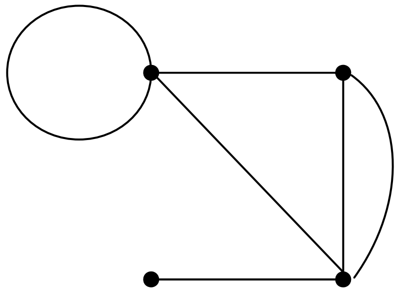
  - both graphs are connected or both graphs are not connected, and

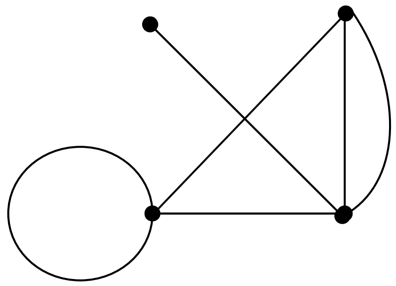
  - pairs of connected vertices must have the corresponding pair of vertices connected.

* In general, it is easier to prove two graphs are not isomorphic by proving that one of the above properties fails.

Exercises:

- Show that the following graphs are isomorphic.





- Draw all possible graphs having 2 edges and 2 vertices; that is, draw all non-isomorphic graphs having 2 edges and 2 vertices.

Since isomorphic graphs are “essentially the same”, we can use this idea to classify graphs.

1.8. Regular, Complete and Complete Bipartite.

1.8.1. Definition: Regular.

A simple graph $G = \{V, E\}$ is said to be *regular of degree k*, or simply *k-regular* if for each $v \in V$, $\delta(v) = k$.

That is, if a graph is *k-regular*, every vertex has degree *k*.

Exercises:

- Draw all 0-regular graphs with

1 vertex; 2 vertices; 3 vertices.

1 vertex: 2 vertices: 3 vertices:

- Draw all 1-regular graphs with

1 vertex; 2 vertices; 3 vertices; 4 vertices.

1 vertex and 3 vertices not possible

◆ Draw all 2-regular graphs with 2 vertices; 3 vertices; 4 vertices.

1.8.2. Definition: Complete.

A simple graph $G = \{V, E\}$ is said to be *complete* if each vertex of *G* is connected to every other vertex of *G*.

The complete graph with *n* vertices is denoted K_n.

Notes:

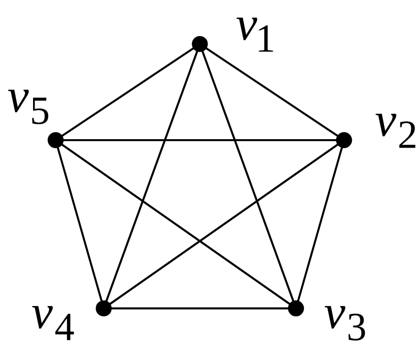
* A complete graph is connected

* $\forall n \in \mathbb{N}$, two complete graphs having *n* vertices are isomorphic

* For complete graphs, once the number of vertices is known, the number of edges and the endpoints of each edge are also known

Example:

Draw K_5



Exercises:

- Draw K_1, K_2, K_3 and K_4. What is the degree of each vertex in each of the graphs drawn?

- Complete the following sentences:
  - A complete graph, K_n, is an ________-regular graph.
  - A complete graph, K_n, has ________ edges. [Use Euler's First Law: $\sum_{v \in V} \delta(v) = 2n(E)$.]

1.8.3. Definition: Bipartite.

A simple graph $G = \{V, E\}$ is said to be *bipartite* if there exists sets $U \subset V$ and $W \subset V$, such that;

1. $U \cup W = V$ and $U \cap W = \emptyset$
2. every edge of G connects a vertex in U with a vertex in W.

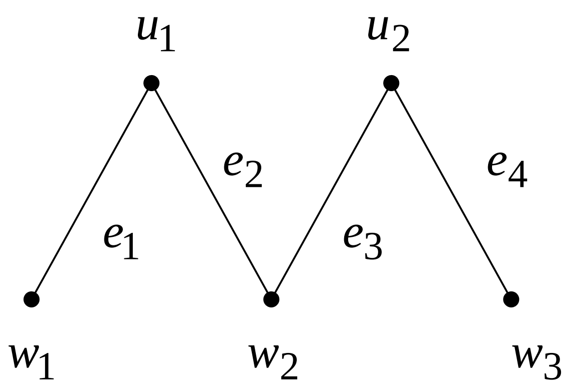
Notes:

