Please staple your assignment together, with this cover sheet. Give full working for all answers, unless the question says otherwise. Untidy or badly set-out work will not be marked, and will be recorded as unsatisfactory. This assignment must be submitted before the end of your tutorial in week 2.

**Question 1.**

Statements may be defined as declarative sentences which are true or false but not both.

What distinguishes declarative sentences from other kinds of sentences?

**Question 2.**

Complete the following sentences:

a) A statement which is true requires a ______________

b) A statement which is false requires a ______________

**Question 3.**

Are the following sentences statements? Where a sentence is a statement, what is its truth value? Briefly justify your answers.

a) For all real numbers \( x \) and \( y \), \( x + y = y + x \)

b) For every real number \( x \) there exists a real number \( y \) such that \( xy = 1 \)

b) This sentence is false.

**Question 4.**

Write the following compound statements in terms of simple statements and appropriate connectives. Make sure that all statements variables \( p, q, r \), and so on are clearly defined.

a) WUCT121 is a 6 credit point subject implies all students are expected to do twelve hours of work in this subject each week of session.

b) If you attend all lectures and tutorials, attempt all tutorial and assignment questions, and ask your teachers for help when you don’t understand something, you should pass this course.
**Question 5.**

a) What does it mean to say that an operation on \( \mathbb{N} \) is closed?

b) Is Exponentiation a closed operation on \( \mathbb{N} \)? Briefly justify your answer.

c) An identity, \( i \), is an element of a set which under an operation with any member, say \( a \), of a set will return that member, \( a \).

Does an identity element exist for Exponentiation on \( \mathbb{N} \)? Briefly justify your answer.

d) An inverse of an element, say \( a \), is an element of a set, say \( b \), which under an operation with \( a \), will return the identity.

Do inverse elements exist for Exponentiation on \( \mathbb{N} \)? Briefly justify your answer.

**Question 6.**

State the commutative, associative, and distributive properties of natural numbers.

**Question 7.**

State the law of trichotomy, the three transitivity properties, and the well ordering property of the set of natural numbers.

**Question 8.**

The set of natural numbers \( \mathbb{N} \) is a subset of the set of integers \( \mathbb{Z} \). \( \mathbb{N} \) is a well ordered set. Is \( \mathbb{Z} \) also a well ordered set? Briefly justify your answer.
Question 1.

Write the following compound statements in terms of simple statements and appropriate connectives. Make sure that all statements variables $p$, $q$, $r$, and so on are clearly defined.

a) Both mathematics and statistics are not easy.

b) Not both mathematics and statistics are easy.

Question 2.

Write full truth tables for the statements in question 1. Are these statements logically equivalent? Justify your answer.

Question 3.

Write full truth tables for the following statements. Are these statements tautologies, contradictions, or contingent statements? Justify your answers.

a) $(p \land q) \Rightarrow \neg(p \lor \neg q)$

b) $((p \lor q) \land (p \lor r)) \Rightarrow (p \land (q \lor r))$

Question 4.

Use the quick method to determine whether the statements in question 3 are tautologies or not. Justify your answers.
Question 5.

Any rational number may be written as the ratio of two integers. Most programming languages have mathematical functions that return the integer part of the ratio (called Mod) and the integer remainder after the division (called Div), for example

\[ \text{Mod}(13, 4) = 3 \]
\[ \text{Div}(13, 4) = 1 \]

Are either of these functions closed operations on \( \mathbb{Z} \)? Briefly justify your answer.

Question 6.

Is zero an odd number, an even number, or neither? State definitions for the odd and even numbers and use them to justify your answer.

Question 7.

Is one a prime number, a composite number, or neither? State definitions for the prime and composite numbers and use them to justify your answer.

Question 8.

Is Pi (\( \pi = 3.14159265 \) rounded to eight decimal places) a rational number or an irrational number? State definitions for the rational and irrational numbers and use them to justify your answer.
Question 1.

Write full truth tables for the following statements. Are these statements tautologies, contradictions, or contingent statements? Justify your answers.

a) \((p \lor q) \Rightarrow \neg(p \land \neg q)\)

b) \((p \lor (q \lor \neg q)) \Rightarrow (q \land (r \land \neg r))\)

Question 2.

Use the quick method to determine whether the statements in question 3 are contradictions or not. Justify your answers.

Question 3.

a) State the Rules of Substitution and Substitution of Equivalence.

b) Using these rules and the following logical equivalences

\[\neg(p \lor q) \iff \neg(p \land \neg q)\] De Morgan’s Laws

\[(p \land (q \land r)) \iff ((p \land q) \land r)\] Associative Laws

\[(p \land q) \iff (q \land p)\] Commutative Laws

\[\neg
\] Double Negation

prove that the following statement must be a tautology.

\[\neg(p \lor (q \lor \neg r)) \iff (r \land (q \land p))\]

[Do not use a full truth table or the quick method to answer this question.]
**Question 4.**

Write the following statements in the notation of predicate logic

a) All students dislike some of their subjects.

b) No student has completed all assigned work in this subject.

**Question 5.**

Find the negations of the statements in Question 4 and determine whether the original statement or its negation is true. Justify your answers.

