Fundamentals Lecture Seven

Remainder Theorem : polynomial division. 1.4.9 (page 1-14)

- Determine the remainder when the polynomial $x^3 + 5x^2 - 2x - 24$ is divided by the polynomial $x - 1$.

Factor Theorem : Finding factors of polynomials. 1.4.9 (page 1-14)

- Show that $(x - 3)$ is a factor of $x^3 - 5x^2 - 2x + 24$ and hence determine the remaining factors.

Consider out previous divisions

i $(x^2 + 5x - 36) \div (x - 4) = x + 9$

ii $\frac{x^3-2x^2-x+2}{x+1} = x^2 - 3x + 2$

iii $(x^3 + 1) \div (x + 1) = x^2 - x + 1$

iv $\frac{3x^3-7x^2+2x+4}{x-3} = 3x^2 + 2x + 8 + \frac{28}{x-3}$

Note that in each case when our given polynomial is divided by $x - a$ the quotient (or answer) is a polynomial of *one degree less* than the original together with a remainder (which was zero in (i), (ii) and (iii) and 28 in (iv)). We can write this as an equation:

$$\frac{f(x)}{x-a} = q(x) + \frac{r}{x-a} \quad \text{OR}$$
$$f(x) = (x - a)\, q(x) + r$$

where $f(x)$ is our given polynomial, $q(x)$ is the polynomial produced by the division and r is the remainder.

If we let $x = a$ in

$$\begin{aligned} f(x) &= (x-a)\,q(x) + r \quad \text{we get} \\ f(a) &= (a-a)\,q(a) + r \\ &= r \quad \text{the remainder} \end{aligned}$$

So, to find the remainder resulting from a division we don't have to perform the division we can simply use the rule

$$f(a) = r$$

For example, we know that $\left(3x^3 - 7x^2 + 2x + 4\right) \div (x-3)$ results in a remainder of 28. We will check if we get the same remainder using this new rule

$$
\begin{aligned}
\text{Now,} \quad f(x) &= 3x^3 - 7x^2 + 2x + 4 \\
\text{and} \quad x - 3 &= x - a \Rightarrow a = 3 \\
f(3) &= \\
&= \\
&= 28
\end{aligned}
$$

Example Determine the remainder in each of the following cases.

1. $(x^3 + 5x^2 - 2x - 24) \div (x - 1)$.

   $a =$ ___

   $\therefore r = f(\qquad)$

   $=$

   $=$

   $=$

2. $(x^3 + 5x^2 - 2x - 24) \div (x + 1)$

   $a =$ ___

   $\therefore r = f(\qquad)$

   $=$

   $=$

   $=$

-20

-18

3. $(x^3 + 5x^2 - 2x - 24) \div (x - 2)$

$a =$ ___

$\therefore r = f(\qquad)$

$=$

$=$

$=$

Key point 1: Let $\mathcal{P}(x)$ be a polynomial. Then the remainder when $\mathcal{P}(x)/x - a$ is $\mathcal{P}(a)$.

Key point 2: If $\mathcal{P}(a) = 0$ then $x - a$ is a *factor* of $\mathcal{P}(x)$.

0

Factor Theorem Why is being able to find the remainder easily important? In our example

$$1224 \div 8 = 153$$

the *remainder is zero* and so

$$1224 = 8 \times 153$$

i.e. 8 and 153 are *factors* of 1224.

Similarly, (i), (ii) and (iii) have *remainders of zero*. So,

i $x^2 + 5x - 36$ has *factors* $(x - 4)$ *and* $(x + 9)$

ii $x^3 - 2x^2 - x + 2$ has *factors* $(x + 1)$ *and* $\left(x^2 - 3x + 2\right)$

iii $x^3 + 1$ has *factors* $(x + 1)$ *and* $\left(x^2 - x + 2\right)$

Therefore, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$ For example, if

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 14x + 12 \\ \text{then } f(2) &= 2^3 + 2.2^2 - 14.2 + 12 \\ &= 0. \end{aligned}$$

Therefore, $(x - 2)$ is a factor of $f(x)$. The remaining factor can be found by long division.

$$x - 2 \,\big|\, \overline{x^3 + 2x^2 - 14x + 12}$$

$$\begin{aligned} \therefore f(x) &= x^3 + 2x^2 - 14x + 12 \\ &= (x - 2) \underline{\qquad\qquad\qquad} \end{aligned}$$

$$\left(x^2 + 4x - 6\right)$$

Example Show that $(x-3)$ is a factor of $f(x) = x^3 - 5x^2 - 2x + 24$ and hence determine the remaining factor(s).

Solution $a =$ ___

$$\therefore \quad r = f(\qquad\quad)$$

$$=$$

$$=$$

$\therefore$ $(x-3)$ is a factor of $f(x)$. To determine further factor(s) use long division

$$x - 3 \,\big|\, \overline{x^3 - 5x^2 - 2x + 24}$$

$$
\begin{aligned}
f(x) &= x^3 - 5x^2 - 2x + 24 \\
&= (x-3)\left(x^2 - 2x - 8\right) \\
&=
\end{aligned}
$$

What happens if we are not given the first factor? Let's look at an example

Example To attempt to find the linear factors of

$$f(x) = x^3 + 5x^2 - 2x - 24$$

we look at the factors of 24

i.e. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8 \pm 12$ and ± 24

By trial and error we substitute these values for x into $f(x)$ until we find a value that gives a zero remainder, i.e. $f(a) = 0$ and so $(x - a)$ is a factor.

$$f(x) = x^3 + 5x^2 - 2x - 24$$

$$f(1) =$$

$$=$$

$\therefore (x-1)$ IS/IS NOT a factor

$$f(-1) =$$

$$=$$

$\therefore (x+1)$ IS/IS NOT a factor

$$f(2) =$$

$$=$$

$\therefore (x-2)$ IS/IS NOT a factor

As before, we use long division to find the other factor(s).

Exercise Verify that

$$x^3 + 5x^2 - 2x - 24 = (x-2)(x+3)(x+4)$$

Example

Factorise $f(x) = 2x^3 - 9x^2 + 7x + 6$.

Solution

1. Find the factors of 6
2. Find one factor by calculating $f(a)$ where a is a factor of 6.
3. Find the remaining factor by long division

Exercises on factoring polynomials

Exercise 1.12.4
page 39.