MATH141 – Autumn 2008
Outline Solutions to Tutorial Sheet – Week 13

1. If $x = 0$ and $y = 0$, then $2 \times 0 - 2 \times 0 + z = 4$, so $z = 4$; hence a point on the plane is $(0, 0, 4)$. A normal to the plane is $(2, -2, 1)$. Therefore

$$\begin{aligned}\text{Distance} &= ((1, -2, 3) - (0, 0, 4)) \cdot \frac{(2, -1, 1)}{\sqrt{4 + 4 + 1}} \\ &= \frac{5}{3} \text{ units}\end{aligned}$$

2.

$$\begin{aligned}\mathcal{I} &= \int \left(\frac{12}{x} + x^5 \right) \mathrm{d}x, \\ &= 12 \ln x + \frac{x^6}{6} + x,\end{aligned}$$

where c is an arbitrary constant.

3. (a) Any plane containing the line of intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$ will be given by

$$x + y + z - 2 + \lambda (y - z + 1) = 0.$$

Now the point $A\,(1, 0, 0)$ lies on $\mathcal{P}_3$, so it must satisfy the equation for the plane:

$$1 + 0 + 0 - 2 + \lambda (0 - 0 + 1 = 0) \Rightarrow -1 + \lambda = 0 \Rightarrow \lambda = 1.$$

Hence, the plane $\mathcal{P}_3$ is given by

$$x + y + z - 2 + (y - z = 1) = 0 \Rightarrow x + 2y - 1 = 0$$

Therefore, $\mathcal{P}_3 : x + 2y - 1 = 0$.

(b) The angle between $\mathcal{P}_1$ and $\mathcal{P}_2$ is the angle between their normals.

$$\mathcal{P}_1 \text{ has normal } \quad \underset{\sim}{n_1} = (1, 1, 1)\,; \qquad \mathcal{P}_1 \text{ has normal } \quad \underset{\sim}{n_2} = (0, 1, -1)$$

$$\text{Now,} \quad \cos\theta = \frac{\underset{\sim}{n_1} \cdot \underset{\sim}{n_2}}{n_1 n_2} \quad \text{and} \quad \underset{\sim}{n_1} \cdot \underset{\sim}{n_2} = (1, 1, 1) \cdot (0, 1, -1) = 0.$$

$$\Rightarrow \theta = \frac{\pi}{2}; \quad \text{i.e., the normals are perpendicular to each other.}$$

Therefore, $\mathcal{P}_1 \perp \mathcal{P}_2$.

4.

$$\mathcal{I} = \int \frac{1}{x\sqrt{10 - x^2}} \, \mathrm{d}x,$$

Use formula 22 from the table of integrals

$$\int \frac{1}{x\sqrt{a^2 - x^2}} \mathrm{d}x = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + c$$

Here $a = \sqrt{10}$. Thus

$$\mathcal{I} = -\frac{1}{\sqrt{10}} \ln \left| \frac{\sqrt{10} + \sqrt{10 - x^2}}{x} \right| + c,$$

where c is an arbitrary constant.

5. In parametric form, the equations of the line are $x = 3 + 8t, y = 4 + 5t$ and $z = -3 - t$.
Substituting into the equation of the plane, we have $(3 + 8t) - 3(4 + 5t) + 5(-3 - t) = 12$, giving $t = -3$.
The point of intersection is therefore $(3 - 24, 4 - 15, -3 - (-3)) = (-21, -11, 0)$.

6.

$$\mathcal{I} = \int_e^{3e} \sqrt{x^2 + e^2}\,\mathrm{d}x.$$

Use formula 28 from the table of integrals.

$$\int \sqrt{x^2 \pm e^2}\,\mathrm{d}x = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right| + c.$$

We have $a = e$. First of all write down the indefinite integral.

$$\int \sqrt{x^2 + e^2}\,\mathrm{d}x = \frac{1}{2}x\sqrt{x^2 + e^2} + e^2 \ln\left|x + \sqrt{x^2 + e^2}\right| + c.$$

Thus

$$\begin{aligned}
\mathcal{I} &= \left[\frac{1}{2}x\sqrt{x^2 + e^2} + e^2 \ln\left|x + \sqrt{x^2 + e^2}\right|\right]_e^{3e}, \\
&= \left(\frac{3e}{2}\sqrt{10e^2} + \frac{1}{2}e^2 \ln\left|3e + \sqrt{10e^2}\right|\right) - \left(\frac{e}{2}\sqrt{2e^2} + \frac{1}{2}e^2 \ln\left|e + \sqrt{2e^2}\right|\right), \\
&= \left(\frac{3\sqrt{10}}{2}e^2 + \frac{1}{2}e^2 \ln\left|\left[3 + \sqrt{10}\right]e\right|\right) - \left(\frac{\sqrt{2}}{2}e^2 + \frac{1}{2}e^2 \ln\left|\left[1 + \sqrt{2}\right]e\right|\right), \\
&= \frac{1}{2}\left(3\sqrt{10} - \sqrt{2}\right)e^2 + \frac{1}{2}e^2 \ln\left(\frac{3 + \sqrt{10}}{1 + \sqrt{2}}\right).
\end{aligned}$$

7.

$$\mathcal{I} = \int \frac{1}{c \ln c}\,\mathrm{d}c,$$

Use the substitution $u = \ln c$. Then $\frac{\mathrm{d}u}{\mathrm{d}c} = \frac{1}{c}$ and consequently $\mathrm{d}c = c\mathrm{d}u$.

$$\begin{aligned}
\mathcal{I} &= \int \frac{1}{u}\,\mathrm{d}u, \\
&= \ln|u| + m, \\
&= \ln|\ln c| + m,
\end{aligned}$$

where m is an arbitrary constant.

8.

$$
\begin{aligned}
\mathcal{I} &= \int \tan x \, \mathrm{d}x \\
&= \int \frac{\sin x}{\cos x} \mathrm{d}x, \\
\text{Let } u &= \cos x, \\
\text{Then } \frac{\mathrm{d}u}{\mathrm{d}x} &= -\sin x \Rightarrow \mathrm{d}x = -\frac{\mathrm{d}u}{\sin x}, \\
\text{Thus } \mathcal{I} &= \int \frac{\sin x}{u} \cdot -\frac{\mathrm{d}u}{\sin x}, \\
&= -\int \frac{1}{u} \mathrm{d}u, \\
&= -\ln |u| + c, \\
&= -\ln |\cos x| + c, \\
&= \ln |\sec x| + c,
\end{aligned}
$$

where c is an arbitrary constant.