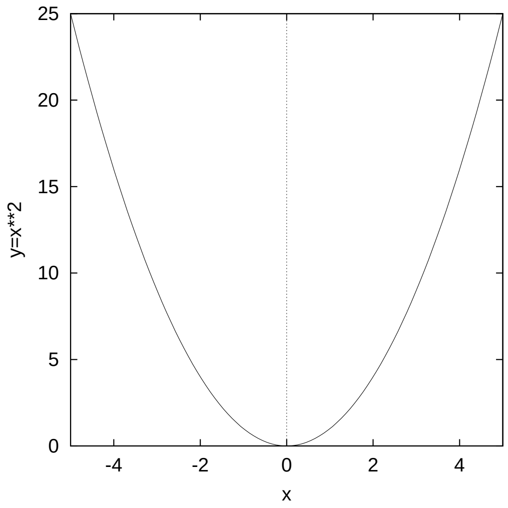
2.5 ONE-TO-ONE AND INVERSE FUNCTIONS

2.5.1 One-to one functions

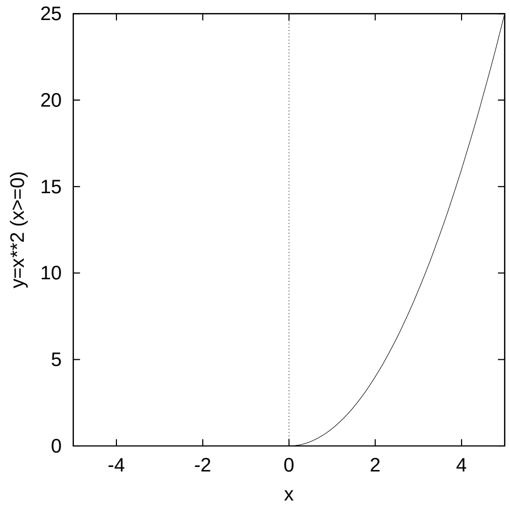
Definition

A function f is one-to-one if its graph is cut only once by any horizontal line.

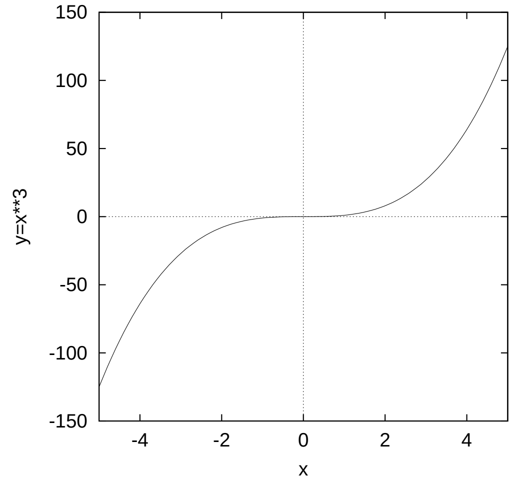
Consider the following graphs:



(a) $f(x)$



(b) $g(x)$



(c) $h(x)$

g and h are examples of one-to-one (often denoted 1-1) functions, f is not 1-1. The functions g and h have for each y value, only one x value.

The function f is a two-to-one function as each y-value has two x-values. e.g. if $y = a^2$, then $x = a$ or $x = -a$. An example of a many-to-one function is $f(x) = \sin x$ e.g. if $y = \frac{1}{2}$, then we have $x = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{13\pi}{6}, \ldots$.

2.5.2 Inverse Functions

The functions $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ have the property that each is the <u>inverse</u> of the other.

$$f(g(x)) = f(\underline{\frac{1}{2}x}) = \underline{2\left(\frac{1}{2}x\right)} = x$$

$$g(f(x)) = g(\underline{2x}) = \frac{1}{2}\underline{2x} = x$$

Similarly, the functions $f(x) = x^{1/3}$ and $g(x) = x^3$ are the reverse of each other.

$$(g(x)) = f(\underline{x^3}) = \underline{\left(x^3\right)^{1/3}} = x$$

$$g(f(x)) = g(\underline{x^{1/3}}) = \underline{\left(x^{1/3}\right)^3} = x$$

We say that f is an inverse of g and g is an inverse of f.
We can also say that f and g are
f and g are inverse functions.
Thus $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ are inverse function s as are $f(x) = x^{1/3}$ and $g(x) = x^3$.

Notation

The *inverse of f* is commonly denoted as f^{-1} (read 'f inverse') thus

$$f\left(f^{-1}(x)\right) = f^{-1}(f(x)) = x$$

So, we can write our inverse functions $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ as

$f(x) = 2x$ and $\underline{f^{-1}(x) = \dfrac{1}{2}x}$ or

$g(x) = \frac{1}{2}x$ and $\underline{g^{-1}(x) = 2x}$.

