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Abstract. Many results in analysis are stated most naturally in terms of measurable sets. For example, if A is measurable subset of $\mathbb{T}$ which is invariant under an irrational rotation, then A has measure 0 or 1. However, the fact that such results assume a knowledge of measure theory creates a barrier to the student who has not encountered measure theory before.

The approach in the talk is to consider the problem of finding the *outer* measure of an invariant set of an irrational rotation on $\mathbb{T}$. This avoids the need to consider the notion of measurable set. Rather, we are led to the Carathédory definition of a measurable set by the problem of trying to say something about the outer measures of the invariant sets of an irrational rotation.

Caratheodory's definition. Let μ_* be an *outer measure* on a set X. That is,

(i) $\mu_*(\emptyset) = 0$,

(ii) $\mu_*(C) \in [0, \infty]$ for all $C \subseteq X$, and

(iii) if (C_n) is a sequence of subsets of X, then

$$\mu_*(\cup_{n=1}^{\infty} C_n) \leq \sum_{n=1}^{\infty} \mu_*(C_n).$$

Various definitions of the notion of a measurable set have been used, but perhaps the most famous is the one given by Carathéodory in 1914.

A subset A of X is called *measurable* if, for all subsets B of X,

$$\mu_*(B) = \mu_*(A \cap B) + \mu_*(A^c \cap B).$$

(A^c denotes the complement of A.)

Carathéodory's definition has frequently been the subject of comment or defence in a way unusual for the definition of a mathematical concept.

Edwin Hewitt and Karl Stromberg write:

> How Carathéodory came to think of this definition seems mysterious, since it is not in the least intuitive. Carathéodory's definition has many useful implications.

Paul Halmos comments

> It is rather difficult to get an understanding of the meaning of ...... measurability except through familiarity with its implications...The greatest justification of this apparently complicated concept is, however, its possibly surprising but absolutely complete success as a tool in proving the important and useful extension theorem.

The ergodicity problem on the unit circle

The unit circle, denoted by $\mathbb{T}$, is

$$\mathbb{T} \;=\; \{z : z \in \mathbb{C} \;\; \text{and} \;\; |z| = 1\}.$$

A function $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ such that for some $z \in \mathbb{T}$

$$\rho(w) = zw, \quad \text{for all} \quad w \in \mathbb{T}.$$

is called a *rotation*. When z is a root of unity, the rotation is called *rational* otherwise it is *irrational*. Let ρ be a rotation. The subset A of $\mathbb{T}$ is *invariant* for ρ if $\rho(A) = A$.

Theorem 1. *Let $\alpha = p/q$ be a rational number in $[0, 1)$, where $p, q \in \mathbb{N}$ and have no common factors. Let $z = \exp(2\pi i p/q)$, and let $\rho(w) = zw$ for all $w \in \mathbb{T}$. Let A be a subset of $\mathbb{T}$. Then the following are equivalent.*

(1) A is ρ-invariant.

(2) There is a subset B of the arc $\{\exp(2\pi i t) : 0 \le t < 1/q\}$ such that $A = \cup_{j=1}^{q} z^{j-1} B$.

Theorem 2. *Let ρ be an irrational rotation. If A is a measurable and ρ-invariant subset of the unit circle then A has measure* 0 *or* 1*.*

That is, an irrational rotation on the unit circle is *ergodic*. There are two approaches for proving an irrational rotation ρ is ergodic:

- Fourier analysis in $L^2(\mathbb{T})$

- Points of density in an invariant set

Both assume a background in measure theory.

The approach here is to consider the problem of finding the *outer* measure of an invariant set of an irrational rotation on $\mathbb{T}$. By considering only the outer measure, we avoid the need to consider the notion of measurable set.

