SOME INTRINSIC PROPERTIES OF SIMPLE GRAPH C^*-ALGEBRAS

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ABSTRACT. To a directed graph E is associated a C^*-algebra $C^*(E)$ called a graph C^*-algebra. There is a canonical action γ of $\mathbb{T}$ on $C^*(E)$, called the gauge action. In this paper we present necessary and sufficient conditions for the fixed point algebra $C^*(E)^\gamma$ to be simple. Our results also yield some structure theorems for simple graph algebras.

1. INTRODUCTION

This paper brings together ideas from the theory of nonnegative matrices associated to strongly connected graphs and topological graph theory to prove some structure results for graph C^*-algebras. Because of the diverse backgrounds involved, we have made an effort to make this paper self-contained by including a little relevant background from each of these areas.

We begin by establishing our notation and conventions for directed graphs and their C^*-algebras (we are aware that graph theorists use quite different terminology (see [5] for example)). Next we bring together two sets of results on the simplicity of graph C^*-algebras due to Paterson ([17]) and Szymański ([22]). We show that up to Morita equivalence, a simple graph C^*-algebra is either AF or the C^*-algebra of a strongly connected graph.

We then give some results about the finite path space of a strongly connected graph. These results are essentially restatements of standard facts about irreducible nonnegative matrices. In the following section we describe the construction of a relative skew product graph from E (cf. [7, 8]). Essentially, if Γ is a group with subgroup H, a relative skew product graph $E \times_{\mathrm{c}} (H \setminus \Gamma)$ is an extension of E by the homogeneous space $H \setminus \Gamma$ using a labelling of the edges in E by elements of Γ. This construction generalizes the usual skew product graph used in [13, 9]. In Theorem 5.2 we show that any connected covering graph of a given connected graph can be written as a relative skew product graph of the base graph, a generalisation of the results in [7, 8]. We describe an invariant, called the local voltage group, which enables us to write the connected components of an ordinary skew product graph as skew product graphs in their own right. For a row-finite graph E there is a canonical labelling of the edges in E with the integer 1 such that $C^*(E \times_{\mathrm{c}} \mathbb{Z})$ is isomorphic to $C^*(E) \times_\gamma \mathbb{T}$, where γ denotes the gauge action of $\mathbb{T}$ on $C^*(E)$.

In the final section we apply our results to graph C^*-algebras. In various stages we prove that the fixed point algebra $C^*(E)^\gamma$ of the gauge action γ on $C^*(E)$ is simple if and

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only if either E consists of a single vertex, or E is row-finite and has a cofinal subgraph with finitely many vertices which is strongly connected with period one. If E is strongly connected with period d, then $C^*(E)^\gamma$ is the direct sum of d mutually isomorphic AF algebras. If in addition E^0 is finite, then $C^*(E)$ is stably isomorphic to a crossed product of a simple AF algebra by a $\mathbb{Z}$-action.

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2. Graphs and their C^*-algebras

A *directed graph* E consists of a sets E^0, E^1 of vertices and edges respectively, together with maps $r, s : E^1 \to E^0$ giving the direction of each edge. A *subgraph* F of E consists of subsets $F^i \subseteq E^i$ for $i = 0, 1$ such that $s(F^1) \subset F^0$ and $r(F^1) \subset F^0$. A *path* in the directed graph E is a finite sequence $\alpha = \alpha_1 \cdots \alpha_n$ of edges such that $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq n-1$. For a finite path $\alpha = \alpha_1 \ldots \alpha_n$, its *length* $|\alpha| := n$ is the number of edges in the sequence α. An *infinite path* in E is an infinite sequence $(x_i)_{i\geq 1}$ of edges such that $r(x_i) = s(x_{i+1})$ for $i \geq 1$; the set of infinite paths in E is denoted E^∞. For $n \geq 0$, let E^n denote all those paths in the directed graph E of length n. Let $E^* = \bigcup_{n\geq 0} E^n$ denote the set of all finite paths in E. The range and source maps extend naturally to E^*; for $\alpha = \alpha_1 \cdots \alpha_n \in E^*$ define $r(\alpha) = r(\alpha_n)$ and $s(\alpha) = s(\alpha_1)$. The graph E is called *row-finite* if every vertex emits finitely many edges. A vertex which does not emit any edges is called a *sink*. A vertex which does not receive any edges is called a *source*.

Let E be a directed graph. For $e \in E^1$ we formally denote by e^{-1} the edge e traversed backwards, so that $s(e^{-1}) = r(e)$ and $r(e^{-1}) = s(e)$. The set of reverse edges is denoted E^{-1}. It is then natural to define $(e^{-1})^{-1} = e$ for $e^{-1} \in E^{-1}$. A *walk* in the directed graph E is a sequence $a = a_1 \cdots a_n$, where $a_i \in E^1 \cup E^{-1}$ are such that $r(a_i) = s(a_{i+1})$ for $i = 1, \ldots, n-1$; we write $s(a) = s(a_1)$ and $r(a) = r(a_n)$. A walk $a = a_1 \cdots a_n$ is said to be *reduced* if it does not contain the subword $a_i a_{i+1} = aa^{-1}$ for any $a \in E^1 \cup E^{-1}$. Given a reduced walk $a = a_1 \cdots a_n$ the reverse walk is written $a^{-1} := a_n^{-1} \cdots a_1^{-1}$, which is also reduced. If a, b are reduced walks with $r(a) = s(b)$, then $a \cdot b$ will be understood to be the reduced walk obtained by concatenation and then cancellation using the relations $ee^{-1} = s(e)$, $e^{-1}e = r(e)$, $e^{-1}s(e) = e^{-1} = r(e)e^{-1}$ and $er(e) = e = s(e)e$ for $e \in E^1$. With composition and inverse operations defined above, the set $\pi_1(E)$ of reduced walks, forms a groupoid with unit space E^0 and is referred to as the *fundamental groupoid* of E (note that the roles of the range and source map must be reversed to make $\pi_1(E)$ a category, cf. [14]).

Let E, F be directed graphs. A *graph morphism* $\varphi : F \to E$ consists of maps $\varphi^i : F^i \to E^i$ for $i = 0, 1$ such that $\varphi^0(r(f)) = r(\varphi^1(f))$ and $\varphi^0(s(f)) = s(\varphi^1(f))$ for all $f \in E^1$.

Definition 2.1. Let E, F be directed graphs and $p : F \to E$ be a graph morphism. Then p is *covering map* if for each $v \in F^0$, p maps $r^{-1}(v)$ bijectively onto $r^{-1}(p(v))$ and $s^{-1}(v)$ bijectively onto $s^{-1}(p(v))$ (cf. [21, Section 2.1.6]).

Let $p : F \to E$ be a covering map, and for $a = a_1 \cdots a_n \in \pi_1(F)$ let $p(a) = p(a_1) \cdots \cdots p(a_n)$ where $p(f^{-1}) := p(f)^{-1}$ for $f \in E^1$. In this way, the covering map p induces a morphism $p_* : \pi_1(F) \to \pi_1(E)$. A graph morphism $p : F \to E$ has the *unique walk lifting property* (see [7, Theorem 2.1.1]) if given $u \in F^0$ and $a \in \pi_1(E)$ with $s(a) = p^0(u)$, there is a unique $\widetilde{a} \in \pi_1(F)$ such that $s(\widetilde{a}) = u$ and $p(\widetilde{a}) = a$. One may also lift reduced walks which end at $p^0(u)$. Likewise one may lift infinite paths from E to F. The following result is routine (see [10, Lemma 17.4] for instance):

Lemma 2.2. *Let E, F be directed graphs and $p : F \to E$ be a graph morphism. Then p has the unique walk lifting property if and only if p is a covering map.*

The directed graph E is said to be *connected* if, given any two distinct vertices in E, there is a reduced walk between them. A directed graph T is a *tree* if and only if there is precisely one reduced walk between any two vertices (so a tree is connected). For a connected graph E and $v \in E^0$ we may define

$$\pi_1(E, v) = \{a \in \pi_1(E) : s(a) = v = r(a)\},$$

so that $\pi_1(E, v)$ is the isotropy group of the unit v in $\pi_1(E)$. If E is connected, then the groupoid $\pi_1(E)$ is connected and so all its isotropy groups are isomorphic. Our definition of the fundamental group matches the usual one given in [21, Section 2.1.6] because taking the reduction of a walk coincides with the notion of path equivalence used there.

