Differentiate and Make Waves

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1. INTRODUCTION. If we differentiate a function, what sort of function do we obtain? Consider the function whose value at \( x \) is \( e^{-x^2} \). If we think of its graph as a wave, then it has a crest; that is, a local maximum, at the origin. It has no trough; that is, no local minimum. However, differentiation of \( e^{-x^2} \) produces the function \( -2xe^{-x^2} \), which has a crest at \( 1/2 \) and a trough at \( -1/2 \). Thus, in this case, differentiation has maintained the crest present in the original wave, but also has produced a trough which was not there before. In this sense, differentiation of the function \( e^{-x^2} \) has “made waves”, as indicated in Figures 1 and 2.

Another way of “making waves” is by taking finite differences. Let \( L^2(\mathbb{R}) \) denote the usual space of complex valued, measurable, square integrable functions on the real line \( \mathbb{R} \). That is, if \( f \) is complex valued and measurable on \( \mathbb{R} \), \( f \in L^2(\mathbb{R}) \) if and only if \( \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \). If \( y \in \mathbb{R} \) and \( f \in L^2(\mathbb{R}) \), \( \delta_y * f \) denotes the translation of \( f \) by \( y \). That is, \( (\delta_y * f)(x) = f(x-y) \), for \( x \in \mathbb{R} \). Translation of a function by \( y \) maintains the shape of the graph of the function, but it changes the position of the graph, which moves through a horizontal amount \( y \). As indicated by the notation, \( \delta_y * f \) is the convolution of \( f \) with the Dirac measure \( \delta_y \) at \( y \). A first order difference is a function in \( L^2(\mathbb{R}) \) which is of the form \( f - \delta_y * f \), for some \( f \in L^2(\mathbb{R}) \) and \( y \in \mathbb{R} \).

Consider the function whose value at \( x \) is \( e^{-x^2} - e^{-(x-1)^2} \). This is a first order difference obtained from the function \( e^{-x^2} \). Like the derivative of \( e^{-x^2} \), this function has a crest and a trough: the crest is at \( x_0 \) and the trough is at \( x_1 \), where \( x_0 < x_1 \) and \( x_0 \) and \( x_1 \) are the solutions of \( x/(x-1) = e^{2x-1} \). Thus, in the same sense as differentiation, forming a first order difference from the function \( e^{-x^2} \) has “made waves”. This is indicated in Figure 3.

In an opposite vein, however, note that if \( f \) is a polynomial function of degree \( n \), it may have as many as \( n - 1 \) crests and troughs, while its derivative is a polynomial function of degree \( n - 1 \) which can have at most \( n - 2 \) crests and troughs. In such a case, differentiation has produced a less wave-like function. Also, if \( g \) denotes for the moment the usual sine function, \( g - \delta_2 * g \) is the zero function, which obviously has no crests or troughs whatsoever. In this case, taking one finite difference has
Figure 1. The function $x \mapsto e^{-x^2}$

Figure 2. The function $x \mapsto -2xe^{-x^2}$, the derivative of $x \mapsto e^{-x^2}$

Figure 3. The function $x \mapsto e^{-x^2} - e^{-(x-1)^2}$, a first order difference obtained from $x \mapsto e^{-x^2}$
completely eliminated the waves in the original function.

The point about such examples as the preceding is that neither the functions involved nor their derivatives are restricted in their behaviour towards \( \infty \) or \(-\infty\). To gain some insight into what happens in general when some such restriction is made, we use the fact that a continuous real valued function which vanishes at, or towards, the end points of an interval attains a maximum or a minimum on the interval. Now consider a real valued continuously differentiable function \( f \) on \( \mathbb{R} \). Let \( D(f) \) denote the derivative of \( f \) and assume that \( \lim_{n \to \infty} D(f)(x) = 0 \) and \( \lim_{n \to -\infty} D(f)(x) = 0 \). These conditions may be thought of as saying “\( D(f)(\infty) = 0 \)” and “\( D(f)(-\infty) = 0 \)”. Assume further that there are \( n \) distinct points in \( \mathbb{R} \), at each of which \( f \) has a crest or a trough. At these points, \( D(f) \) vanishes but, as well, \( D(f) \) also vanishes at \( \infty \) and \(-\infty\) in the sense described above. Thus, there is a total of \( n + 2 \) points, including \( \infty \) and \(-\infty\), at each of which \( D(f) \) vanishes. As \( D(f) \) is continuous, \( D(f) \) will have a crest or a trough between any two such points, and there are \( n + 1 \) pairs of consecutive such points. This produces \( n + 1 \) distinct points, at each of which \( D(f) \) has a crest or a trough. Thus, whereas \( f \) had a crest or a trough at \( n \) distinct points in \( \mathbb{R} \), \( D(f) \) has a crest or a trough at \( n + 1 \) distinct points in \( \mathbb{R} \). Hence the total number of crests and troughs has increased under differentiation from \( n \) to \( n + 1 \).