Definitions. If J is an arc of $\mathbb{T}$, $\mu(J)$ denotes its length. If $A \subseteq \mathbb{T}$, the *outer measure* of A is $\mu_*(A)$ and equals the infimum of

$$\left\{ \sum_{n=1}^{\infty} \mu(J_n) : A \subseteq \bigcup_{n=1}^{\infty} J_n \text{ and each } J_n \text{ is an arc} \right\}$$

Theorem 3. *Let ρ be an irrational rotation on $\mathbb{T}$ and let A be a ρ-invariant subset of $\mathbb{T}$. Suppose that there is $\theta < 2$ such that for all arcs J of $\mathbb{T}$,*

$$\mu_*(A \cap J) + \mu_*(A^c \cap J) \leq \theta \mu(J). \qquad (*)$$

Then, either A or A^c is a set of outer measure zero.

Note that it is *always* the case that

$$\mu_*(J) \leq \mu_*(A \cap J) + \mu_*(A^c \cap J) \leq 2\mu(J).$$

Hence, the "simplest" way in which $(*)$ may be satisfied is to have

$$\mu(A \cap J) + \mu_*(A^c \cap J) = \mu(J),$$

for all arcs J of $\mathbb{T}$. However, if A is any set for which this happens, it happens also that

$$\mu_*(A \cap B) + \mu_*(A^c \cap B) = \mu(B), \qquad (**)$$

for *all* subsets B of $\mathbb{T}$, not just for arcs, so that A must be measurable in this case. So, the problem of calculating the outer measure of an invariant set of an irrational rotation has motivated the definition of the notion of a measurable set.

Idea of the proof. In an analysis talk, we let $\varepsilon > 0$. Assume that $\mu_*(A) > 0$ and $\mu_*(A^c) > 0$. Then, there are arcs J, K of $\mathbb{T}$ such that

$$\mu_*(A \cap J) \geq (1-\varepsilon)\mu(J) \quad \text{and}$$

$$\mu(A^c \cap K) \geq (1-\varepsilon)\mu(K).$$

These arcs can be chosen to have equal length. Adding gives

$$\mu_*(A \cap J) + \mu_*(A^c \cap K) \geq 2(1-\varepsilon)\mu(J).$$

Rotating in the second term on the left of the inequality gives

$$\mu_*(A \cap J) + \mu_*(A^c \cap J) \geq 2(1-2\varepsilon)\mu(J).$$

The proof of Theorem 3 is based on an elementary notion of "arcs of density", and should be compared with the proof of ergodicity of an irrational rotation given by Sinai which is based on the standard notion of "points of density". The notion of an *arc* of density can be regarded as a crude notion of a *point* of density, and the proof here may be regarded as a cruder approach than Sinai's to the problem of the ergodic behaviour of an irrational rotation. But it is precisely the use of these cruder tools which has led to the additional information and to Carathéodory's definition of a measurable set.

Now, if A is a ρ_z-invariant set which satisfies $(*)$, the theorem implies that either A or A^c is a set of outer measure zero. It then follows that A must be measurable. Thus, a ρ_z-invariant set A which satisfies $(*)$ must be measurable, so that in fact it satisfies condition $(**)$ which is formally stronger than $(*)$.

The theorem leaves open the question as to whether, for some given irrational rotation ρ, there exits a ρ-invariant, non-measurable set. However, Peter Nickolas has given an example of a set A which is ρ-invariant and non-measurable. Such a set must have the property that $\mu_*(A) > 0$ and $\mu_*(A^c) > 0$.

Any non-measurable set A which is also invariant for some irrational rotation must have the property that $\mu_*(A) = \mu_*(A^c) = 1$, as shown in the following result.