**Question 6.**

Write the following statements in English

a) \( \forall x \in \mathbb{Z} \) (\( x \) is odd \( \Rightarrow \) \( \exists y \in \mathbb{Z} \) (\( x = 2y + 1 \))

b) \( \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z} \) (\( x + y \neq 0 \))

**Question 7.**

Find the negations of the statements in Question 6 and determine whether the original statement or its negation is true. Justify your answers.
Student name: ___________________________________________ Student number: ____________

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**Question 1.**

Using the Principle of Mathematical Induction, prove

\[ 1.2 + 2.3 + 3.4 + \cdots + (n - 1)n = \frac{n(n - 1)(n + 1)}{3} \]

for each natural number \( n \).

**Question 2.**

Using the Principle of Mathematical Induction, prove

\[ n(n^2 + 5) \text{ is divisible by } 6 \]

for each natural number \( n \).

[Hint: Consider the definition of divisibility: For all integers \( a \) and \( b \), if \( b \) divides \( a \), there exists an integer \( c \) such that \( a = bc \), that is, \( \forall a, b \in \mathbb{Z} \ (a|b \Rightarrow \exists c \in \mathbb{Z}, a = bc) \) ]

**Question 3.**

Using the Generalised Principle of Mathematical Induction, prove

\[ \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n + 1}{2n} \]

for each natural number \( n \geq 2 \)

**Question 4.**

Using the Generalised Principle of Mathematical Induction, prove

\[ n^3 > 2n + 1 \]

for each natural number \( n \geq 2 \)
Question 5.

Suppose that $a_1, a_2, a_3, \ldots$ is a sequence defined as follows:

$$a_1 = \frac{9}{10}, \ a_2 = \frac{10}{11}, \text{ and } a_n = a_{n-1} \cdot a_{n-2} \text{ for all natural numbers } n \geq 3$$

Use the Strong Principle of Mathematical Induction to prove that

$$a_n \leq 1$$

for all natural numbers $n$.

Question 6.

Observe that

$$\left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) = \frac{1}{3}$$

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) = \frac{1}{4}$$

Guess a general formula and prove it by Mathematical Induction.

Question 7.

a) State the Law of Syllogism and the rules of Modus Ponens and Modus Tollens.

b) Express the following argument in the notation of Predicate Logic

Julia Gillard and Tony Abbott are politicians.
No politician can be trusted.
Therefore neither Julia Gillard nor Tony Abbott can be trusted.

c) Is this a valid argument? Justify your answer.

Question 8.

a) Briefly explain how you would disprove a Universal statement.

b) Briefly explain how you would prove an Existential statement.
Question 1.

a) Briefly explain how you would prove a Universal statement.

b) Briefly explain how you would disprove an Existential statement.

Question 2.

Prove or disprove the following statements:

a) \( \forall a, b \in \mathbb{Z} \ (a|b \Rightarrow \exists c \in \mathbb{Z} \ (b = ac)) \)

b) \( \exists a, b \in \mathbb{Z} \ (a|b \land b|a \land a \neq b) \)

Question 3.

State which methods of proof use the following tautologies:

a) \( (p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p) \)

b) \( ((p \lor q) \Rightarrow r) \equiv ((p \Rightarrow r) \land (q \Rightarrow r)) \)

c) \( (p \land (q \land \sim q)) \Rightarrow \sim p \)

Question 4.

Using the tautologies given above in question (3), prove the following statements:

a) For all natural numbers \( n \), if \( n^3 \) is odd, then so is \( n \).

b) If \( n = 1, 2, \) or \( 3 \), then \( (n-1)(n-2)(n-3) = 0 \).

c) There is no smallest positive real number.
**Question 5.**

a) State the Quotient-Remainder Theorem.

b) Using this theorem or otherwise, prove that any integer can be written in the form $4k, 4k + 1, 4k + 2, \text{ or } 4k + 3$, for some integer $k$.

**Question 6.**

a) State the Fundamental Theory of Arithmetic.

b) Using this theorem or otherwise, identify all of the prime factors of 17493.

**Question 7.**

a) State the algorithm used in the Sieve of Eratosthenes.

b) Using the Sieve of Eratosthenes, find all prime numbers between 300 and 400.

c) Identify all the twin primes between 300 and 400.
Student name: ___________________________ Student number: ______________

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**Question 1.**

Let $U = \{1, 2, 3, 4, 5, 6\} = \{x \in \mathbb{Z} : 1 \leq x \leq 6\}$ be the universal set.
Let $A = \{2, 3, 4\} = \{x \in \mathbb{Z} : 2 \leq x \leq 4\}$ and $B = \{3, 4, 5\} = \{x \in \mathbb{Z} : 3 \leq x \leq 5\}$ be subsets of $U$.

a) Write down the following sets:
   i) $A \cup B$
   ii) $A \cap B$
   iii) $A - B$
   iv) $B - A$
   v) $\overline{A}$
   vi) $\overline{B}$

b) Draw a Venn diagram showing $U$, $A$, and $B$

**Question 2.**

Let $U = [1, 6] = \{x \in \mathbb{R} : 1 \leq x \leq 6\}$ be the universal set.
Let $A = [2, 4] = \{x \in \mathbb{R} : 2 \leq x \leq 4\}$ and $B = [3, 5] = \{x \in \mathbb{R} : 3 \leq x \leq 5\}$ be subsets of $U$.