BEWARE: $f^{-1}(x)$ denotes the inverse of $f(x)$ *not* the reciprocal.

Question. Do all functions have inverses?

Definition A function f has an inverse if and only if it is one-to-one.

2.5.2.1 Exercises Do the following functions have inverses?

(a) $f(x) = x^2, \; x \leq 0$

(b) $g(x) = \sqrt{3x - 2}$

Hint: Sketch the functions.

2.5.3 Domain and Range of an Inverse

Consider the function $f(x) = 2x$, let $\text{Dom}\, f = \underline{\{x : x = 2, 4, 6, 8\}}$ then $\text{Range}\, f = \underline{\{y : y = 4, 8, 12, 16\}}$.

Consider also the function $f^{-1}(x) = \frac{1}{2}x$ and let $\text{Dom}\, f^{-1} = \underline{\{x : x = 4, 8, 12, 16\}}$ then $\text{Range}\, f^{-1} = \underline{\{y : y = 2, 4, 6, 8\}}$.

If we compare we see that

$$\underline{\text{Dom}\, f^{-1}} = \underline{\text{Range}\, f}$$
$$\underline{\text{Range}\, f^{-1}} = \underline{\text{Dom}\, f}$$

This is true for all f and f^{-1}

$$\begin{array}{rcl} \text{Dom}\, f^{-1} & = & \text{Range}\, f \\ \text{Range}\, f^{-1} & = & \text{Dom}\, f \end{array}$$

2.5.3.1 Exercises Find Dom f^{-1} and Range f^{-1} given f

(a) $f(x) = 2x - 4$

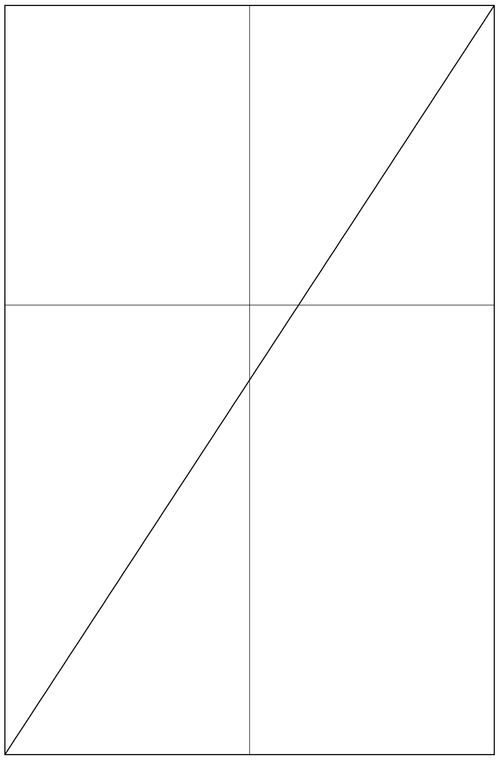
(b) $f(x) = x^2 - 1,\ x \geq 0$

(c) $f(x) = \sqrt{x - 3},\ x > 3$

$$\boxed{\begin{aligned} \text{Dom}\, f^{-1} &= \text{Range}\, f \\ \text{Range}\, f^{-1} &= \text{Dom}\, f \end{aligned}}$$

Find Dom f^{-1} and Range f^{-1}:

(a) $f(x) = 2x - 4$. Draw the function.



$$\text{Dom}\, f(x) = x \in (-\infty, +\infty)$$

$$\text{Range}\, f(x) = y \in (-\infty, +\infty).$$ Thus

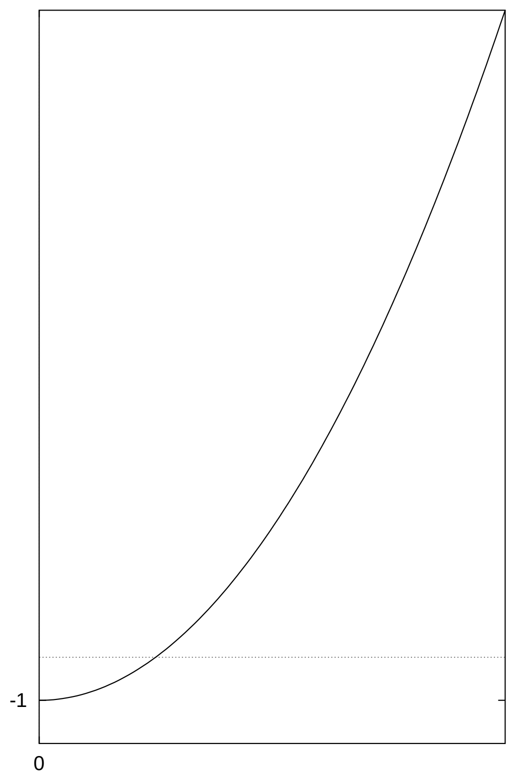
$$\text{Dom}\, f^{-1}(x) = x \in (-\infty, +\infty)$$

$$\text{Range}\, f^{-1}(x) = y \in (-\infty, +\infty)$$

$$\boxed{\begin{aligned} \text{Dom}\, f^{-1} &= \text{Range}\, f \\ \text{Range}\, f^{-1} &= \text{Dom}\, f \end{aligned}}$$