Using the axiom of choice it may be shown that every connected graph E contains a spanning tree T (cf. [21, Section 2.1.5]): a subgraph T which is itself a tree with $T^0 = E^0$. Fix a given spanning tree T of E and a vertex $v \in E^0$. Then for $w \in T^0$ we set b_w to be the unique reduced walk in T from v to w. If E has a spanning tree, then it is connected.

Let E be a directed graph. Then a *Cuntz-Krieger E-family* (or a *representation* of E) consists of a family $\{P_v : v \in E^0\}$ of mutually orthogonal projections and a family $\{S_e : e \in E^1\}$ of partial isometries with mutually orthogonal ranges such that

$$S_e^* S_e = P_{r(e)}, \quad S_e S_e^* \leq P_{s(e)}, \quad P_v = \sum_{s(e)=v} S_e S_e^* \text{ if } 0 < |s^{-1}(v)| < \infty.$$

The graph C^*-algebra $C^*(E)$ is generated by a universal Cuntz-Krieger E-family $\{s_e, p_v\}$ (see [18, Section 1]). The class of graph C^*-algebras is quite broad. It includes, up to Morita equivalence, all AF algebras and all Cuntz-Krieger algebras (see [12], [11] amongst others).

3. Simple graph algebras

A *loop* in E is a path α with $|\alpha| \geq 1$ such that $s(\alpha) = r(\alpha)$. The loop $\alpha \in E^n$ is *simple* if the vertices $\{r(\alpha_i) : 1 \leq i \leq n\}$ are distinct. The graph E satisfies condition (K) if no vertex lies on exactly one simple loop. The directed graph E is *cofinal* if, given $x \in E^\infty$ and $v \in E^0$, there is a path α such that $s(\alpha) = v$ and $r(\alpha) = r(x_n)$ for some $n \geq 1$.

If $v, w \in E^0$ then we write $v \geq w$ if there is a path from v to w. A subset H of E^0 is *hereditary* if $v \in H$ and $v \geq w$ implies that $w \in H$. A hereditary subset H is *saturated* if there is no vertex $v \in E^0 \setminus H$ with $0 < |s^{-1}(v)| < \infty$ such that $r(e) \in H$ for all $e \in s^{-1}(v)$. For $X \subseteq E^0$, define L_X to be the hereditary subset consisting of all vertices

which X connects to. Then $\Sigma(X)$, the smallest saturated hereditary subset containing X, consists of all those vertices of $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$ such that every infinite path starting at v eventually uses vertices in L_X. Recall from [2] that a nontrivial saturated hereditary subset H of E gives rise to a nontrivial gauge invariant ideal I_H of $C^*(E)$.

Theorem 3.1. *Let E be a directed graph. Then $C^*(E)$ is simple if and only if*

(i) *E is cofinal.*
(ii) *E satisfies condition (K).*
(iii) *If vertex v emits infinitely many edges, then every vertex connects to v.*

Proof. See [17, Theorem 4] or [22, Theorem 12]. In [17, Theorem 4] the proof of the only if direction is omitted, but as we see below it is not hard.

Suppose that E is not cofinal. Then there are $x \in E^\infty$ and $v \in E^0$ such that v does not connect to any vertex used by x. Hence $s(x) \notin \Sigma(L_{\{v\}})$ because the infinite path x does not enter $L_{\{v\}}$. Therefore $\Sigma(L_{\{v\}})$ gives rise to a nontrivial gauge invariant ideal $I_{\Sigma(L_{\{v\}})}$ of $C^*(E)$ in which case $C^*(E)$ is not simple.

Suppose that E does not satisfy condition (K). Then there is a simple loop L in E such that each vertex of L lies on no other loop. Either E is cofinal or it is not. If E is not cofinal then $C^*(E)$ is not simple by the argument above. If E is cofinal, then by [12, Theorem 2.4] $C^*(E)$ is strongly Morita equivalent to $C(\mathbb{T})$ which is not simple.

If E has a vertex v of infinite valency and a vertex w which does not connect to v, then $v \notin L_{\{w\}}$. Moreover $v \notin \Sigma(L_{\{w\}})$, since v has infinite valency. Hence $\Sigma(L_{\{w\}})$ is a nontrivial saturated hereditary subset of E^0 and so gives rise to a nontrivial gauge invariant ideal $I_{\Sigma(L_{\{w\}})}$ of $C^*(E)$. Hence $C^*(E)$ is not simple, which concludes the proof. □

The directed graph E is *strongly connected* if for every pair of vertices v, w there is a path α with $|\alpha| \geq 1$ such that $s(\alpha) = v$, $r(\alpha) = w$. A strongly connected graph is sometimes said to be transitive or irreducible (since its vertex incidence matrix A_E is irreducible (cf. [20, Section 1.3])). If E is strongly connected, then it is cofinal. A subgraph F of E is said to be *cofinal* if for every $x \in E^\infty$ there exists $N(x)$ such that $r(x_n) \in F^0$ for $n \geq N(x)$.

Theorem 3.2. *If the directed graph E is cofinal, then either E has no loops, or there exists a strongly connected cofinal subgraph $F \subseteq E$.*

Proof. Under the relation of mutual connectivity (i.e. $v \sim w$ if and only if $v \geq w$ and conversely), there exists an equivalence class X of vertices which contains any vertex which lies on a loop. If E has a loop and is cofinal there can only be one such equivalence class. Let F be the subgraph with $F^0 = X$ and $F^1 = \{e : s(e) \in F^0\}$. We claim that $r(F^1) \subseteq F^0$. Suppose that $e \in F^1$ is such that $r(e) \notin F^0$. Since $s(e) \in F^0$, the vertex $s(e)$ connects to itself via α. Let $y = \alpha\alpha\alpha\cdots \in E^\infty$. Then since E is cofinal, $r(e)$ must connect to some vertex on α and hence to $s(e)$. But this contradicts our assumption that $r(e) \notin F^0$. By its definition F is strongly connected. It remains to show that $F \subseteq E$ is cofinal.

Let $x \in E^\infty$. Then if $s(x) \in F^0$, then we may take $N(x) = 1$. So we suppose that $s(x) \notin F^0$. Let $v \in F^0$. Then there is a loop α with $s(\alpha) = v$ which yields

$y = \alpha\alpha\cdots \in E^\infty$. Since E is cofinal there exists $\beta \in E^*$ such that $s(\beta) = v$ and $r(\beta) = r(x_m)$ some $m \geq 1$. Let $w = r(x_{m+1})$. Then by cofinality there must be a path $\gamma \in E^*$ with $s(\gamma) = w$ and $r(\gamma) = r(y_{\ell|\alpha|}) = v$ for some $\ell \geq 1$. Then v connects to w via βx_{m+1} and w connects to v via γ, and so $w \in F^0$ by definition. □

Corollary 3.3. *Suppose that $C^*(E)$ is simple. Then either $C^*(E)$ is an* AF *algebra or there exists a strongly connected subgraph $F \subseteq E$ such that $C^*(E)$ is strongly Morita equivalent to $C^*(F)$.*

Proof. If $C^*(E)$ is simple, then by Theorem 3.1 the graph E is cofinal. If E has no loops then by Theorem 3.1 E cannot have any vertices of infinite valency and then $C^*(E)$ is an AF algebra by [12, Theorem 2.4]. On the other hand if E has loops, then by Theorem 3.2 there exists a strongly connected cofinal subgraph F of E. Let $P = \sum_{v \in F^0} p_v$. Then by a similar argument to the one given in [3, Section 1] P defines a projection in the multiplier algebra of $C^*(E)$ such that $PC^*(E)P \cong C^*(F)$. To see that $PC^*(E)P$ is full, just observe that $P \neq 0$ and that $C^*(E)$ is simple. Hence $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ as required. □

4. Strongly connected graphs

In this section we briefly give some results about the path space of a strongly connected graph. Essentially, they are reformulations of results about finite, nonnegative matrices. For more details about nonnegative matrices see [20, Section 1.3] or [15, Section 4]. Let E be a directed graph. Then for $v \in E^0$ we define its *period* $d(v)$ to be the greatest common divisor of the lengths of all loops which begin at v. If there are no such loops, then we set $d(v) = 0$.