a) Write down the following sets:
   i) $A \cup B$
   ii) $A \cap B$
   iii) $A - B$
   iv) $B - A$
   v) $\overline{A}$
   vi) $\overline{B}$

b) Draw number lines showing each of the sets in (i) to (vi) above.
Question 3.

a) State down the definition of the Power set of a set \( A \), that is, \( \mathcal{P}(A) \).

b) Let \( A = \{1, 2\} \). Write down the following sets:
   i) \( \mathcal{P}(A) \)
   ii) \( \mathcal{P}(\mathcal{P}(A)) \)

c) Draw a Hasse diagram of \( \mathcal{P}(A) \)

Question 4.

a) State down the definition of the Greatest Common Divisor of two integers \( a \) and \( b \).

b) Find \( \text{gcd}(512, 172) \)

c) Find \( \text{gcd}(612, 372) \)

Question 5.

a) State down the definition of the Least Common Multiple of two integers \( a \) and \( b \).

b) Find \( \text{lcm}(512, 172) \)

c) Find \( \text{lcm}(612, 372) \)

Question 6.

a) State down the Euclidean Algorithm.

b) Use the Euclidean algorithm to find \( \text{gcd}(512, 172) \)

c) Use the Euclidean algorithm to find \( \text{gcd}(612, 372) \)
Question 1.

Determine integers $m$ and $n$ such that
a) $\gcd(512, 172) = 512m + 172n$

b) $\gcd(612, 372) = 612m + 372n$

Question 2.

What is the minimum number of students that must be enrolled at the WCA for there to be two students who have the same birthday?

Question 3.

Suppose we have 200 language students are enrolled at the WCA, and we intend to allocate them to 15 classes such that no class has less than 12 students or more than 16 students. Show that there must be at least five classes of at least 13 students.

Question 4.

a) State appropriate definitions for Div and Mod.

b) Write down the following results

i) $1632 \text{ div } 51$

ii) $1632 \text{ mod } 51$

iii) $2783 \text{ div } 29$

iv) $2783 \text{ mod } 29$
**Question 5.**

a) State appropriate definitions for equivalence class and set of residues.

b) Write down addition and multiplication tables for $\mathbb{Z}_7$.

c) Does an additive identity exist for $\mathbb{Z}_7$? Justify your answer.

d) Write down the additive inverses of each element of $\mathbb{Z}_7$ or explain why they do not exist.

e) Does a multiplicative identity exist for $\mathbb{Z}_7$? Justify your answer.

f) Write down the multiplicative inverses of each element of $\mathbb{Z}_7$ or explain why they do not exist.

**Question 6.**

Use the set theorems on pages 159-161 of your notes to prove \((A - B) - C = A \cap (\overline{B} \cup \overline{C})\)

**Question 7.**

Use a typical element argument to prove \((A - B) - C = A \cap (\overline{B} \cup \overline{C})\)

**Question 8.**

Use any method to prove or disprove the following statements:

a) \(A - (B - C) = (A - B) - C\)

b) \((C \subseteq A \land C \subseteq B) \Rightarrow C \subseteq (A \cap B)\)
Question 1.

State the definition of
a) an ordered pair \((x, y)\),
b) a Cartesian product \(A \times B\), and
c) a Binary relation \(T\) from \(A\) to \(B\)

Question 2.

Let \(A = \{1, 2, 3\}\) and \(B = \{4, 5, 6\}\). List all of the elements of the following sets
a) \(A \times B\)
b) \(T = \{(x, y) \in A \times B : x + y \text{ is a prime number}\}\)

Question 3.

State the definition of
a) the domain and range of a binary relation \(T\) from \(A\) to \(B\)
b) the inverse of a binary relation \(T\) from \(A\) to \(B\)

Question 4.

Sketch the following relations and write down their domains and ranges
a) \(T_1 = \{(x, y) : x \in \mathbb{R} \land y \in \mathbb{R} \land x^2 + y^2 < 4\}\)
b) \(T_2 = \{(x, y) : x \in \mathbb{R} \land y \in \mathbb{Z} \land x^2 + y^2 < 4\}\)
c) \(T_3 = \{(x, y) : x \in \mathbb{Z} \land y \in \mathbb{R} \land x^2 + y^2 < 4\}\)
d) \(T_4 = \{(x, y) : x \in \mathbb{Z} \land y \in \mathbb{Z} \land x^2 + y^2 < 4\}\)
**Question 5.**

State the definition of

a) a simple graph $G$,

b) a non-connected graph $G$,

c) adjacent vertices $v_1$ and $v_2$,

d) adjacent edges $e_1$ and $e_2$,

e) incident edge $e_1$ and vertex $v_1$.

