MATH141 – Autumn 2008
Outline Solutions to Tutorial Sheet – Week 6

1.

(a)

$$f(x) = \frac{1}{\sqrt{9-x^2}} \quad \text{is defined when } \sqrt{9-x^2} \text{ is defined and not zero; that is, when}$$

$$\begin{aligned} 9-x^2 &\iff x^2 < 9 \\ &\iff |x| < 3 \quad \text{or} \quad -3 < x < 3. \\ \text{Thus,} \quad &\operatorname{Dom} f = (-3, 3) \end{aligned}$$

(b)

$$g(x) = \frac{x^4-1}{x^3-2x} \quad \text{is defined except when}$$

$$\begin{aligned} x^3 - 2x = 0 &\iff x\left(x^2-2\right) = 0 \\ &\iff x = 0, \pm\sqrt{2}. \\ \text{Thus, } \operatorname{Dom} g &= \left\{x \in \mathbb{R} : x \neq 0, \pm\sqrt{2}\right\} \end{aligned}$$

2.

(a)
$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right) \xrightarrow{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{2} R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\begin{aligned} \text{Therefore, } x_3 &= 2; \\ x_2 &= 5 - 3x_3 = 5 - 6 = -1; \\ x_1 &= 9 + 2x_2 - 3x_3 = 9 - 2 - 6 = 1. \end{aligned}$$

(b)
$$\left(\begin{array}{ccc|c} 2 & 8 & 3 & 2 \\ 1 & 3 & 2 & 5 \\ 2 & 7 & 4 & 8 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & 8 & 3 & 2 \\ 2 & 7 & 4 & 8 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & -1 & -6 \\ 2 & 7 & 4 & 8 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & -1 & -6 \\ 0 & 1 & 0 & -2 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

$$\begin{aligned} \text{Therefore } x_3 &= 4; \\ x_2 &= -6 + x_3 = -2; \\ x_1 &= 5 - 3x_2 - 2x_3 = 5 + 6 - 8 = 3. \end{aligned}$$

3.

$$\begin{aligned} \operatorname{Dom} f &= \left\{x \in \mathbb{R} : x^2 - 1 \geq 0\right\} \\ &= \left\{x \in \mathbb{R} : x^2 \geq 1\right\} \\ &= (-\infty, -1] \cup [1, \infty). \end{aligned}$$

$$\begin{aligned} \operatorname{Dom} g &= \left\{x \in \mathbb{R} : 4 - x^2 > 0\right\} \\ &= \left\{x \in \mathbb{R} : x^2 < 4\right\} \\ &= (-2, 2) \end{aligned}$$

(a) (i) $(f+g)(x) = f(x) + g(x)$

$$= \sqrt{x^2-1} + \frac{1}{\sqrt{4-x^2}}$$

(ii) $(fg)(x) = f(x) \cdot g(x)$

$$= \frac{\sqrt{x^2-1}}{\sqrt{4-x^2}}$$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$= \sqrt{x^2-1} \cdot \sqrt{4-x^2}$$

(b) (i) $\text{Dom}(f+g) = \{x : x \text{ in } \text{Dom} f \text{ and } x \in \text{Dom} g\}$

$$= (-2,-1] \cup [1,2)$$

(ii) $\text{Dom}(fg) = \text{Dom} f \cap \text{Dom} g$

$$= (-2,-1] \cup [1,2).$$

(iii) $\text{Dom}\left(\frac{f}{g}\right) = \text{Dom} f \cap \text{Dom} g$

$$= (-2,-1] \cup [1,2), \text{since } g(x) \text{ is never } 0.$$

4.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right) \xrightarrow{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 4R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$

(a) There will be no solution if $a^2 - 16 = 0$ but $a - 4 \neq 0$, i.e., if $a = -4$.

(b) There will be a unique solution if $a^2 - 16 \neq 0$, i.e., $a \neq \pm 4$.

(c) There will be infinitely many solutions if $a^2 - 16 = 0$ and $a - 4 = 0$, i.e., if $a = 4$.