Find Dom f^{-1} and Range f^{-1}

(b) $f(x) = x^2 - 1,\ x \geq 0$. Draw the function.



$$\text{Dom}\, f\,(x) = x \in [0, +\infty)$$

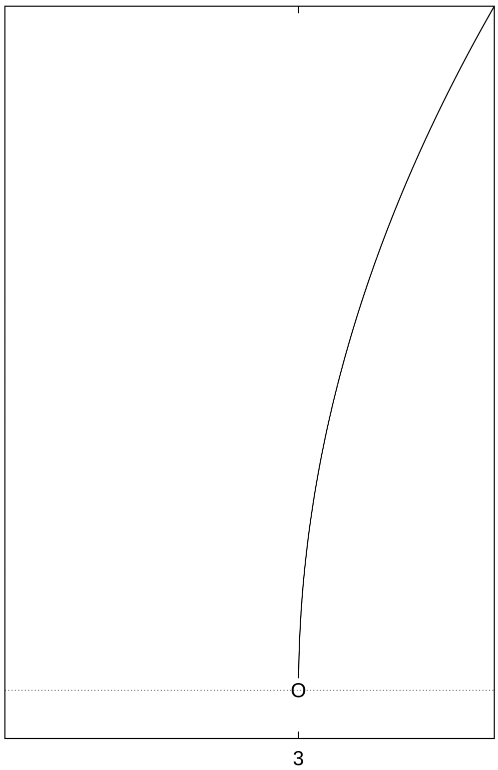
$$\text{Range}\, f\,(x) = y \in [-1, +\infty)\,. \qquad \text{Thus}$$

$$\text{Dom}\, f^{-1}\,(x) = x \in [-1, +\infty)$$

$$\text{Range}\, f^{-1}\,(x) = y \in [0, +\infty)$$

$$\boxed{\begin{aligned} \text{Dom}\, f^{-1} &= \text{Range}\, f \\ \text{Range}\, f^{-1} &= \text{Dom}\, f \end{aligned}}$$

Find Dom f^{-1} and Range f^{-1} (c)
$f(x) = \sqrt{x-3},\ x > 3$. Draw the function.



$$\text{Dom}\, f(x) = x \in (3, +\infty)$$

$$\text{Range}\, f(x) = y \in (0, +\infty)\,. \qquad \text{Thus}$$

$$\text{Dom}\, f^{-1}(x) = x \in (0, +\infty)$$

$$\text{Range}\, f^{-1}(x) = y \in (3, +\infty)$$

2.5.4 Finding $f^{-1}(x)$

If we let $y = f(x)$, we can find the inverse, f^{-1}, in terms of y, by solving for x.

$$
\begin{aligned}
\text{Let} \quad & y = f(x) \\
& \text{taking } \underline{f^{-1}} \text{ of both sides} \\
& f^{-1}(y) = \underline{f^{-1}\left(f\left(x\right)\right)} = x
\end{aligned}
$$

i.e. $\underline{x = f^{-1}\left(y\right) \text{ if and only if } y = f\left(x\right)}$.

To find the inverse of the function $f(x)$.

1. Check f is 1-1. Why?
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x.

2.5.4.1 Exercises

1. Find the inverse, if it exists, of each of the following:

   (a) $f(x) = 2x + 4$

   (b) $f(x) = 7x - 6$

   (c) $f(x) = 3x^3 - 5$

   (d) $f(x) = \dfrac{3}{x^2}$

   (e) $f(x) = \dfrac{3}{x^2},\ x < 0$

1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x.

(1a) Find the inverse, if it exists of:
$f(x) = 2x + 4$

$$
\begin{aligned}
y &= 2x + 4 \\
y - 4 &= 2x \\
x &= \frac{1}{2}(y - 4) \\
f^{-1}(x) &= \frac{x - 4}{2}.
\end{aligned}
$$

1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x.