* A bipartite graph $G = \{V, E\}$ is one whose vertices can be separated into two disjoint sets, where every edge joins a vertex in one set to a vertex in the other

* $\forall u \in U, \ \delta(u) \le n(W)$ and $\forall w \in W, \ \delta(w) \le n(U)$ [$n(W)$ is the number of elements in W]

Example:

Consider the following graph $G = \{V, E\}$, where $V = \{u_1, u_2, w_1, w_2, w_3\}$, is it bipartite? Yes

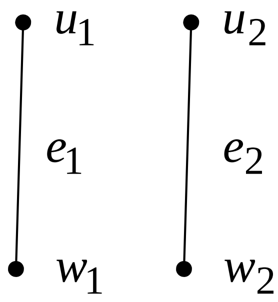


List the sets U and W. $U = \{u_1, u_2\}$; $W = \{w_1, w_2, w_3\}$

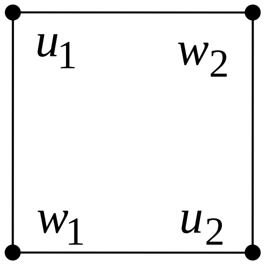
Exercises:

- Consider the following graphs, are they bipartite? If so write down the sets U and W.

  o



o



- Draw two bipartite graphs with 5 vertices.
    - o Where U has 1 element and W has 4 elements.
    - o Where U has 2 elements and W has 3 elements.

1.8.4. Definition: Complete Bipartite.

A simple graph $G = \{V, E\}$, is said to be *complete bipartite* if;

1. G is bipartite and
2. every vertex in U is connected to every vertex in W.

Notes:

* A complete bipartite graph is one whose vertices can be separated into two disjoint sets where every vertex in one set is connected to every vertex in the other but no vertices within either set are connected.

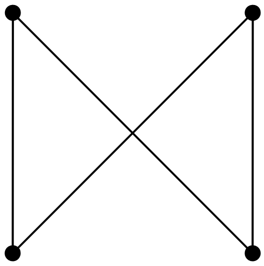
* The symbol $K_{n,m}$ is used to denote the complete bipartite graph having *n* vertices in one set and *m* in the other. [Note that the comma is necessary!]

* $\forall u \in U,\ \delta(u) = n(W)$ and $\forall w \in W,\ \delta(w) = n(U)$

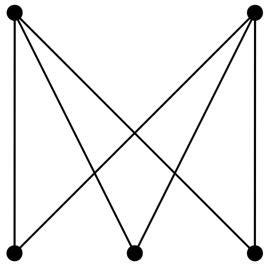
* $n(E) = n \times m$

Examples:

- Draw $K_{2,2}$
- Draw $K_{2,3}$



$K_{2,2}$



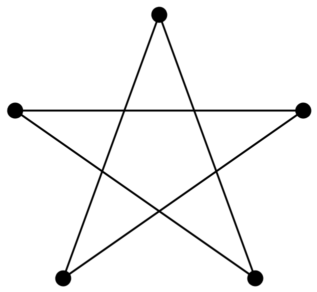
$K_{2,3}$

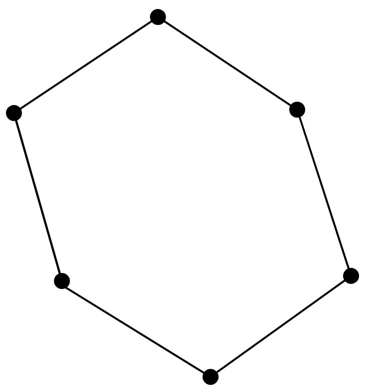
Exercises:

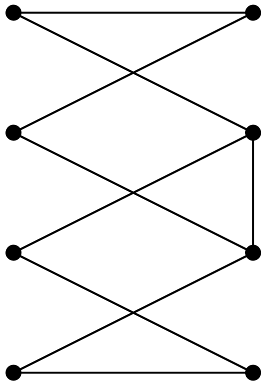
- Draw $K_{1,3}$

- Draw $K_{2,4}$

- Which of these graphs are bipartite.

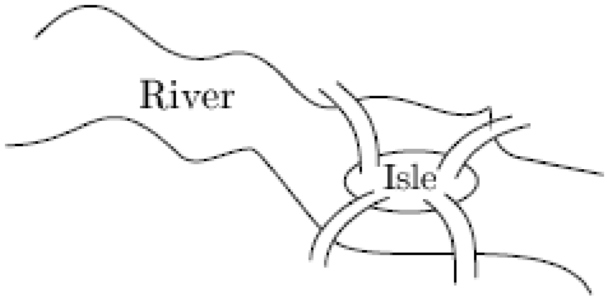






1.9. Eulerian Graphs.

Problem 1:



If you are walking in the park, can you cross each bridge exactly once?