5. (a) $\text{Dom} f = \{x \in \mathbb{R} : x \geq 0\}$; $\text{Dom} g = \left\{x \in \mathbb{R} : x \neq \frac{1}{2}\right\}$

(b)

(i) $(f+g)(x) = \sqrt{x} + \frac{1}{2x-1}$ (ii) $(fg)(x) = \frac{\sqrt{x}}{2x-1}$ (iii) $\left(\frac{g}{f}\right)(x) = \frac{1}{\sqrt{x}(2x-1)}$

(iv) $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{1}{2x-1}\right)$$

$$= \sqrt{\frac{1}{2x-1}}$$

$$= \frac{1}{\sqrt{2x-1}}$$

(v) $(g \circ f)(x) = g(f(x))$

$$= g(\sqrt{x})$$

$$= \frac{1}{2\sqrt{x}-1}$$

(c) (i) $\text{Dom}(f+g) = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ (ii) $\text{Dom}(fg) = \text{Dom} f + g$

$$\text{(iii)}\quad \operatorname{Dom}\left(\frac{g}{f}\right) = \operatorname{Dom} f \cap \operatorname{Dom} g - \{x : f(x) = 0\}$$
$$= \operatorname{Dom} f \cap \operatorname{Dom} g - \{0\}$$
$$= \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$$

$$\text{(iv)}\quad \operatorname{Dom} f \circ g = \{x : x \in \operatorname{Dom} g \quad \text{and} \quad g(x) \in \operatorname{Dom} f\}$$
$$= \left\{x \in \mathbb{R} : x \neq \frac{1}{2} \quad \text{and} \quad \frac{1}{2x-1} \geq 0\right\}$$
$$= \left\{x \in \mathbb{R} : x \neq \frac{1}{2} \quad \text{and} \quad x \geq \frac{1}{2}\right\}$$
$$= \left\{x \in \mathbb{R} : x > \frac{1}{2}\right\}$$

$$\text{(v)}\quad \operatorname{Dom} g \circ f = \{x : x \in \operatorname{Dom} f \quad \text{and} \quad f(x) \in \operatorname{Dom} f\}$$
$$= \left\{x \in \mathbb{R} : x \geq 0 \quad \text{and} \quad \sqrt{x} \neq \frac{1}{2}\right\}$$
$$= \left\{x \in \mathbb{R} : x \geq 0 \quad \text{and} \quad x \neq \frac{1}{4}\right\}$$
$$= \left[0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \infty\right)$$

6.

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & -3 & -2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to -R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & -3 & -2 & 1 & -2 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & -8 & 1 & -2 & -3 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{8}R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{array}\right) \xrightarrow{R_2 \to R_2 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{8} & \frac{3}{4} & -\frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{array}\right)$$

$$(a) \quad \text{The inverse exists and}$$

$$A^{-1} = \begin{pmatrix} \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix}$$

$$\text{Check} \quad AA^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(b) \quad \underset{\sim}{x} = A^{-1}\underset{\sim}{b} = \begin{pmatrix} \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix}$$

7. $f(x) = x^2 + 1$, $\text{Dom}\, f = \mathbb{R}$; $g(x) = \dfrac{1}{x^2+1}$, $\text{Dom}\, g = \mathbb{R}$.

$$(a) \quad (f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2+1}\right) = \left(\frac{1}{x^2+1}\right)^2 + 1$$

$\text{Dom}\, f \circ g = \mathbb{R}$

$$(b) \quad (g \circ f)(x) = g(f(x)) = g\left(x^2+1\right) = \frac{1}{(x^2+1)^2 + 1}$$

$\text{Dom}\, g \circ f = \mathbb{R}$

8.

$$(A\,|I) = \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -2 & -1 & 1 & 0 \\ 0 & 5 & -5 & -4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{3}R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 5 & -5 & -4 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 5R_2]{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{5}{3} & -\frac{7}{3} & -\frac{5}{3} & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{3}{5}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{7}{5} & 1 & -\frac{3}{5} \end{array}\right) \xrightarrow[R_2 \to R_2 + \frac{2}{3}R_3]{R_1 \to R_1 - \frac{4}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{6}{5} & -1 & \frac{4}{5} \\ 0 & 1 & 0 & \frac{3}{5} & 1 & -\frac{2}{5} \\ 0 & 0 & 1 & \frac{7}{5} & 1 & -\frac{3}{5} \end{array}\right)$$

Hence

$$A^{-1} = \begin{pmatrix} -\frac{6}{5} & -1 & \frac{4}{5} \\ \frac{3}{5} & 1 & -\frac{2}{5} \\ \frac{7}{5} & 1 & -\frac{3}{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -6 & -5 & 4 \\ 3 & 5 & -2 \\ 7 & 5 & -3 \end{pmatrix}.$$

9.