(1b) Find the inverse, if it exists of:
$f(x) = 7x - 6$

$$
\begin{aligned}
y &= 7x - 6 \\
y + 6 &= 7x \\
x &= \frac{1}{7}\left(y + 6\right) \\
f^{-1}\left(x\right) &= \frac{x + 6}{7}.
\end{aligned}
$$

1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x.

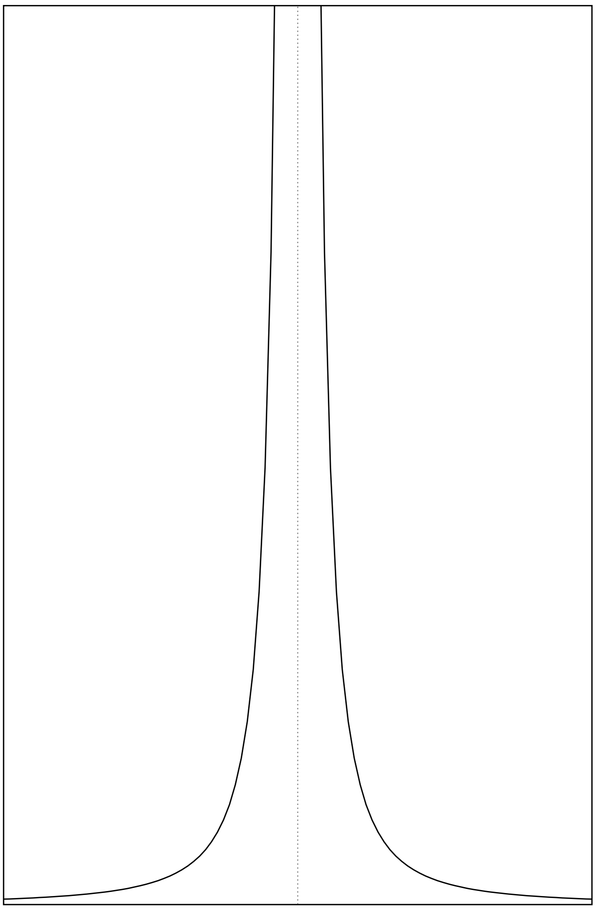
(1c) Find the inverse, if it exists of:
$f(x) = 3x^3 - 5$

$$
\begin{aligned}
y &= 3x^3 - 5 \\
y + 5 &= 3x^3 \\
x^3 &= \frac{y+5}{3} \\
x &= \left(\frac{y+5}{3}\right)^{1/3} \\
f^{-1}(x) &= \left(\frac{x+5}{3}\right)^{1/3}
\end{aligned}
$$

1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x.

(1d) Find the inverse, if it exists of: $f(x) = \dfrac{3}{x^2}$

The inverse does not exist because the function is not one-to-one.



1. Check f is 1-1.
2. Let $y = f(x)$.
3. Solve for x to obtain $x = f^{-1}(y)$.
4. Write f^{-1} in terms of x.

(1e) Find the inverse, if it exists of:

$$f(x) = \frac{3}{x^2}, \ x < 0$$

$$y = \frac{3}{x^2}, \quad x < 0$$

$$x^2 = \frac{3}{y}, \quad x < 0$$

$$x = -\sqrt{\frac{3}{y}}$$

$$f^{-1}(x) = -\sqrt{\frac{3}{x}}$$

2. The temperature at which water boils will, to a point, decrease linearly as the altitude increases. Let the boiling point be a function of altitude such that $T_B = f(h)$, where T_B is the boiling point in ^{o}C and h is the altitude in metres. What is the meaning in practical terms of $f^{-1}(90)$? If $f^{-1}(90) = 3000$ evaluate $f^{-1}(85)$.

Hint. What is the boiling point of water at sea (ground) level ($h = 0$)?

We know that

$$T_B = A - Bh.$$

The boiling point of water at sea level is $100^o C$. Thus

$$100 = A.$$

The question tells us that

$$f^{-1}(90) = 3000.$$

Thus

$$90 = A - B(3000),$$
$$\Rightarrow 90 = 100 - B(3000),$$
$$\Rightarrow B = \frac{10}{3000} = \frac{1}{300}.$$

Hence

$$T_B = 100 - \frac{1}{300}h$$

The solution $f^{-1}(85)$ is given by

$$85 = 100 - \frac{1}{300}h,$$
$$\Rightarrow h = 15\,(300) = 4500.$$

2.5.5 Revision of key ideas

The following questions are about the key ideas in this section.

1. What is meant by the expression that 'f is a 1-to-1 function'?
2. Suppose that f and g are functions. What is meant by the expression 'f is the inverse of g'?
3. What is the requirement for a function f to have an inverse?
4. How do the domain and range of a function f relate to the domain and range of its inverse function f^{-1}?