**Question 6.**

Draw the following graphs or state why such a graph cannot exist.

a) simple graph with four vertices of degree 1, 1, 1, and 2

b) simple graph with four vertices of degree 1, 1, 1, and 3

c) simple graph with four vertices of degree 1, 1, 1, and 5

**Question 7.**

Given $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{(v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_3), (v_2, v_6), (v_3, v_6)\}$,

a) Sketch the graph $G = \{E, V\}$

b) Is $G$ a simple graph? If yes, justify your answer. If no, find a simple sub-graph $H$ using the same set of vertices.

c) Is $G$ a connected graph? If yes, justify your answer. If no, find a connected graph $H$ using the same set of vertices and which contains $G$ as a sub-graph.
Question 1.
State the definition of the reflexivity, symmetry, and transitivity properties for a binary relation $T$ from $A$ to $B$.

Question 2.

a) State the definition of a binary function $f$ from $A$ to $B$.
b) Is the domain of $f$ always equal to $A$? Justify your answer.
c) Is the range of $f$ always equal to $B$? Justify your answer.

Question 3.
Let $H$ be the set of all human beings. Consider the following relation on $H$
$$T = \{(x, y) : x \text{ is the brother or sister of } y\}$$
a) Write down the domain and range of $T$.
b) Write down the inverse relation $T^{-1}$.
c) Is $T$ reflexive? Justify your answer.
d) Is $T$ symmetric? Justify your answer.
e) Is $T$ transitive? Justify your answer.
f) Is $T$ an equivalence relation? Justify your answer.
g) Is $T$ a one-to-one relation? Justify your answer.
h) Is $T$ a function? Justify your answer.
i) Is $T$ onto $H$? Justify your answer.
**Question 4.**

State the definition of a permutation $f$ on $A$.

**Question 5.**

Let $A = \{1, 2, 3, 4, 5\}$. Let $f$ and $g$ be permutations on $A$ such that $f = (1 \ 4 \ 3) \ (2 \ 5)$ and $g = (3 \ 4 \ 5)$.

Write down the following permutations:

a) $f \cdot g$

b) $f^{-1}$

c) $g^{-1}$

d) $g^{-1} \cdot f^{-1}$

**Question 6.**

a) How could you tell whether two graphs with the same numbers of vertices and edges are isomorphic or not?

b) Draw three graphs each with three vertices and three edges and explain whether or not any of them are isomorphic graphs. Justify your answers.

**Question 7.**

a) State the definition of Bipartite and Complete Bipartite Graphs.

b) Draw a Bipartite Graph $G$ with 5 vertices and 5 edges (or explain why no such graph exists), where 2 vertices belong to $U$, 3 vertices belong to $W$, and $U$ and $W$ are subsets of the set of all vertices $V$.

c) Draw a Complete Bipartite Graph $H$ with 5 vertices where 2 vertices belong to $U$ and 3 vertices belong to $W$.

d) Is $G$ a sub-graph of $H$? Are $G$ and $H$ actually the same graph? Justify your answers.

**Question 8.**

a) State the definition of a circuit.

b) State the definition of an Eulerian circuit.

c) State the definition of an Eulerian graph?

d) Draw the graph $\{V, E\}$, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{(v_1, v_2), (v_1, v_7), (v_2, v_3), (v_3, v_4), (v_3, v_5), (v_4, v_6), (v_5, v_6), (v_6, v_7)\}$, and determine whether or not it is an Eulerian graph. Justify your answer.

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WUCT121: Discrete Mathematics
Assignment 9, Spring 2010
Submission Receipt

Student name: ___________________________ Student number: __________

Date submitted: ___________________________ Tutor initials: __________
Question 9.

a) State the definition of a tree.

b) State the definition of a spanning tree.

c) Find all of the spanning trees for the graph defined in Question 9(d) above. You may either draw the trees or explain what edges should be removed from the graph to result in each spanning tree.

Question 10.

Use Kruskal’s algorithm to find a minimal spanning tree for the graph below.

```
  a
 /|\
 b 6/ |
 |6|
 7/ |
 9|
  a
 ```
Question 1.

A declarative sentence basically describes a property of a thing or a relationship between two or more things. For example, \( x = 1 \) describes a property of a number \( x \), while \( x = y \) describes a relationship between numbers \( x \) and \( y \).

Question 2.

a) Proof.

b) Demonstration.

Question 3.

a) True statement – this is the Commutivity property for addition of real numbers.

b) False statement – when \( x = 0 \), no real number \( y \) exists such that \( xy = 1 \).

c) Not a statement – this is a declarative statement which does not have a truth value.

If we say that “This sentence is false” is true, we have a problem in that the sentence would be both true and false, which is not possible, so it cannot be true.

If we say that “This sentence is false” is false, we are forced to conclude that the sentence must be true, which is the same problem as above, so it cannot be false.

Therefore, this sentence cannot be true and cannot be false, and so is not a statement.

Question 4.

a) Statement form is \( p \land q \), where

\( p \) WUCT121 is a 6 credit point subject.

\( q \) All students are expected to do twelve hours of work in this subject each week of session.

b) Statement form is \((p \land q) \land (r \land s) \land t \Rightarrow u\), where

\( p \) You attend all lectures.

\( q \) You attend all tutorials.

\( r \) You attempt all tutorial questions.