Lemma 4.1. *If E is strongly connected, then for any $v, w \in E^0$ one has $d(v) = d(w)$.*

Proof. Let $v, w \in E^0$. Then since E is strongly connected there are paths $\alpha \in E^k$ and $\beta \in E^\ell$ from v to w and w to v, respectively. So, if w is the source of a loop of length s then v is the source of loops of length $k + \ell + s$ and $k + \ell + 2s$. Hence $d(v)$ divides $(k + \ell + 2s) - (k + \ell + s) = s$. Therefore $d(v)$ divides s for every loop with source w of length s. Since $d(w)$ is the greatest common divisor of such numbers s, we must have $d(v) \leq d(w)$. But since the argument can be repeated with v and w exchanged, one has $d(w) \leq d(v)$, and so $d(v) = d(w)$ as required. □

Hence we may define the *period* of a strongly directed graph to be the period of any one of its vertices. If E is strongly connected with period 1 and has finitely many vertices, then its vertex incidence matrix A_E is aperiodic in the sense that there is a $k \geq 1$ such that every entry of A_E^k is strictly positive (cf. [4, p.253]).

Lemma 4.2. *Let E be a strongly connected graph with period d. For each $v \in E^0$ there is a positive integer $N(v)$ such that for $k \geq N(v)$, v is the source of a loop of length kd.*

Proof. Suppose that $\alpha \in E^{kd}$ and $\beta \in E^{sd}$ are loops with source v. Then $\alpha\beta \in E^{(k+s)d}$ is a loop with source v. Hence the set

$$V = \{kd : \text{there is a loop of the length } kd \text{ in } E \text{ with source } v\}$$

of positive integers is closed under addition; moreover their greatest common divisor is d. But then by [20, Lemma A3] V must contain all but a finite number of positive multiples of d, and the result follows. □

Next we examine the structure of the finite path space of a strongly connected graph with period d (see [20, Theorem 1.3 in Part I]):

Lemma 4.3. *Let E be a strongly connected graph with period d, and $v \in E^0$. For any $w \in E^0$ there exists r_w with $0 \leq r_w < d$ such that:*

(i) *if $\mu \in E^s$ is a path from v to w, then $s \equiv r_w \pmod d$;*

(ii) *there is a positive integer $N(w)$ such that for $k \geq N(w)$ there is a path of length $kd + r_w$ from v to w.*

Proof. For (i), let $\alpha \in E^m$ and $\beta \in E^{m'}$ be paths from v to w. Since E is strongly connected, there is a path $\gamma \in E^p$ from w to v. Hence $\alpha\gamma \in E^{m+p}$ and $\beta\gamma \in E^{m'+p}$ are loops with source v. Since E has period d, it follows that d divides $m+p$ and $m'+p$, and so d divides $(m+p)-(m'+p) = m - m'$, that is, $m - m' \equiv 0 \pmod d$. Set $r_w = m \pmod d$. If $\mu \in E^s$ is a path from v to w, then $s \equiv r_w \pmod d$.

For (ii), let $\alpha \in E^n$ be a path from v to w. Then from (i) there is a positive integer m such that $n = md + r_w$. By Lemma 4.2 there is a positive integer N_0 such that for all $s \geq N_0$, v is the source of a loop of length sd. Set $N(w) = N_0 + m$. If $k \geq N(w)$, then $k - m \geq N_0$, and so there is a loop $\delta \in E^{(k-m)d}$ with source v. Hence $\alpha\delta$ is a path of length $n + (k-m)d = (m+k-m)d + r_w = kd + r_w$ from v to w. □

Note that if $w = v$ in the above lemma, then $r_v = 0$. Note also that r_w in Lemma 4.3 is called a *residue class* of w with respect to v (cf. [20, Definition 1.7 in Part I]).

5. Skew product graphs

In this section we describe the relative skew product graph construction and show how it can be used to describe all connected coverings of a given connected graph. We introduce an invariant, called the *local voltage group*, which enables us to describe the connected components of ordinary skew product graphs and exhibit them as ordinary skew product graphs in their own right. Finally we specialize to a certain skew product graphs formed from the integers with a view to applications to graph C^*-algebras in the next section.

The following definition generalizes the ones given in [8, Section 4] and [7, Section 2.3.2] for row-finite graphs with finitely many vertices and groups which are finite. Let E be a directed graph, Γ a countable group with subgroup H, and $c : E^1 \to \Gamma$ a function which we think of as a labelling of the edges in E by elements of Γ. From this data we may form the *relative skew product graph* $E \times_{\mathrm{c}} (H \setminus \Gamma)$ with vertex set $E^0 \times (H \setminus \Gamma)$, edge set $E^1 \times (H \setminus \Gamma)$, and the range and source of the the edge (e, Hg) given by

$$s(e, Hg) = (s(e), Hg) \quad \text{and} \quad r(e, Hg) = (r(e), Hgc(e)) \quad \text{respectively.}$$

If H is the trivial subgroup of Γ, then the above definition reduces to that of the ordinary skew product graph $E \times_{\mathrm{c}} \Gamma$ used in [13, Definition 2.1] and [7, Section 2.1.1] (as opposed to those used in [9, 6] where the labelling gives rise to a coaction of Γ on $C^*(E)$). If H is trivial, there is a free action λ of Γ on $E \times_{\mathrm{c}} \Gamma$ defined for $g, h \in \Gamma$ by

$\lambda_h^i(x,g) = (x,hg)$ where $x \in E^i$ for $i = 0,1$. If H is a subgroup of Γ, then H also acts freely on $E \times_{\mathrm{c}} \Gamma$ and the quotient graph $H \setminus (E \times_{\mathrm{c}} \Gamma)$ is isomorphic to $E \times_{\mathrm{c}} (H \setminus \Gamma)$ via the maps $[x,g] \mapsto (x,Hg)$ where $x \in E^i$ for $i = 0,1$.

For functions $c_1, c_2 : E^1 \to \Gamma$, we say that c_1 and c_2 are *cohomologous* and write $c_1 \sim c_2$ if there is a function $b : E^0 \to \Gamma$ such that $c_1(e)b(r(e)) = b(s(e))c_2(e)$ for all $e \in E^1$. The relation $\sim$ amongst all functions from E^1 to Γ is an equivalence relation. If $c_1, c_2 : E^1 \to \Gamma$ are cohomologous then $E \times_{c_1} \Gamma$ is equivariantly isomorphic to $E \times_{c_2} \Gamma$ and $E \times_{c_1} (H \setminus \Gamma) \cong E \times_{c_2} (H \setminus \Gamma)$ via the maps $(v,g) \mapsto (v, gb(v))$ and $(e,g) \mapsto (e, gb(s(e)))$.

The map $p_{\mathrm{c}} : E \times_{\mathrm{c}} (H \setminus \Gamma) \to E$ defined by $p_{\mathrm{c}}(x,Hg) = x$, where $x \in E^i$ for $i = 0,1$, is a covering map. If $E, E \times_{\mathrm{c}} (H \setminus \Gamma)$ are connected and H is normal in Γ, then p_{c} is a *regular* covering since the image of $\pi_1(E \times_{\mathrm{c}} (H \setminus \Gamma), (v,Hg))$ under $(p_{\mathrm{c}})_*$ is a normal subgroup of $\pi_1(E,v)$. In fact, we shall show that every connected covering of a given connected graph E is a relative skew product graph formed from E (cf. [7, 8]).