Theorem 4. *Let ρ be an irrational rotation on $\mathbb{T}$ and let A be a subset of $\mathbb{T}$ which is ρ-invariant. Then if A is non-measurable, $\mu_*(A) = \mu_*(A^c) = 1$.*

Idea of the proof. If A is not measurable, for each $1 < \eta < 2$, there is an arc J such that

$$\mu_*(A \cap J) + \mu_*(A^c \cap J) \geq \eta \ \mu(J). \qquad (***)$$

We can show that this arc J may have a length as small as we please. Then, if K is any arc of the form $\rho^{[n]}(J)$, $(***)$ will hold with K in place of J, because A is ρ-invariant and the outer measure does not change under rotations. By taking a suitable finite disjoint union of such arcs K, we get a set B such that $\mu(B) > \eta - 1$ and

$$\mu_*(A \cap B) + \mu_*(A^c \cap B) \geq \eta \ \mu(B).$$

We deduce that

$$\begin{aligned} \mu_*(A) + \mu_*(A^c) &\geq \mu_*(A \cap B) + \mu_*(A^c \cap B) \\ &\geq \eta \ \mu(B) \\ &\geq \eta(\eta - 1). \end{aligned}$$

Since this is true for all $1 < \eta < 2$, it follows that

$$\mu_*(A) + \mu_*(A^c) = 2.$$

Since $\mu_*(A) \leq 1$ and $\mu_*(A^c) \leq 1$, this gives $\mu_*(A) = \mu_*(A) = 1$. $\square$

The standard way of looking at the invariant sets of an irrational rotation is to restrict attention to the measurable sets. When the set A is measurable, denote the outer measure $\mu_*(A)$ of A by $\mu(A)$.

Theorem 5. *Let z be an element of $\mathbb{T}$ which is not a root of unity, and let A be a measurable subset of $\mathbb{T}$ which is ρ_z-invariant. Then either $\mu(A) = 0$ or $\mu(A) = 1$.*

Proof. Theorem 4 shows that $\mu(A) = 0$ or $\mu(A^c) = 0$. If $\mu(A) = 0$, there is nothing to prove; while if $\mu(A^c) = 0$, the additivity of μ on the measurable sets gives

$$\mu(A) = \mu(\mathbb{T}) - \mu(A^c) = 1 - 0 = 1.$$

□

Theorem 5 is the standard way of stating that an irrational rotation on the unit circle is ergodic, and assumes the knowledge of what a measurable set is and the knowledge that the outer measure is additive when it is restricted to the measurable sets. Theorem 3 can be viewed as the result which can be obtained in place of Theorem 5 when one does not assume any knowledge of measurable sets, nor any knowledge of the additivity of the outer measure on the measurable sets.

The approach to ergodicity, discussed here for rotations, has been carried out in terms of the outer measure only. This approach may also be carried out in some other contexts. For example, let ξ be the function on $\mathbb{T}$ given by $z \longmapsto z^2$. Then the following result is a standard one for *measurable* sets.

Theorem 6. *If A is a subset of $\mathbb{T}$ which is ξ-invariant, then $\mu_*(A) = 0$ or $\mu_*(A) = 1$.*

Note that if the ξ-invariant set A in Theorem 6 is non-measurable, then A^c is also non-measurable. Then, because any set of outer measure zero is measurable it follows that, in this case, $\mu_*(A) = \mu_*(A^c) = 1$.

It is interesting to note the difference in behaviour between an irrational rotation ρ and the function ξ. Whereas the conclusion in Theorem 3 that $\mu_*(A) = 0$ or $\mu_*(A^c) = 0$ depends upon some assumption about the ρ_z-invariant subset A which goes beyond the mere fact that it is a subset, the conclusion in Theorem 6 that $\mu_*(A) = 0$ or $\mu_*(A) = 1$ requires no particular assumption about the ξ-invariant set A. This difference can be seen as arising from the facts that ρ "preserves" the distance between points, but ξ "stretches" the distance between points. Similar differences are seen also in other contexts. For example, ρ is not weak mixing but ξ is strong mixing.

Also, when $[0, 1)$ is identified with $\mathbb{T}$ under the map $t \longmapsto \exp(2\pi i t)$, and when ρ is then regarded as a (discontinuous) function on $[0, 1)$ instead of $\mathbb{T}$, then ρ has a "weak" form of chaotic behaviour compared with more familiar examples of chaos, such as those exemplified by the function ξ when *it* is also regarded as a function on $[0, 1)$. These differences in behaviour are also related to the one-to-one nature of ρ, while ξ is not one-to-one.