2.7 INVERSE HYPERBOLIC FUNCTIONS

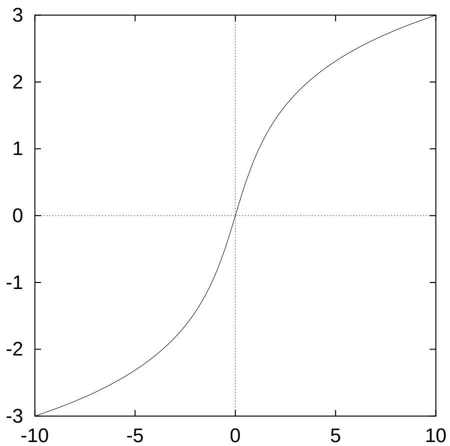
You were asked to read through sections 2.7.1–2.7.3 before the lecture. I will not go through this material in the lecture, but will assume that you know it. Equations (1)–(3) are particularly useful and you are advised to make a note of them.

$$\sinh^{-1} x = \ln\left[x + \sqrt{x^2+1}\right], x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left[x + \sqrt{x^2-1}\right], x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, -1 < x < 1$$

2.7.1 The Inverse Hyperbolic Sine Function



The Graph of $y = \sinh^{-1} x$ (or $y = \operatorname{arcsinh} x$)

Figure 15: $y = \sinh^{-1} x$

From the graph note

1. $\operatorname{Dom} \sinh^{-1} x = \underline{\mathbb{R}}$
2. $\operatorname{Range} \sinh^{-1} x = \underline{\mathbb{R}}$

Definition If $y = \sinh^{-1} x$, then $\underline{x = \sinh y}$, for $x \in \underline{\mathbb{R}}$ and $y \in \underline{\mathbb{R}}$.

Properties

$$\sinh\left(\sinh^{-1} x\right) = x, x \in \mathbb{R}$$

$$\sinh^{-1}(\sinh x) = x, x \in \mathbb{R}$$

$$\sinh^{-1} x = \ln\left[x + \sqrt{x^2 + 1}\right] \quad (1)$$

2.7.2 The Inverse Hyperbolic Cosine Function

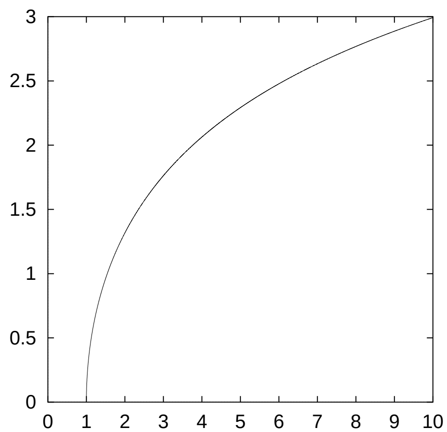


Figure 16: The graph $y = \cosh^{-1} x$

From the graph note

1. Dom $\cosh^{-1} x = \underline{[1, \infty)}$
2. Range $\cosh^{-1} x = \underline{[0, \infty)}$

Definition

If $y = \cosh^{-1} x$, then $\underline{x = \cosh y}$, for $\underline{x \geq 1}$ and $\underline{y \geq 0}$.

Properties

$$\cosh\left(\cosh^{-1} x\right) = x, x \geq 1$$

$$\cosh^{-1}\left(\cosh x\right) = x, x \geq 0$$

$$\cosh^{-1} x = \ln\left[x + \sqrt{x^2 - 1}\right] \quad (2)$$

2.7.3 The Inverse Hyperbolic Tangent Function

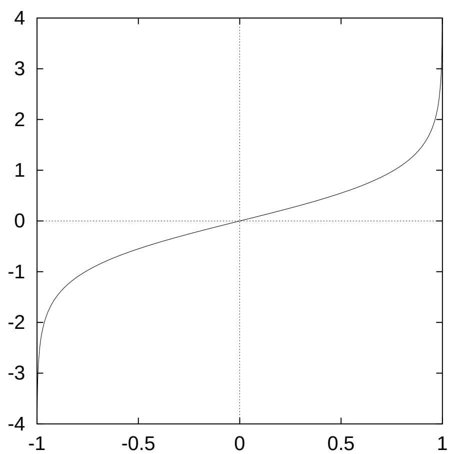


Figure 17: The graph $y = \tanh^{-1} x$

From the graph note

1. $\operatorname{Dom} \tanh^{-1} x = \underline{(-1, 1)}$
2. $\operatorname{Range} \tanh^{-1} x = \underline{\mathbb{R}}$

Definition If $y = \tanh^{-1} x$, then $\underline{x = \tanh y}$, for $-1 < x < 1$ and $y \in \mathbb{R}$.

Properties

$$\tanh\left(\tanh^{-1} x\right) = x, -1 < x < 1$$

$$\tanh^{-1}(\tanh x) = x, x \in \mathbb{R}$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|, -1 < x < 1 \tag{3}$$

2.7.4 Examples

1. Evaluate:

(a) $\sinh 1.275$ (b) $\tanh^{-1} 0.75$

2. Prove that

$\sinh^{-1} x = \ln\left[x + \sqrt{x^2+1}\right]$

(Ex 2.7.6 Q2)

(1a) Evalute $\sinh 1.275$.

(1b) Evaluate $\tanh^{-1} 0.75$

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\sinh 1.275 = \frac{1}{2}\left(e^{1.275} - e^{-1.275}\right)$$

$$\approx 1.649$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

$$\tanh^{-1} 0.75 = \frac{1}{2}\ln\left|\frac{1+0.75}{1-0.75}\right|$$

$$\approx 0.973$$

2. Prove that $\sinh^{-1} x = \ln\left[x + \sqrt{x^2+1}\right]$ (Ex 2.7.6 Q2)

Let $y = \sinh x$ and solve for x to find the inverse.

$$\begin{aligned} y &= \sinh x \\ &= \frac{1}{2}\left(e^x - e^{-x}\right) \\ 2y &= e^x - e^{-x} \\ 2ye^x &= e^{2x} - 1 \\ e^{2x} - 2ye^x - 1 &= 0 \\ \left(e^x\right)^2 - 2ye^x - 1 &= 0 \end{aligned}$$

Let $u = e^x$.

$$\therefore \quad u^2 - 2yu - 1 = 0.$$

$$\begin{aligned} \therefore u &= \frac{-(-2y) \pm \sqrt{(-2y)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{2y \pm \sqrt{4y^2+4}}{2} \\ &= \frac{2y \pm 2\sqrt{y^2+1}}{2} \\ &= y \pm \sqrt{y^2+1} \end{aligned}$$

But $u = e^x$

$$\therefore e^x = y \pm \sqrt{y^2+1}$$

Now, $e^x > 0$ and $y - \sqrt{y^2+1} < 0$

$$\begin{aligned} \therefore \quad e^x &= y + \sqrt{y^2+1} \\ \therefore x &= \ln\left(y + \sqrt{y^2+1}\right) \\ &= f^{-1}(y) \end{aligned}$$

$$\therefore \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

2.7.5 Revision Questions

The following questions are about the key ideas in this section.

1. Sketch the following graphs: (a) $y = \sinh^{-1} x$ (b) $y = \cosh^{-1} x$ (c) $y = \tanh^{-1} x$.
2. What are the domain and range of: (a) $y = \sinh^{-1} x$ (b) $y = \cosh^{-1} x$ (c) $y = \tanh^{-1} x$.
3. Why do we need to restrict the domain of $\cosh^{-1}$ and $\tanh^{-1}$ but not $\sinh^{-1}$?
4. **Exercise 2.7.6** Questions 1–2.