Lemma 5.1. *Let E and F be connected graphs, $p : F \to E$ a covering map, T a spanning tree for E and $v \in E^0$. Then there is a spanning forest $\{\widetilde{T}_u : u \in p^{-1}(v)\}$ of F such that $p(\widetilde{T}_u) = T$ for all $u \in p^{-1}(v)$.*

Proof. For $w \in E^0$ let b_w be the unique reduced walk in T from v to w. For $u \in p^{-1}(v)$ let $\widetilde{b}_w(u)$ be the unique lift of b_w to a reduced walk beginning at $u \in F^0$. Put $\widetilde{T}_u^0 = r(\widetilde{b}_w(u))$, and let $\widetilde{T}_u^1$ consist of all the edges comprising each $\widetilde{b}_w(u)$ as w runs through E^0. By the unique walk lifting property $\widetilde{T}_u$ is a tree such that $p(\widetilde{T}_u) = T$.

To show that $\{\widetilde{T}_u : u \in p^{-1}(v)\}$ is a forest, we must show that the $\widetilde{T}_u$ are disjoint. Suppose that w is a vertex in $\widetilde{T}_u \cap \widetilde{T}_{u'}$, where $u, u' \in p^{-1}(v)$. Let a be the unique reduced walk in $\widetilde{T}_u$ from u to w, and b be the unique reduced walk in $\widetilde{T}_{u'}$ from u' to w. Then ab^{-1} is a reduced walk in F from u to u', and so $p(ab^{-1}) = p(a)p(b)^{-1}$ is a loop in T with source v. Hence $p(a) = p(b)$ and then $a^{-1} = b^{-1}$ by unique walk lifting at w. Therefore we have that $u = s(a) = s(b) = u'$. A similar argument shows that $\widetilde{T}_u$ and $\widetilde{T}_{u'}$ cannot have an edge in common.

To show that $\{\widetilde{T}_u : u \in p^{-1}(v)\}$ spans F, suppose that $w \in F^0$. Then there exists a unique reduced walk $b_{p(w)}$ in T from v to $p(w)$. By the unique walk lifting property there is a reduced walk $\widetilde{b}_{p(w)}^{-1}$ in F such that $s(\widetilde{b}_{p(w)}^{-1}) = w$ and $p(\widetilde{b}_{p(w)}^{-1}) = b_{p(w)}^{-1}$. Since $p(r(\widetilde{b}_{p(w)}^{-1})) = v$, we must have $r(\widetilde{b}_w^{-1}) \in p^{-1}(v)$, and so w lies in $\widetilde{T}_{r(\widetilde{b}_w^{-1})}$. □

Theorem 5.2. *Let E, F be connected graphs, $p : F \to E$ a covering map, T a spanning tree for E and $v \in E^0$. Let $w \in F^0$ be such that $p(w) = v \in E^0$. Then there exists a map $c = c_{v,T} : E^1 \to \pi_1(E,v)$ such that $F \cong E \times_c (p_*\pi_1(F,w) \setminus \pi_1(E,v))$.*

Proof. Let $H = p_*\pi^1(F,w)$ and $\{a_i : i \in I\} \subset \pi^1(E,v)$ be a right transversal for H in $\pi^1(E,v)$. For $i \in I$, let $\widetilde{a}_i$ be the lift of a_i in F with $s(\widetilde{a}_i) = w$, and put $\widetilde{v}_i = r(\widetilde{a}_i)$. Then $p^{-1}(v) = \{\widetilde{v}_i : i \in I\}$. Let $c = c_{v,T} : E^1 \to \pi^1(E,v)$ be defined by $c(e) = b_{s(e)}eb_{r(e)}^{-1}$, where $b_{s(e)}$ and $b_{r(e)}$ are the unique reduced walks in T from v to $s(e)$ and $r(e)$ respectively.

For $i \in I$, let $\widetilde{T}_i$ be the tree $\widetilde{T}_{\widetilde{v}_i}$ in F described in Lemma 5.1. Define a map $\varphi : F \to E \times_c (H \setminus \pi^1(E,v))$ as follows: if $u \in \widetilde{T}_i^0$, then set $\varphi^0(u) = (p(u), Ha_i)$. If $f \in F^1$ is such that $s(f) \in \widetilde{T}_j^0$ and $r(f) \in \widetilde{T}_i^0$, then we put $\varphi^1(f) = (p(f), Ha_i c(p(f))^{-1})$.

For $f \in F^1$ we have $\varphi^0(s(f)) = (p(s(f)), Ha_j)$ whereas

$$s(\varphi^1(f)) = s(p(f), Ha_i c(p(f))^{-1}) = (s(p(f)), Ha_i c(p(f))^{-1}),$$

so $s(\varphi^1(f)) = \varphi^0(s(f))$ provided that $Ha_i c(p(f))^{-1} = Ha_j$. To see this, let

$$h = \widetilde{a}_i \widetilde{b}_{r(f)}^{-1} f^{-1} \widetilde{b}_{s(f)} \widetilde{a}_j^{-1},$$

where $\widetilde{b}_{r(f)}$ is the lift of $b_{p(r(f))}$ with $s(\widetilde{b}_{r(f)}) = \widetilde{v}_i$, and $\widetilde{b}_{s(f)}$ is the lift of $b_{p(s(f))}$ with $s(\widetilde{b}_{r(f)}) = \widetilde{v}_j$. So h is a closed walk in F with $s(h) = w$. Hence

$$p(h) = a_i b_{p(r(f))}^{-1} p(f)^{-1} b_{p(s(f))} a_j^{-1} = a_i c(p(f))^{-1} a_j^{-1} \in H,$$

and so $Ha_i c(p(f))^{-1} = Ha_j$ as required. It is straightforward to show that $r \circ \varphi^1 = \varphi^0 \circ r$, and hence φ is a graph morphism.

We now show that φ is injective. Suppose that for some $\widetilde{e}, \widetilde{f} \in F^1$ we have $\varphi^1(\widetilde{e}) = \varphi^1(\widetilde{f})$, where $r(\widetilde{e}) \in \widetilde{T}_i$, $r(\widetilde{f}) \in \widetilde{T}_{i'}$, $s(\widetilde{e}) \in \widetilde{T}_j$ and $s(\widetilde{f}) \in \widetilde{T}_{j'}$. Then

$$(p(\widetilde{e}), Ha_i c(p(\widetilde{e}))^{-1}) = (p(\widetilde{f}), Ha_{i'} c(p(\widetilde{f}))^{-1}),$$

in which case $p(\widetilde{e}) = p(\widetilde{f})$ and then $i = i'$. Let $\widetilde{b}$ be a reduced walk in $\widetilde{T}_i$ from $s(\widetilde{e})$ to $s(\widetilde{f})$. Then $b = p(\widetilde{b})$ is a closed reduced walk in T, which implies that $s(\widetilde{e}) = s(\widetilde{f})$. So $\widetilde{e} = \widetilde{f}$ by unique walk lifting of $p(\widetilde{e}) = p(\widetilde{f})$ at $s(\widetilde{e})$. The proof that φ is injective on F^0 follows similarly.

Finally we claim that φ is surjective. Given (e, Ha_j) in $(E \times_c (H \setminus \pi_1(E,v)))^1$, let $\widetilde{a} = \widetilde{a}_j \widetilde{b}_{s(e)}$ be the lift of a_j at w followed by the lift of $b_{s(e)}$ at $\widetilde{v}_j$. Set $\widetilde{e}$ to be the lift of e in F based at $r(\widetilde{a})$. Then $s(\widetilde{e}) \in \widetilde{T}_j$ and suppose that $r(\widetilde{e}) \in \widetilde{T}_i$. Then since $p(\widetilde{e}) = e$, one has $\varphi^1(\widetilde{e}) = (e, Ha_i c(e)^{-1})$ and $Ha_i c(e)^{-1} = Ha_j$ fllows by a similar argument to the one given to show that $\varphi^0 \circ s = s \circ \varphi^1$ above. The proof that φ^0 is surjective follows similarly. □

The function $c = c_{v,T} : E^1 \to \pi_1(E,v)$ given in the proof of Theorem 5.2 depends on the transversal $p_* \pi_1(F,w)$ in $\pi_1(E,v)$, and on the choice of v and T. In Remarks 5.9 we show that the cohomology class of c is independent of the choice of v and T. By unique walk lifting, a different transversal has no effect other than to permute the trees in the spanning forest of F.