\( s \) You attempt all assignment questions.

\( t \) You ask your teachers for help when you don’t understand something

\( u \) You should pass this course.

Strictly speaking, \( t \) is also a compound statement, as “when” and “don’t” can be interpreted in terms of connectives and simpler statements, and could be written as \( \sim t_1 \Rightarrow t_2 \), where

\( t_1 \) You do understand something.

\( t_2 \) You ask your teachers for help.

Other alternate ways of expressing conditional statements were discussed in your week 2 logic lecture. These expressions are all examinable.

Notice that all these simple statements must begin with “You …”. If you leave this word out, the sentence cannot be assigned a truth value, and so it would not be a statement.
Question 5.

a) An operation on the set of natural numbers is said to be closed if and only if the result of that operation is always a natural number. Formally, we could write something like “The operation \( op \) is closed on the set of natural numbers if, for all natural numbers \( a \) and \( b \), there exists a natural number \( c \) such that \( op(a, b) = c \).”

b) Exponentiation is a closed operation on \( \mathbb{N} \), that is, for all \( a, b \in \mathbb{N} \), \( a^b \in \mathbb{N} \), as \( a^b \) is simply a shorthand way of writing the product of \( a \) times itself \( b \) times, and as multiplication is a closed operation of \( \mathbb{N} \), then \( a^b \) must be a natural number, as so exponentiation must also be a closed operation on \( \mathbb{N} \).

c) The identity is 1 as for all natural number \( a \), \( a^1 = a \).

d) The inverse of 1 under exponentiation is itself. No other natural numbers have inverses under exponentiation.

Question 6.

For all natural numbers \( x, y, \) and \( z \), the following properties are true

1) Commutative properties:
   \( x + y = y + x \)
   \( xy = yx \)

2) Associative properties:
   \( x + (y + z) = (x + y) + z \)
   \( x(yz) = (xy)z \)

3) Distributive property:
   \( x(y + z) = xy + xz \)

Question 7.

Law of Trichotomy:
Given two natural numbers \( x \) and \( y \), one and only one of the following three relationships are true:

a) \( x > y \)

b) \( x = y \)

c) \( x < y \)

Laws of Transitivity:
Given three natural numbers \( x, y, \) and \( z \), the following statements are true:

a) If \( x > y \) and \( y > z \) then \( x > z \)

b) If \( x = y \) and \( y = z \) then \( x = z \)

c) If \( x < y \) and \( y < z \) then \( x < z \)
Well ordering principle:
A set of numbers is deemed to be a well ordered set if and only if the set has a minimum value and every subset of that set also has a minimum value.

Question 8.

\( \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \) is not a well ordered set as it does not contain a minimum value. However, some subsets of \( \mathbb{Z} \), such as \( \mathbb{N} \), are well ordered sets.
Question 1.

Let \( p \) be “Mathematics is easy” and \( q \) be “Statistics is easy”.

a) \(~p \land \sim q\)

b) \(~(p \land q)\)

Question 2.

a) Statement is contingent.

\[
\begin{array}{cccc}
 p & q & \sim p & \sim q \\
 T & T & F & F \\
 T & F & F & T \\
 F & T & T & F \\
 F & F & T & T \\
\end{array}
\]

b) Statement is contingent.

\[
\begin{array}{cccc}
 p & q & \sim (p \land q) \\
 T & T & F & T \\
 T & F & T & F \\
 F & T & T & F \\
 F & F & T & F \\
\end{array}
\]

However, the two statements are not logically equivalent as their truth values are not the same for each possible combination of truth values of \( p \) and \( q \).

Question 3.

a) Statement is always true and so is a tautology

\[
\begin{array}{cccccccc}
 p & q & (p \land q) & \Rightarrow & \sim (\sim p \lor \sim q) \\
 T & T & T & T & T & T & F & F \\
 T & F & F & T & T & F & F & F \\
 F & T & F & T & T & T & T & T \\
 F & F & F & T & T & T & T & T \\
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\]
b) Statement is neither always true nor always false and so is contingent.

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**Question 4.**

a) Statement cannot be false and so is a tautology

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<tr>
<th></th>
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<th>((p \land q))</th>
<th>(\Rightarrow)</th>
<th>(\neg(p \lor \neg q))</th>
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</table>

Step 1: Place an F under the main connective.

Step 2: This requires that connective #1 is true and connective #5 is false.

Step 3: Connective #1 is true requires that the first occurrences of both p and q are true.

Step 4: Connective #5 is false requires that connective #4 is true.

Step 5: We now have a problem – there are three possible ways for connective #4 to be true: T \lor T, T \lor F, and F \lor T. We could test each of these cases separately, but since we already know the truth values of p and q, we will use them here.

Step 6: This requires that connectives #2 and #3 must both be false.