In a very entertaining article [18], Professor Robert Strichartz described “How to make wavelets”. Any readers of his article who may have wished to go even further, by making waves, may be reassured to know that both differentiation and taking finite differences of functions suffice not only to make wavelets, but waves as well! In fact, in a certain precise sense, the procedures of differentiation and taking first order differences produce exactly the same space of functions.

**THEOREM 1.** A function in \( L^2(\mathbb{R}) \) is the derivative of a function in \( L^2(\mathbb{R}) \) if and only if it is a finite sum of first order differences.

Here, as explained in Section 2, the derivative of an \( L^2(\mathbb{R}) \) function is taken in the sense of the theory of distributions, as developed by Laurent Schwartz [17]. If a function in \( L^2(\mathbb{R}) \) is differentiable in the usual sense and has a derivative in \( L^2(\mathbb{R}) \), the usual derivative equals the distributional derivative.

When higher order derivatives are considered, it might be expected that these are expressible in terms of higher order differences, and this is indeed the case, as now indicated. A **second order difference** is a function in \( L^2(\mathbb{R}) \) which is of the form \( f - \frac{1}{2}(\delta_y + \delta_{-y}) * f \), for some \( f \in L^2(\mathbb{R}) \) and \( y \in \mathbb{R} \). Figures 4 and 5 indicate that taking the second derivative, or taking a second order difference, again “makes waves” in a similar sense to taking one derivative or a first order difference. More specifically, taking either the second derivative of \( e^{-x^2} \) or the given second order difference of \( e^{-x^2} \) produces an extra trough as occurred in the case of first derivatives and differences, but produces as well an additional crest or trough.

**THEOREM 2.** A function in \( L^2(\mathbb{R}) \) is the second derivative of some function in \( L^2(\mathbb{R}) \) if and only if it is a finite sum of second order differences.

Theorem 1 can be regarded as describing the range of the differentiation oper-
Figure 4. The function \( x \mapsto 2(2x^2 - 1)e^{-x^2} \), the second derivative of \( x \mapsto e^{-x^2} \).

Figure 5. The function \( x \mapsto e^{-x^2} - 2^{-1}\left( e^{-((x+1)^2 + e^{-((x-1)^2)}} \right) \), a second order difference obtained from \( x \mapsto e^{-x^2} \).

ator \( D \) on the subspace of \( L^2(\mathbb{R}) \) comprising those functions whose derivatives are in \( L^2(\mathbb{R}) \). Theorem 2 gives a corresponding description of the range of \( D^2 \). One aim of this paper is to present a more general description of the relationship between derivatives and differences, and this description depends upon Fourier transform techniques. The crux of the matter is indicated by the following observation: if \( g \) is in \( L^2(\mathbb{R}) \), and if either \( g \) is the derivative of a function in \( L^2(\mathbb{R}) \) or if \( g \) is a first order difference, then the Fourier transform \( \hat{g} \) of \( g \) has a precise behaviour near the origin,
as expressed by the fact that \( \int |\hat{g}(x)|^2 |x|^{-2} dx < \). In fact, as shall be seen, this behaviour characterizes both the derivatives of \( L^2(\mathbb{R}) \)-functions and the functions which are finite sums of first order differences, which leads to the conclusion of Theorem 1.

The classical definition of the derivative suggests that the derivative of a function at a point may be approximated by the value of a first order difference at that point. This idea underlies various methods for calculating approximate solutions of differential equations. A typical illustration of this idea is the following: points are selected and the derivatives of the solution of the equation at these points are approximated by finite difference expressions – these approximations are substituted into the original equation, and the new equation so obtained may be solved to obtain approximate values of the solution at the selected points. Such methods are discussed, for example, in Chapter 5 of the book by Uri Ascher, Robert Mattheij and Robert Russell [1]. The underlying idea in these methods is the local approximation of derivatives by finite difference expressions. In contradistinction, a main theme here is to describe the relationship between derivatives of functions and finite differences when these are considered as functions on \( \mathbb{R} \) – that is, the concern is with how derivatives of functions, and finite differences of functions, are related globally to each other.

REFERENCES


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