We shall now fix our attention on ordinary skew product graphs. Let E be a directed graph and $c : E^1 \to \Gamma$ a function where Γ is a countable group. Then there is a map (which we shall also denote by c) from $\pi_1(E)$ to Γ defined by

$$c(a_1 \cdots a_n) = c(a_1) \cdots c(a_n),$$

where $c(e^{-1}) = c(e)^{-1}$. It is straightforward to show that c is a functor from $\pi_1(E)$ to Γ. For $v \in E^0$ let

$$\Gamma_v(c) = \{c(a) : a \in \pi^1(E,v)\}.$$

Since it is the range of a homomorphism, $\Gamma_v(c)$ is a subgroup of Γ, called the *local voltage group* of c based at v (in [7, Section 2.5] these groups are referred to as the local group). The following facts are not difficult to establish (see [19, Lemma 6.5]):

Lemma 5.3. *Let E be a directed graph and $c : E^1 \to \Gamma$ a function where Γ is a countable group.*

(i) *If there is a walk between $v, w \in E^0$, then $\Gamma_v(c)$ and $\Gamma_w(c)$ are conjugate subgroups of Γ.*

(ii) *If $c_1, c_2 : E^1 \to \Gamma$ are cohomologous, then for all $v \in E^0$, $\Gamma_v(c_1)$ and $\Gamma_v(c_2)$ are conjugate subgroups of Γ.*

Remark 5.4. Let E be a directed graph and $c : E^1 \to \Gamma$ a function where Γ is a countable group. If $\widetilde{a}$ is a reduced walk in $E \times_c \Gamma$ from (v, g) to (u, h), then $h = gc(a)$ and $a = p_c(\widetilde{a})$ is a reduced walk in E from v to u (cf. [7, Theorem 2.1.2]).

Proposition 5.5. *Let E be a connected graph, $c : E^1 \to \Gamma$ a function where Γ is a countable group. Then the vertices (v, g) and (u, h) lie in the same connected component of $E \times_c \Gamma$ if and only if there is a reduced walk a in E from v to u such that $g^{-1}h$ lies in the coset $\Gamma_v(c)c(a)$ of the local voltage group of c based at v.*

Proof. If (v, g) and (u, h) lie in the same connected component of $E \times_c \Gamma$, then there exists a reduced walk $\widetilde{a}$ in $E \times_c \Gamma$ from (v, g) to (u, h). By Remark 5.4 one has $h = gc(a)$, where $a = p_c(\widetilde{a})$. Then $g^{-1}h = c(a)$ lies in $\Gamma_v(c)c(a)$ as required.

Conversely, let a be a reduced walk in E from v to u such that $g^{-1}h \in \Gamma_v(c)c(a)$. Hence there is $b \in \pi^1(E, v)$ such that $g^{-1}h = c(b)c(a) = c(ba)$. Let $\widetilde{d}$ be the unique lift of $d = ba$ in $E \times_c \Gamma$ starting at (v, g). Then $\widetilde{d}$ is a reduced walk in $E \times_c \Gamma$ with $s(\widetilde{d}) = (v, g)$ and $r(\widetilde{d}) = (r(ba), gc(ba)) = (u, h)$, so that (v, g) and (u, h) are in the same connected component. □

Corollary 5.6. *Let E be a connected graph and $c : E^1 \to \Gamma$ a function where Γ is a countable group. Then $E \times_c \Gamma$ consists of $|\Gamma : \Gamma_v(c)|$ mutually isomorphic connected components, where $\Gamma_v(c)$ is the local voltage group of c based at $v \in E^0$. In particular, $E \times_c \Gamma$ is connected if and only if $\Gamma_v(c) = \Gamma$ for some $v \in E^0$.*

Proof. By Proposition 5.5 the number of connected components of $E \times_c \Gamma$ is equal to $|\Gamma : \Gamma_v(c)|$, the number of cosets of $\Gamma_v(c)$ in Γ. Since the natural action λ of Γ on $E \times_c \Gamma$ is transitive on $p_c^{-1}(v) = \{(v, g) : g \in \Gamma\}$ for all $v \in E^0$ and E is connected, it follows that the connected components of $E \times_c \Gamma$ are mutually isomorphic. □

There appears to be an error on [7, p.88] where it is shown that the connected components of $E \times_c \Gamma$ are mutually isomorphic by the transitivity of the natural action of Γ on $E \times_c \Gamma$ — which is certainly not the case in general.

Definition 5.7. Let E be a directed graph, $c : E^1 \to \Gamma$ a function where Γ is a countable group. Fix T a spanning tree of E and vertex $v \in E^0$. Then define the *T-voltage based at v* to be the function $c_{v,T} : E^1 \to \Gamma$ given by

$$c_{v,T}(e) := c(b_{s(e)} e b_{r(e)}^{-1}) = c(b_{s(e)})c(e)c(b_{r(e)})^{-1}, \tag{1}$$

where for $w \in E^0$, b_w is the unique reduced walk in T from v to w.

The function $c_{v,T} : E^1 \to \Gamma$ is the same as the one used in the proof of Theorem 5.2. The following result is not difficult to establish

Proposition 5.8. *Let E be a directed graph with spanning tree T and $c : E^1 \to \Gamma$ be a function where Γ is a countable group. Then for $v \in E^0$, one has $c_{v,T} \sim c$, and furthermore $\Gamma_v(c) = \Gamma_v(c_{v,T})$.*

Remarks 5.9. Proposition 5.8 shows that if we use another base vertex to compute the T-voltages or if we use a different spanning tree, then we get a cohomologous function from E^1 to Γ. Moreover, if T is a spanning tree for E, then the local voltage group of c at v is generated by $\{c_{v,T}(e) : e \in E^1 \setminus T^1\}$ (cf. [7, Theorem 2.5.3]).

Proposition 5.10. *Let E be a connected directed graph with spanning tree T, $v \in E^0$ and $c : E^1 \to \Gamma$ a function where Γ is a countable group. Then each connected component of $E \times_{\mathrm{c}} \Gamma$ is isomorphic to $E \times_{c_{v,T}} \Gamma_v(c)$.*

Proof. For $w \in E^0$ let b_w be the unique reduced walk in T from v to w. Since each of the connected components of $E \times_{\mathrm{c}} \Gamma$ are isomorphic by Corollary 5.6, we shall deal with the connected component F which contains the vertex $(v, 1_\Gamma)$. Define $\varphi : E \times_{c_{v,T}} \Gamma_v(c) \to E \times_{\mathrm{c}} \Gamma$ by

$$\varphi^0(v, c(a)) = (v, c(a)c(b_v)) \text{ and } \varphi^1(e, c(a)) = (e, c(a)c(b_{s(e)})),$$

where $a \in \pi_1(E, v)$. We check that φ is a graph morphism:

$$\begin{aligned}\varphi^0(r(e, c(a)) &= \varphi^0(r(e)c(a)c(b_{s(e)}eb_{r(e)}^{-1})) = (r(e), c(a)c(b_{s(e)}e)) \\ &= r(e, c(a)c(b_{s(e)}) = r(\varphi^1(e, c(a)),\end{aligned}$$

and

$$\varphi^0(s(e, c(a))) = (s(e), c(a)b_{s(e)}) = s(e, c(a)b_{s(e)}) = s(\varphi^1(e, c(a)).$$