$$
\begin{aligned}
(a)\quad f(x) &= \frac{1}{1+x^2},\\
\text{Dom}\, f &= \mathbb{R}\\
g(x) &= \tan x,\\
\text{Dom}\, g &= \left\{x \in \mathbb{R} : x \neq \pm\frac{\pi}{x}, \pi\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}\\
(f \circ g)(x) &= f(g(x))\\
&= f(\tan x)\\
&= \frac{1}{1+\tan^2 x}\\
&= \frac{1}{\sec^2 x}\\
&= \cos^2 x\\
\text{Dom}\,(f \circ g) &= \{x : x \in \text{Dom}\, g \quad \text{and} \quad g(x) \in \text{Dom}\, f\}\\
&\text{However,}\quad \text{Dom}\, h = \mathbb{R} \neq \text{Dom}\, f \circ g\\
&\therefore, f \circ g \neq h.
\end{aligned}
$$

$$
\begin{aligned}
(b)\quad h(x) &= \cos x.\\
&\text{Now}\quad 1-\cos^2 x = \sin^2 x, \quad \text{so} \quad (f \circ h)(x) = 1 - (h(x))^2.\\
\text{Let}\quad f(x) &= 1 - x^2. \quad \text{Then} \quad (f \circ h)(x) = 1 - \cos^2 x = \sin^2 x
\end{aligned}
$$

10.

$$
A\,|I|\,\underset{\sim}{b} = \left(\begin{array}{ccc|ccc|} 3 & 2 & 2 & 1 & 0 & 0\\ 2 & 2 & 2 & 0 & 1 & 0\\ -4 & 4 & 3 & 0 & 0 & 1 \end{array} \; -1 \;\; 2 \;\; 0\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc|} 1 & 0 & 0 & 1 & -1 & 0\\ 2 & 2 & 2 & 0 & 1 & 0\\ -4 & 4 & 3 & 0 & 0 & 1 \end{array} \; -3 \;\; 2 \;\; 0\right)
$$

$$
\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc|} 1 & 0 & 0 & 1 & -1 & 0\\ 0 & 2 & 2 & -2 & 3 & 0\\ -4 & 4 & 3 & 0 & 0 & 1 \end{array} \; -3 \;\; 8 \;\; 0\right) \xrightarrow{R_2 \to \frac{1}{2}R_2} \left(\begin{array}{ccc|ccc|} 1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 1 & -1 & \frac{3}{2} & 0\\ -4 & 4 & 3 & 0 & 0 & 1 \end{array} \; -3 \;\; 4 \;\; 0\right)
$$

$$
\xrightarrow{R_3 \to R_3 + 4R_1} \left(\begin{array}{ccc|ccc|} 1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 1 & -1 & \frac{3}{2} & 0\\ 0 & 4 & 3 & 4 & -4 & 1 \end{array} \; -3 \;\; 4 \;\; 12\right) \xrightarrow{R_3 \to R_3 - 4R_2} \left(\begin{array}{ccc|ccc|} 1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 1 & -1 & \frac{3}{2} & 0\\ 0 & 0 & -1 & 8 & 10 & 1 \end{array} \; -3 \;\; 4 \;\; -28\right)
$$

$$
\xrightarrow{R_3 \to (-1)R_3} \left(\begin{array}{ccc|ccc|} 1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 1 & -1 & \frac{3}{2} & 0\\ 0 & 0 & 1 & -8 & -10 & -1 \end{array} \; -3 \;\; 4 \;\; 28\right) \xrightarrow{R_2 \to R_2 - 4R_3} \left(\begin{array}{ccc|ccc|} 1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 0 & 7 & -\frac{17}{2} & 1\\ 0 & 0 & 1 & -8 & -10 & -1 \end{array} \; -3 \;\; -24 \;\; 28\right)
$$

(a) The inverse exists: $A^{-1} = \begin{pmatrix} 1 & -1 & 0\\ 7 & -\frac{17}{2} & 1\\ -8 & -10 & -1 \end{pmatrix}$

(b) The solution is given by $\underset{\sim}{b} = \begin{pmatrix} -3 & -24 & 28 \end{pmatrix}$.

11. (a) In matrix form, we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{aligned} A^{-1} \text{ exists, so } \quad \underset{\sim}{x} &= A^{-1}\underset{\sim}{b} \\ &= \begin{pmatrix} 0 & 2 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\ &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

(b) In matrix form, we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}.$$

$$\begin{aligned} A^{-1} \text{ exists, so } \quad \underset{\sim}{x} &= A^{-1}\underset{\sim}{b} \\ &= \begin{pmatrix} 0 & 2 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \\ &= \begin{pmatrix} 13 \\ -10 \\ -11 \end{pmatrix} \end{aligned}$$