Step 7: This requires that connective #4 is false. However, we have already determined in step 4 that connective #4 is true. It is impossible for it to be both true and false. Therefore, our original assumption in step 1, that the main connective could be false, was wrong, and so this statement cannot be false. It is therefore a tautology.
b) Statement can be false and so is not a tautology.

\[
\begin{array}{cccccc}
((p \lor q) \land (p \lor r)) & \Rightarrow & (p \land (q \lor r)) \\
1 & 3 & 2 & 6 & 5 & 4 \\
\text{Step 1} & & & F \\
\text{Step 2} & T & & F \\
\text{Step 3} & T & T & \\
\text{Step 4} & T & T & T \\
\text{Step 5} & & & F \\
\text{Step 6} & F & F & \\
\text{Step 7} & F & F & \\
\text{Step 8} & T & T & \\
\end{array}
\]

Step 1: Place an F under the main connective.
Step 2: This requires that connective #3 is true and connective #5 is false.
Step 3: Connective #3 is true requires that connectives #1 and #2 must both be true.
Step 4: We now have a problem – there are three possible ways for connectives #1 and #2 to be true: T \lor T, T \lor F, and F \lor T. Similarly, there are three possible ways for connective #5 to be false: T \land F, F \land T, and F \land F. We may have to test each possible combination of truth values until we find a combination which works or we have exhausted all of the possible combinations. Place a T under all occurrences of p.
Step 5: This requires that connectives #4 is false.
Step 6: This requires that q and r must both be false.
Step 7: Since we now know the truth values of q and r, we will use them here.
Step 8: This requires that connectives #1 and #2 must both be true. We have already determined in step 3 that they are both true. There is no inconsistency. Therefore, our original assumption in step 1, that the main connective could be false, has been shown to be correct, and so this statement cannot always be true. It is therefore not a tautology.

**Question 5.**

If Mod and Div were closed operations on \( \mathbb{Z} \) we would require

For all \( a, b \in \mathbb{Z}, \text{Mod}(a, b) \in \mathbb{Z} \)

For all \( a, b \in \mathbb{Z}, \text{Div}(a, b) \in \mathbb{Z} \)

However, the ratio \( a/b \) is always undefined when \( b = 0 \), so Mod, the integer part of the ratio, and Div, the integer remainder after the division, are also undefined. Therefore Mod and Div cannot be closed operations on \( \mathbb{Z} \).

**Question 6.**

\( x \) is an even number if and only if there exists an integer \( y \) such that \( x = 2y \).
\( x \) is an odd number if and only if there exists an integer \( y \) such that \( x = 2y - 1 \).

Therefore, zero is an even number since \( 0 = 2 \times 0 \)
Question 7.

$x$ is a prime number if and only if $x$ is greater than 1 and has no positive factors other than itself and 1.

$x$ is a composite number if and only if $x$ is greater than 1 and has positive factors other than itself and 1.

Therefore, one is neither a prime nor composite number.

Question 8.

$x$ is a rational number if and only if it can be written as the ratio of two integers $a$ and $b$ where $b \neq 0$.

$x$ is an irrational number if and only if it cannot be written as the ratio of two integers $a$ and $b$ where $b \neq 0$.

In school you were told that all rational numbers could be written as a terminating decimal number for example, $1/4 = 0.25$ or as a repeating decimal number, for example $1/7 = 0.142857142857...$

You were also told that Pi was equal to $22/7 = 3.142857142857...$, which is a rational number. However, this is only a close approximation (to within 0.05%) of the true value of Pi. The true value of Pi has been determined to millions of decimal places and has not been found to be a terminating or a repeating decimal. Therefore, Pi seems to be an irrational number. ☺
Question 1.

a) Statement is a tautology.

\[
\begin{array}{c|c|c|c|c|c|c|}
 p & q & (p \lor q) & \Rightarrow & \neg (\neg p \land \neg q) \\
\hline
 T & T & T & T & T & 6 & 5 & 2 & 4 & 3 \\
 T & F & T & T & F & F & T & 5 & 2 & 4 \\
 F & T & T & T & T & T & F & 5 & 2 & 3 \\
 F & F & T & T & F & F & F & 5 & 2 & 3 \\
\end{array}
\]

b) Statement is a contradiction.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|}
 p & q & r & (p \lor (q \lor \neg q)) & \Rightarrow & (q \land (r \land \neg r)) \\
\hline
 T & T & T & T & T & T & T & T & T & T \\
 T & T & F & T & F & T & T & T & T & T \\
 T & F & T & T & F & T & T & T & T & T \\
 T & F & F & T & F & T & T & T & T & T \\
 F & T & T & T & F & F & F & T & T & T \\
 F & T & F & T & F & F & F & T & T & T \\
 F & F & T & T & F & F & F & T & T & T \\
 F & F & F & T & F & F & F & T & T & T \\
\end{array}
\]

Question 2.

a) Statement can be true and so is not a contradiction

\[
\begin{array}{c|c|c|c|c|c|c|}
 (p \lor q) & \Rightarrow & \neg (\neg p \land \neg q) \\
\hline
 1 & 6 & 5 & 2 & 4 & 3 \\
\end{array}
\]

Step 1: Place a T under the main connective.

Step 2: We now have a problem – there are three possible ways for connective #6 to be true: T \Rightarrow T, F \Rightarrow T, and F \Rightarrow T. We may have to test each possible combination of truth values until we find a combination which works or we have exhausted all of the possible combinations. Place an F under connective #1. Connective #5 can be either true or false but not both.