To see that φ is injective, suppose that $\varphi^1(e, c(a)) = \varphi^1(f, c(b))$. Then $(e, c(a)c(b_{s(e)})) = (f, c(b)c(b_{s(f)}))$ in which case $e = f$ and then $c(a) = c(b)$. To show that φ^0 is injective is similar. It only remains to show that the image of φ is F. Suppose that $(u, g) \in F^0$. Then there is a reduced walk $\widetilde{b}$ from $(v, 1_\Gamma)$ such that $b = p_{\mathrm{c}}(\widetilde{b})$ is a reduced walk in E from v to u with $g = c(b)$. Since bb_u^{-1} is a closed reduced walk in E which begins at v it follows that $c(bb_u^{-1}) \in \Gamma_v(c)$ and then $(u, c(bb_u^{-1})$ is a vertex in $E \times_{c_{v,T}} \Gamma_v(c)$ such that $\varphi^0(u, c(bb_u^{-1})) = (u, c(b)) = (u, g)$. A similar proof shows that φ^1 is surjective, which completes our proof. □

Remark 5.11. Let E be a directed graph and $c : E^1 \to \Gamma$ a function. If $E \times_{\mathrm{c}} \Gamma$ is cofinal, then E is cofinal. However, the converse is not true.

We shall be interested in the case when each edge in E is labelled by the integer 1. The skew product graph $E \times_{\mathrm{c}} \mathbb{Z}$ then has no loops (see [13, Proposition 2.6]). The next result can be deduced from the definitions and the results from section 4.

Proposition 5.12. *Let E be strongly connected and let $c : E^1 \to \mathbb{Z}$ given by $c(e) = 1$ for all $e \in E^1$. If E has period d, then $\Gamma_v(c) \cong d\,\mathbb{Z}$ for all $v \in E^0$.*

If E^0 is finite then we can say a little more.

Proposition 5.13. *Let E be a strongly connected graph of period d with finitely many vertices. Let $c : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$. Then each connected component of $E \times_c \mathbb{Z}$ is cofinal.*

Proof. Suppose that $\widetilde{x}$ is an infinite path in one connected component of $E \times_c \mathbb{Z}$, and (v, m) is a vertex in the same component. Since E^0 is finite, there is $u \in E^0$ and an increasing sequence $n_1 \geq n_2 \geq n_3 \geq \dots$ of integers such that (u, n_i) is the range of some edge in $\widetilde{x}$. Let α be a shortest path from $v = p_c(v, m)$ to $w = s(p_c(\widetilde{x}))$ in E. Then $|\alpha| = r_w$, the residue class of w in E with respect to v (see Lemma 4.3). Let $\widetilde{\alpha}$ be the lift of α in $E \times_c \mathbb{Z}$ with source (v, m). Then by Remark 5.4, $r(\widetilde{\alpha}) = (w, m + r_w)$. Since $(w, n) := s(\widetilde{x})$ and $(w, m+r_w)$ belong to the same connected component, by Proposition 5.5 we must have

$$n - (m + r_w) = k'd \tag{2}$$

for some $k' \in \mathbb{Z}$. Since (w, n) connects to (u, n_i), it also follows from Lemma 4.3 (ii) that for all $i \geq 1$ one has

$$n_i = n + k_i d + s_u \tag{3}$$

for some $k_i \geq 1$, where s_u is the residue class of u in E with respect to w. By Lemma 4.3 (ii) there is a positive integer $N(u)$ such that for all $k \geq N(u)$ there exists a path of length $kd + s_u$ from w to u. Since the k_i's become arbitrarily large we may assume that $k' + k_i \geq N(u)$ for sufficiently large i, and so by Lemma 4.3 (ii) there exists paths β_i in E from w to u of length $(k' + k_i)d + s_u$. Let $\widetilde{\beta}_i$ be the lift of β_i in $E \times_c \mathbf{Z}$ with source $(w, m + r_w)$. Then by Remark 5.4

$$\begin{aligned} r(\widetilde{\beta}_i) &= (u, m + r_w + (k' + k_i)d + s_u) \\ &= (u, m + r_w + (n - (m + r_w)) + k_i d + s_u) && \text{by (2)} \\ &= (u, n + k_i d + s_u) = (u, n_i) && \text{by (3).} \end{aligned}$$

Hence $\widetilde{\alpha}\widetilde{\beta}_i$ is a path in the connected component of $E \times_c \mathbb{Z}$ containing (v, m) and (w, n), beginning at (v, m), and ending at (u, n_i) for sufficiently large i. Since $\widetilde{x}$ passes through (u, n_i) for all i, it follows that this component is cofinal by definition. □

6. The AF core

Let E be a directed graph, and $\{s_e, p_v\}$ a Cuntz-Krieger E-family generating $C^*(E)$. There is a strongly continuous action γ of $\mathbb{T}$ on $C^*(E)$ called the *gauge action* which is characterized by

$$\gamma_z s_e = z s_e, \quad \text{and} \quad \gamma_z p_v = p_v,$$

where $z \in \mathbb{T}$. The fixed point algebra $C^*(E)^\gamma$ of the gauge action is AF and is normally referred to as the AF core (see [16, Section 2.2] and [3, Section 2]). We wish to establish conditions which guarantee that $C^*(E)^\gamma$ is simple. To do this we shall use the following Morita equivalence which was established in [13, Proposition 2.8].

Lemma 6.1. *Let E be a row-finite directed graph with no sinks and sources, and $c : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$. Then $C^*(E)^\gamma$ is strongly Morita equivalent to $C^*(E \times_c \mathbb{Z})$.*

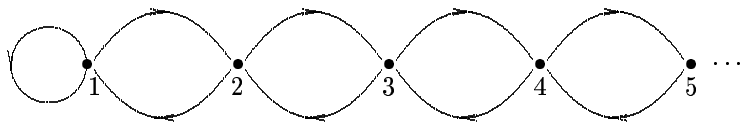
The case when there are finitely many vertices is straightforward.

Theorem 6.2. *Let E be a row-finite directed graph with no sinks or sources and E^0 finite. Then $C^*(E)^\gamma$ is simple if and only if E is strongly connected with period* 1.

Proof. Let E be strongly connected with period one, and $c : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$. By Proposition 5.13 the skew product graph $E \times_c \mathbb{Z}$ is cofinal. Since $E \times_c \mathbb{Z}$ has no loops and E is row-finite, by Theorem 3.1 $C^*(E \times_c \mathbb{Z})$ is simple, and then so is $C^*(E)^\gamma$ by Lemma 6.1.

Now suppose that $C^*(E)^\gamma$ is simple. Then by Lemma 6.1 $C^*(E \times_c \mathbb{Z})$ is simple. From Theorem 3.1 $E \times_c \mathbb{Z}$ is cofinal, and so E is cofinal by Remark 5.11. We claim that E is strongly connected. Let $u, v \in E^0$. Then since E^0 is finite and E has no sources, there is a loop $\alpha \in E^*$ and a path $\beta \in E^*$ with $s(\alpha) = s(\beta)$ and $r(\beta) = v$. Let $x = \alpha\alpha \cdots \in E^\infty$. Then since E is cofinal there is a path $\mu \in E^*$ with $s(\mu) = u$ and $r(\mu) = s(\alpha)$. Hence there is a path $\mu\beta \in E^*$ from u to v, which establishes our claim. From Proposition 5.12 and Corollary 5.6 it follows that E has period 1. □

Remark 6.3. Theorem 6.2 is not true if E^0 is infinite. The graph E shown below



is clearly strongly connected with period 1. However for the function $c : E^1 \to \mathbb{Z}$ given by $c(e) = 1$ for all $e \in E^1$, the skew product graph $E \times_c \mathbb{Z}$ is not cofinal.