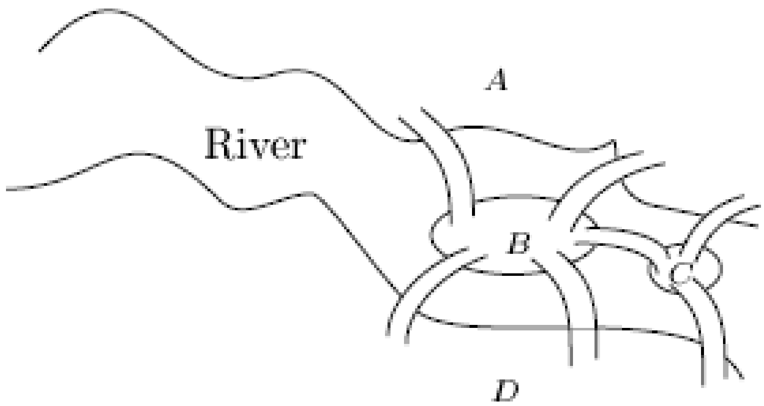
Does it depend on where you start?

Could you do it with less bridges?

Could you do it with more bridges?

Problem 2:

The Koenigsberg Bridge Problem.



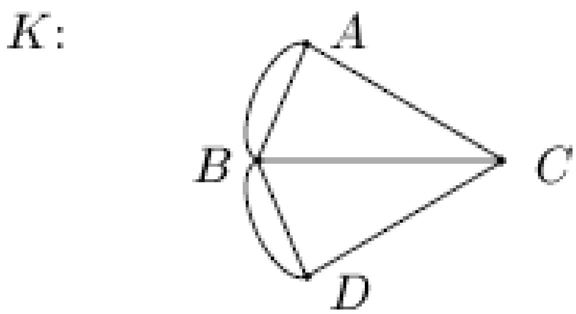
Can you stroll around, crossing each bridge exactly once and return to your starting point?

No one can do it.

How do you prove it can't be done?

If you're Leonhard Euler, you make some definitions, make the problem precise and prove a theorem!

Redraw the problem as a graph:



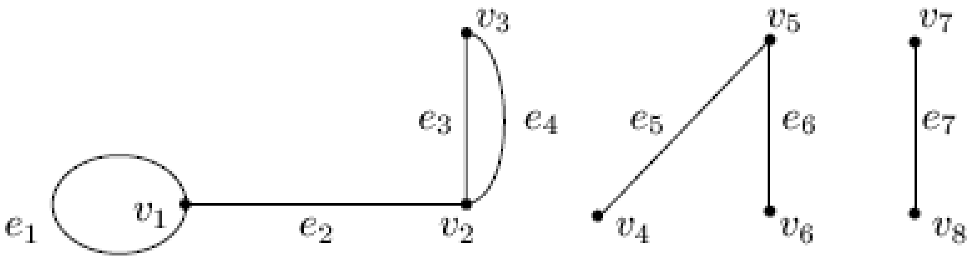
1.9.1. Definition: Circuit.

Let $G = \{V, E\}$ be a graph and let $v \in V$. A circuit in G is a path from v to v in which no edge is repeated.

Note: A circuit is a closed path and in many books is called a cycle.

Exercises:

- Write down the circuits in the following graph, starting at the given vertices.



- v_2 :

- v_1 :

- v_3 :

- v_7 :

1.9.2. Definition: Eulerian Path

Let $G = \{V, E\}$ be a graph. A path in G is an Eulerian path if every edge of G is included once and only once in the path.

1.9.3. Definition: Eulerian Circuit

Let $G = \{V, E\}$ be a graph. A circuit in G is an Eulerian circuit if every edge of G is included exactly once in the circuit.

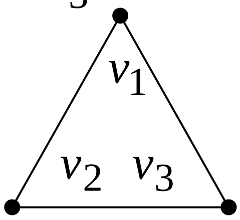
1.9.4. Definition: Eulerian Graph

Let $G = \{V, E\}$ be a graph. G is an Eulerian graph if G has an Eulerian circuit.