Step 3: This requires that the first occurrences of both p and q are false.

Step 4: Since we now know the truth values of p and q, we will use them here.

Step 5: This requires that both connectives #2 and #3 are true.

Step 6: This requires that connective #4 is true.
Step 7: This requires that connective #5 is false. We have already determined in step 2 that connective #5 could be true or false but not both. Therefore, there is no inconsistency. Therefore, our original assumption in step 1, that the main connective could be true, has been shown to be correct, and so this statement cannot always be false. It is therefore not a contradiction.

b) Statement cannot be true and so is a contradiction.

<table>
<thead>
<tr>
<th>(p ∨ (q ∨ ~ q)) ⇒ (q ∧ (r ∧ ~ r))</th>
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<tbody>
<tr>
<td>3 2 1 7* 6 5 4</td>
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</table>

**Step 1**: Place a T under the main connective.

**Step 2**: We now have a problem – there are three possible ways for connective #7 to be true: T ⇒ T, F ⇒ T, and F ⇒ T. We may have to test each possible combination of truth values until we find a combination which works or we have exhausted all of the possible combinations. Place an F under connective #3. Connective #6 can be either true or false but not both.

**Step 3**: This requires that both the first occurrence of p and connective #2 are false.

**Step 4**: Connective #2 is false requires that both the first occurrence of q and connective #1 are false.

**Step 5**: This requires that the second occurrence of q is true. However, we have already determined in step 3 that q is false. It is impossible for it to be both true and false. Therefore, our assumption in step 2, that connective #3 could be false, was wrong. However, we have not shown that the statement cannot be false. We must now redo these steps for a different combination of truth values.

<table>
<thead>
<tr>
<th>(p ∨ (q ∨ ~ q)) ⇒ (q ∧ (r ∧ ~ r))</th>
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</thead>
<tbody>
<tr>
<td>3 2 1 7* 6 5 4</td>
</tr>
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</table>

**Step 6**: Place a T under connective #3.

**Step 7**: This requires that connective #6 is true.

**Step 8**: This requires that both the last occurrence of q and connective #5 are true.

**Step 9**: This requires that both the first occurrence of r and connective #4 are true.
Step 10: This requires that the last occurrence of $r$ is false. However, we have already determined in step 8 that $r$ is true. It is impossible for it to be both true and false. Therefore, our assumption in step 6, that connective #3 could be true, was wrong. We have now shown that all of the possible combinations of truth values discussed in step 2 are impossible and so the statement cannot be true. It is therefore a contradiction.

Question 3.

a) Substitution: If in a tautology we replace every occurrence of a statement variable by some statement, the resulting statement is also a tautology.

For example, if we replace every occurrence of $p$ in the De Morgan Law $\neg(p \lor q) \iff (\neg p \land \neg q)$ with $\neg p$, the resulting statement $\neg(\neg p \lor q) \iff (\neg \neg p \land \neg q)$ is also a tautology.

Substitution of Equivalence: If in a tautology we replace the occurrence of some part of the statement by a logically equivalent statement, the resulting statement is also a tautology.

For example, if we use the Double Negation Law to replace the $\neg \neg p$ in the above example with $p$, the resulting statement $\neg(\neg p \lor q) \iff (p \land \neg q)$ is also a tautology.

b) $\neg(\neg p \lor (\neg q \lor \neg r)) \iff (\neg \neg p \land (\neg \neg q \land \neg r))$ De Morgan’s Law

$\iff (\neg p \land (\neg q \land \neg r))$ De Morgan’s Law

$\iff (p \land (\neg q \land \neg r))$ Double Negation

$\iff (p \land (q \land \neg r))$ Double Negation

$\iff (p \land (q \land r))$ Double Negation

$\iff ((p \land q) \land r)$ Associative Laws

$\iff (r \land (p \land q))$ Commutative Laws

$\iff (r \land (q \land r))$ Commutative Laws

A more formal approach would be to derive the required statement using the four given tautologies as assumptions. This is somewhat more difficult.

Starting with De Morgan’s Law, substitute all occurrences of $q$ with $(q \lor r)$,

(1) $\neg(p \lor (q \lor r)) \iff (\neg p \land (q \lor r))$

Now substitute all occurrences of $p$ and $q$ in De Morgan’s Law with $q$ and $r$ respectively,

(2) $\neg(q \lor r) \iff (\neg q \land \neg r)$

Using (1), (2), and Substitution of Equivalence,

(3) $\neg(p \lor (q \lor r)) \iff (\neg p \land (\neg q \land \neg r))$

Now substitute in all occurrences of $p, q$ and $r$ in (3) with $\neg p, \neg q$ and $\neg r$ respectively,

(4) $\neg(\neg p \lor (\neg q \lor \neg r)) \iff (\neg \neg p \land (\neg \neg q \land \neg \neg r))$

Now substitute all occurrences of $p$ in Double Negation with $q$,

(5) $\neg \neg q \iff q$

Now substitute all occurrences of $r$ in Double Negation with $r$,

(6) $\neg \neg r \iff r$
Using (4), (5), (6), Double Negation, and Substitution of Equivalence,
(7) \(~(\neg p \lor (\neg q \lor \neg r)) \iff (p \land (q \land r))\)