If E has sinks or a vertex of infinite valency, then we will make use of the following result:

Lemma 6.4. *Let E be a directed graph, and suppose that $H \subset E^0$ is a nonempty saturated hereditary subset of E^0 such that $H \neq E^0$, then $C^*(E)^\gamma$ is not simple.*

Proof. Let H be a nontrivial saturated hereditary subset of E^0 and I_H be the gauge invariant ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. Let $\Phi : C^*(E) \to C^*(E)^\gamma$ be the conditional expectation associated to the gauge action. Then Φ is faithful on positive elements (cf. [3, Section 1]). Since H is a proper subset of E^0, and Φ an expectation which is faithful on positive elements, it follows that $\Phi(I_H)$ is a proper ideal of $C^*(E)^\gamma$ and the result follows. □

Let us first deal with the case when there are sinks.

Proposition 6.5. *Let E be a row-finite directed graph which has at least one sink. Then $C^*(E)^\gamma$ is simple if and only if E consists of a single vertex.*

Proof. If E consists of a single vertex, then since there are no edges one has $C^*(E) = C^*(E)^\gamma = \mathbb{C}$, which is simple. Suppose that E has sinks $v_1 \neq v_2$. Then $\Sigma(\{v_1\})$ is a proper saturated hereditary subset of E^0, and so gives rise to a proper ideal in $C^*(E)^\gamma$ by Lemma 6.4. Suppose now that v is the only sink and let J be the subalgebra of $C^*(E)^\gamma$ generated by elements of the form $s_\mu s_\nu^*$, where $\mu, \nu \in E^n$ for some $n \geq 1$ are such that $r(\mu) = r(\nu)$. It is straightforward to show that J is a closed 2-sided ideal in $C^*(E)$. If E

has edges, then J is nontrivial and $J \neq C^*(E)$ since $p_v \notin J$. Hence $C^*(E)^\gamma$ is simple only if there are no edges and $E^0 = \{v\}$. □

The case when there is a vertex of infinite valency is quite similar.

Proposition 6.6. *Let E be a directed graph with at least one vertex of infinite valency. Then $C^*(E)^\gamma$ is not simple.*

Proof. Suppose $v \in E^0$ has infinite valency. Let J be the C^*-subalgebra of $C^*(E)^\gamma$ generated by elements of the form $s_\mu s_\nu^*$, where $\mu, \nu \in E^n$ for some $n \geq 1$ are such that $r(\mu) = r(\nu)$. It is straightforward to show that J is a closed 2-sided ideal in $C^*(E)^\gamma$. Now $J \neq C^*(E)^\gamma$ since $p_v \notin J$, and J is nonzero since E has edges, so the result follows. □

Now suppose that E^0 is infinite and E has no sinks. If E is either strongly connected or has no loops, then we claim that for each $v \in E^0$ there is an infinite path $x(v)$, with source v such that for all n there is no path of length less than n from v to $r(x(v)_n)$. Every vertex on such a path is visited in the shortest distance from v. An infinite path with this property is said to be *v-depth-first*, since it is a path in the depth-first spanning tree of E (see [5, Section 3.3]).

For $v \in E^0$, let $V(0) = \{v\}$ and for $n \geq 1$ let $V(n)$ denote those vertices to which v connects by a path whose length is less than n. Since E^0 is infinite with no sinks and E is strongly connected or has no loops it follows that $V(n)$ is nonempty and the containment $V(n-1) \subset V(n)$ is strict for $n \geq 1$. The set $V(n) \setminus V(n-1)$ denotes those vertices which can be reached from v with shortest path of length n. We say that a path $\alpha \in E^n(v) := \{\alpha \in E^n : s(\alpha) = v\}$ is *v-deep* if it uses different vertices, that is if $r(\alpha_i) \in V(i) \setminus V(i-1)$ for $1 \leq i \leq n$. Let $E_g^n(v)$ denote the collection of v-deep paths in $E^n(v)$.

Lemma 6.7. *Let E be a directed graph with E^0 infinite and no sinks. If E is strongly connected or has no loops, then for $v \in E^0$ the set $E_g^n(v)$ is non-empty for all $n \geq 1$.*

Proof. The result follows immediately from the fact that $r\left(E_g^n(v)\right) = V(n) \setminus V(n-1)$ which is non-empty for all $n \geq 1$. □

Theorem 6.8. *Let E be a row-finite directed graph with E^0 infinite and no sinks. If E is strongly connected or has no loops, then for each $v \in E^0$ there is a v-depth-first path.*

Proof. For integers $m > n \geq 0$, let $E_g^{n,m}(v)$ denote those $\alpha' \in E_g^n(v)$ which appear as the first part of some $\alpha \in E_g^m(v)$. Clearly $E_g^{n,m}(v) \neq \emptyset$ for all $m > n \geq 0$ since $E_g^m(v)$ is non-empty and for any $\alpha = \alpha'\alpha'' \in E_g^m(v)$ with $\alpha' \in E^n(v)$ we have $\alpha' \in E_g^{n,m}(v)$. Moreover, this argument also shows that $E_g^{n,m}(v) \supseteq E_g^{n,p}(v)$ for all integers $p > m > n \geq 0$. Let

$$E_g^{n,\infty}(v) = \bigcap_{m>n} E_g^{n,m}(v).$$

Then $E_g^{n,\infty}(v) \neq \emptyset$, since it is the intersection of a decreasing sequence of finite non-empty sets. Now we may define the infinite path we seek: Since $E_g^{1,\infty}(v) \neq \emptyset$, there is $x(v)_1 \in E^1$ such that there are $\alpha_n \in E^n$ for $n \geq 1$ with $x(v)_1\alpha_n \in E_g^{1,n+1}(v)$. Since

E is row-finite, there is $x(v)_2 \in E^1$ such that there are infinitely many such α_n with $\alpha_n = x(v)_2\alpha''$, where $\alpha'' \in E^{n-1}$. In particular $x(v)_1x(v)_2 \in E_g^{2,\infty}(v)$, since for $n > 2$ there are $\alpha \in E_g^{2,n}(v)$ with $\alpha = x(v)_1x(v)_2\alpha''$ for some $\alpha'' \in E^{n-2}$. In this way we may define $x(v) = (x(v)_i) \in E^\infty$ with $s(x(v)_1) = v$ and $r(x(v)_i) \in V(i) \setminus V(i-1)$ for all $i \geq 1$. □

Corollary 6.9. *Let E be a row-finite directed graph with E^0 infinite and no sinks. If E is strongly connected or has no loops, then $C^*(E)^\gamma$ is not simple.*

Proof. Let $c : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$ and for $v \in E^0$, let $x(v)$ be a v-depth-first path. Let $x' \in (E \times_c \mathbb{Z})^\infty$ be the lift of $x(v)$ with source $(-1, v)$, and suppose that α' is such that $s(\alpha') = (0, v)$ and $r(\alpha') = r(x'_i) = (i-1, r(x(v)_i))$ for some i. Hence $\alpha = p_c(\alpha')$ is a path of length $i-1$ from v to $r(x(v)_i)$, which means that $r(x(v)_i) \in V(i-1)$, which contradictions the definition of $x(v)$. Hence $E \times_c \mathbb{Z}$ is not cofinal, so $C^*(E \times_c \mathbb{Z})$ is not simple by Theorem 3.1, and hence $C^*(E)^\gamma$ is not simple by Lemma 6.1 □

Theorem 6.10. *Let E be a directed graph. Then $C^*(E)^\gamma$ is simple if and only if either E has a cofinal strongly connected subgraph of period* 1 *which has finitely many vertices, or E consists of a single vertex.*

Proof. If E has a vertex infinite valency, then $C^*(E)^\gamma$ is not simple by Proposition 6.6. So we may suppose that E is row-finite. If E is not cofinal then, by the proof of Theorem 3.1, there is a nontrivial saturated hereditary subset of E^0 which gives rise to a nontrivial ideal in $C^*(E)^\gamma$ by Lemma 6.4. So we may suppose that E is cofinal. If E does not satisfy condition (K), then there is a cofinal subgraph L which consists of a loop with no exits. Let $c : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$. Then $E \times_c \mathbb{Z}$ is cofinal if and only if L has one vertex. Hence by Lemma 6.1, $C^*(E)^\gamma$ is simple if and only if E has a cofinal strongly connected subgraph of period 1 which consists of a single vertex (and edge). So we may suppose that E also satisfies condition (K). Therefore we may assume that E satisfies conditions (i)–(iii) of Theorem 3.1, and so $C^*(E)$ is simple.