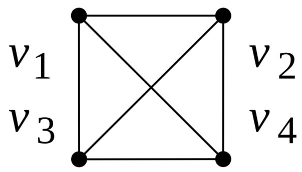
Exercises:

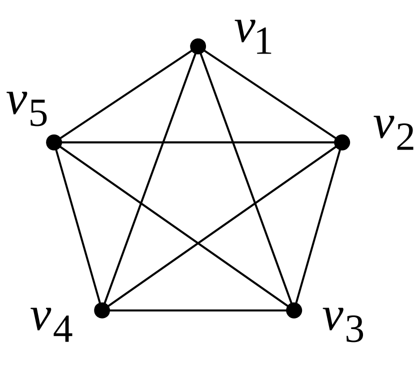
- Which of these graphs are Eulerian?

K_3

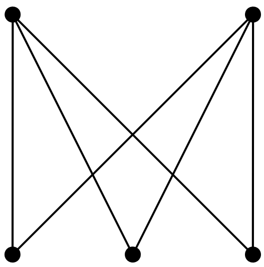


K_4



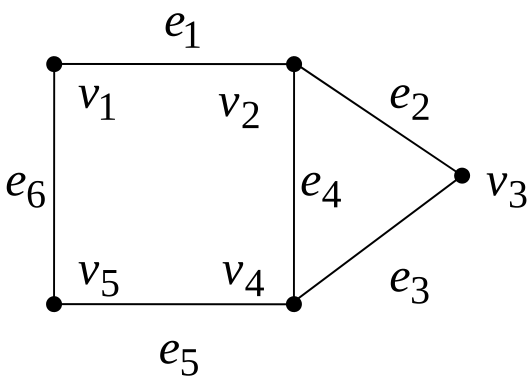


K_5



$K_{2,3}$

◆ Can you find an Eulerian path in the following graph that is not an Eulerian circuit?



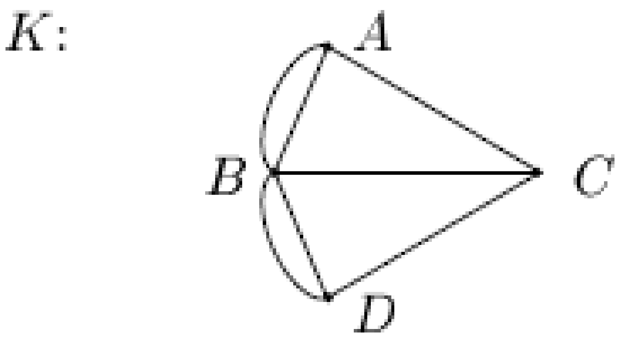
◆ Is the graph Eulerian?

In the Koenisberg Bridge Problem, we wanted to start and end at the same vertex. That is, we need to prove that the graph K is not Eulerian. How?

1.9.5. Theorem: Euler's Theorem.

Let G be a connected graph. G is an Eulerian graph if and only if the degree of each vertex is even.

For the Koenisberg Bridge Problem:

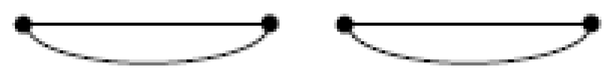


$\delta(C) = 3$, so the graph is not Eulerian.

Discussion:

Is the "connectedness" condition in Euler's Theorem necessary?

Consider the graph



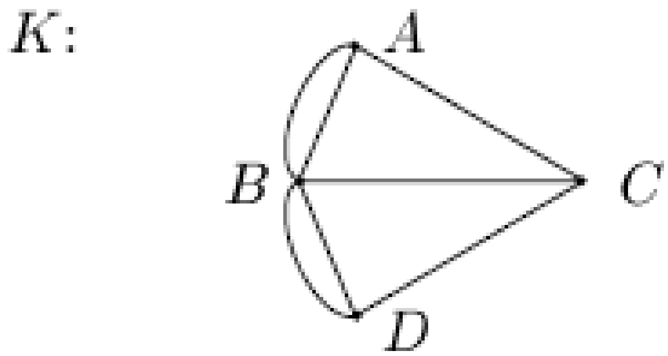
Each vertex has even degree but there is no Eulerian circuit.

In the Koenisberg Bridge setup, is it possible to cross each bridge exactly once without necessarily finishing where you start? That is, is there an Eulerian path in *K*?

1.9.6. Theorem: Existence of Eulerian Path.

A connected graph has an Eulerian path, which is not a circuit, if it has exactly two vertices of odd degree.

Example:



$\delta(A) = 3;\ \delta(B) = 5;\ \delta(C) = 3;\ \delta(D) = 3$

There are more than 2 vertices of odd degree, therefore, *K* does not have an Eulerian path.