If E has a sink, then by Proposition 6.5 $C^*(E)^\gamma$ is simple if and only if E consists of a single vertex. So we suppose that E has no sinks. By Theorem 3.2 either E has no loops, or there is a strongly connected cofinal subgraph $F \subseteq E$. Suppose that E has no loops. Then E^0 cannot be finite (as that would mean there had to be a sink cf. [12, Section 2]). If E^0 is infinite, then $C^*(E)^\gamma$ is not simple by Corollary 6.9. Suppose E has a strongly connected cofinal subgraph F. Then by Corollary 3.3 there is a projection $P \in M(C^*(E))$ such that $C^*(F) \cong PC^*(E)P$. Since this isomorphism is essentially the identity map, it commutes with the usual gauge actions on $C^*(E)$ and $C^*(F)$, which are both denoted by γ. The projection P is the limit of a sum of projections in $C^*(E)$ which are invariant under the gauge action. Hence $P \in M(C^*(E))^\gamma$, and then $C^*(F)^\gamma \cong PC^*(E)^\gamma P$. Since F is cofinal in E, it follows that $PC^*(E)^\gamma P$ is full, and so $C^*(E)^\gamma$ is strongly Morita equivalent to $C^*(F)^\gamma$. The result now follows from Theorem 6.2 and Theorem 6.9. □

If we discount the trivial case when the graph only consists of a single vertex, Theorem 6.10 says the following: Up to Morita equivalence, the only graphs E for which the

AF core of $C^*(E)$ is simple are those which have finitely many vertices and are strongly connected with period one. For strongly connected graphs we prove the following structure results for the AF core.

Theorem 6.11. *Let E be a strongly connected row-finite graph with period d. Then $C^*(E)^\gamma$ is a direct sum of d mutually isomorphic AF algebras. If in addition E^0 is finite, then these AF algebras are simple.*

Proof. Let E be strongly connected, row-finite with period d, and $c : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$. By Corollary 5.6 $E \times_c \mathbb{Z}$ consists of d mutually isomorphic components. Since each component is a subgraph of a graph which has no loops, the first part follows from Lemma 6.1 and [12, Theorem 2.4]. If E^0 is finite, then each component is cofinal by Proposition 5.13, and the last statement follows from Theorem 3.1. □

Theorem 6.12. *Let E be a strongly connected row-finite graph with finitely many vertices. Then $C^*(E)$ is stably isomorphic to a crossed product of a simple AF algebra by an action of $\mathbb{Z}$.*

Proof. Let E be a strongly connected graph with period d, T be a spanning tree for E, $v \in E^0$ and $c : E^1 \to \mathbb{Z}$ be given by $c(e) = 1$ for all $e \in E^1$. By Proposition 5.10 each component of $E \times_c \mathbb{Z}$ is isomorphic to $E \times_{c_{v,T}} \Gamma_v(c)$. Since E is strongly connected with period d, by Proposition 5.12 we have $\Gamma_v(c) = d\,\mathbb{Z} \cong \mathbb{Z}$. Let λ denote the free $\mathbb{Z}$-action on $E \times_{c_{v,T}} d\mathbb{Z}$ which has quotient E. Then by [13, Corollary 3.9] one has

$$C^*(E \times_{c_{v,T}} d\,\mathbb{Z}) \times_\lambda \mathbb{Z} \cong C^*(E) \otimes \mathcal{K}\big(\ell^2(\mathbb{Z})\big).$$

Since $E \times_{c_{v,T}} d\,\mathbb{Z}$ is isomorphic to a subgraph of $E \times_c \mathbb{Z}$ it has no loops, and so $C^*(E \times_{c_{v,T}} d\mathbb{Z})$ is AF by [12, Theorem 2.4]. If E^0 is finite, then $E \times_{c_{v,T}} d\,\mathbb{Z}$ is cofinal by Proposition 5.13. It then follows that $C^*(E \times_{c_{v,T}} d\,\mathbb{Z})$ is simple by Theorem 3.1, which completes the proof. □

REFERENCES

[1] A. AN HUEF, I. RAEBURN. The ideal structure of Cuntz-Krieger algebras, *Ergodic Theory Dynamic. Systems* **17**(1997), 611–624.

[2] T. BATES, J-H. HONG, I. RAEBURN, W. SZYMAŃSKI, The ideal structure of the C^*-algebras of infinite graphs, University of Newcastle, preprint, 2001.

[3] T. BATES, D. PASK, I. RAEBURN, W. SZYMANSKI, The C^*-algebras of row-finite graphs, *New York J. Math.* **6**(2000), 307–324.

[4] J. CUNTZ, W. KRIEGER, A class of C^*-algebras and topological Markov chains, *Invent. Math.* **56**(1980), 251–268.

[5] G. CHARTRAND, O.R. OELLERMANN, *Applied and Algorithmic Graph Theory*, International Series in Pure Applied Mathematics, McGraw-Hill, Ney York, 1993.

[6] K. DEICKE, D. PASK, I. RAEBURN, Covering of directed graphs and crossed products of C^*-algebras by coactions of homogeneous spaces, University of Newcastle, prepint, 2001.

[7] J.L. GROSS, T.W. TUCKER, *Topological Graph Theory*, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley & Sons, New York, 1987.

[8] J.L. GROSS, T.W. TUCKER, Generating all graph coverings by permutation voltages assignments, *Discrete Math.* **18**(1977), 273–283.

[9] S. KALISZEWSKI, J. QUIGG, I. RAEBURN, Skew products and crossed products by coactions, *J. Operator Theory* **46**(2001), 411–433.

[10] C. KOSNIOWSKI, *A First Course in Algebraic Topology*, Cambridge Univ. Press, 1980.

[11] A. KUMJIAN, D. PASK, I. RAEBURN, J. RENAULT, Graphs, groupoids and Cuntz-Krieger algebras, *J. Funct. Anal.* **144**: 505–541, 1997.
[12] A. KUMJIAN, D. PASK, I. RAEBURN, Cuntz-Krieger algebras of directed graphs, *Pacific. J. Math.* **184**(1998), 161–174.
[13] A. KUMJIAN, D. PASK, C^*-algebras of directed graphs and group actions, *Ergodic Theory Dynamic. Systems* **19**(1999), 1503–1519.
[14] A. KUMJIAN, D. PASK, Higher rank graph C^*-algebras, *New York J. Math.*, **6**(2000), 1–20.
[15] D. LIND, H. MARCUS, *An Introduction to Symbolic Dynamical System and Coding*, Cambridge Univ. Press, 1995.
[16] D. PASK, I. RAEBURN, On the K-theory of Cuntz-Krieger algebras, *Publ. Res. Inst. Math. Sci.* **32**(1996), 415–443.
[17] A.L.T. PATERSON, Graph inverse semigroups, groupoids and their C^*-algebras, *J. Operator Theory*, to appear.
[18] I. RAEBURN, W. SZYMAŃSKI, Cuntz-Krieger algebras of infinite graphs and matrices, University of Newcastle, preprint, 2000.
[19] S.-J. RHO, Properties of graphs and their C^*-algebras Masters Thesis, University of Newcastle, 2001.
[20] E. SENETA, *Non-Negative Matrices: An Introduction to Theory and Applications*, Publ. George Allen and Unwin Ltd., Second edition, 1973.
[21] J. STILLWELL, *Classical Topology and Combinatorial Group Theory*, Graduate Texts in Math., vol. 72, Springer-Verlag, (1980).
[22] W. SZYMAŃSKI, Simplicity of Cuntz-Krieger algebras of infinite matrices, *Pacific J. Math.* **199**(2001), 249–256.

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