Using (7), Associative Law, and Substitution of Equivalence,
(8) \(~(\neg p \lor (\neg q \lor \neg r)) \iff ((p \land q) \land r)\)

Now substitute all occurrences of p and q in the Commutative Law with \((p \land q)\) and r respectively,
(9) \(((p \land q) \land r) \iff (r \land (p \land q))\)

Using (9), Commutative Law, and Substitution of Equivalence,
(10) \(((p \land q) \land r) \iff (r \land (q \land p))\)

Using (8), (10), and Substitution of Equivalence, we have derived the require statement
(11) \(~(\neg p \lor (\neg q \lor \neg r)) \iff (r \land (q \land p))\)

**Question 4.**

a) \(\forall\) students \(x\), \(\exists\) subjects \(y\) (\(x\) dislikes \(y\))

b) \(~(\exists\) student \(x\), \(\forall\) assigned work in this subject \(y\) (\(x\) has completed \(y\)))

**Question 5.**

a) \(~(\forall\) students \(x\), \(\exists\) subjects \(y\) (\(x\) dislikes \(y\)))
   \[\equiv \exists\) students \(x\), \(\forall\) subjects \(y\) (\(x\) does not dislike \(y\))\]
   \[\equiv\) Some students do not dislike any of their subjects.\]

   The original statement is (hopefully) false. It could be proved if we surveyed all students and found that every student disliked some of his or her subjects. On the other hand, it would be disproved by finding any student who does not dislike any of his or her subjects.

b) \(~(\exists\) student \(x\), \(\forall\) assigned work in this subject \(y\) (\(x\) has completed \(y\)))
   \[\equiv \exists\) student \(x\), \(\forall\) assigned work in this subject \(y\) (\(x\) has completed \(y\))\]
   \[\equiv\) Some students have completed all assigned work in this subject.\]

   The original statement is (unfortunately) true, but every session Garry, Mia, and I hope that someone will disprove it. 😊

**Question 6.**

a) Every odd natural number is equal to twice some natural number plus one

b) Some integers do not have an additive inverse.
Question 7.

a) \( \neg (\forall x \in \mathbb{Z} \ (x \text{ is odd} \Rightarrow \exists y \in \mathbb{Z} \ (x = 2y + 1))) \)

\[ \equiv \exists x \in \mathbb{Z} \ (x \text{ is odd} \land \neg \exists y \in \mathbb{Z} \ (x = 2y + 1)) \]
\[ \equiv \exists x \in \mathbb{Z} \ (x \text{ is odd} \land \forall y \in \mathbb{Z} \ \neg (x = 2y + 1)) \]
\[ \equiv \exists x \in \mathbb{Z} \ (x \text{ is odd} \land \forall y \in \mathbb{Z} \ (x \neq 2y + 1)) \]
\[ \equiv \text{Some odd integers are not equal to twice some integer plus one.} \]

The original statement is true. This is the definition of the odd integers. Compare this statement with \( \forall x \in \mathbb{N} \ (x \text{ is odd} \Rightarrow \exists y \in \mathbb{N} \ (x = 2y + 1)) \) which is false. Do you see why? ☺

b) \( \neg (\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z} \ (x + y 
eq 0))) \)

\[ \equiv \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \ \neg (x + y 
eq 0) \]
\[ \equiv \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} \ (x + y = 0) \]
\[ \equiv \text{Every integer has an additive inverse.} \]

The negation of the original statement is true. This is a known property of the integers.
Question 1.

Claim($n$) is $1.2 + 2.3 + 3.4 + \cdots + (n - 1)n = \frac{n(n-1)(n+1)}{3}$ for each natural number $n$

Let $n = 1$. Claim(1) is $0.1 = \frac{1(0)(2)}{3}$.
$LHS = 0$, $RHS = 0$, so Claim(1) is true.

Let $n = 2$. Claim(2) is $0.1 + 1.2 = \frac{2(1)(3)}{3}$.
$LHS = 3$, $RHS = 3$, so Claim(2) is true.

Let $n = 3$. Claim(2) is $0.1 + 1.2 + 2.3 = \frac{3(2)(4)}{3}$.
$LHS = 8$, $RHS = 8$, so Claim(3) is true.

Assume that Claim($k$) is true for some $k \geq 1$, that is, $1.2 + 2.3 + 3.4 + \cdots + (k - 1)k = \frac{k(k-1)(k+1)}{3}$.

Now try to show that Claim($k+1$) is also true, that is, $1.2 + 2.3 + 3.4 + \cdots + (k - 1)k + k(k + 1) = \frac{(k+1)(k)(k+2)}{3}$

$LHS = 1.2 + 2.3 + 3.4 + \cdots + (k - 1)k + k(k + 1)$

$= \frac{k(k-1)(k+1)}{3} + k(k + 1)$ \hspace{1cm} using Claim($k$)

$= \frac{k(k-1)(k+1)}{3} + \frac{3k(k+1)}{3}$

$= \frac{k(k+1)}{3}((k - 1) + 3)$

$= \frac{k(k+1)}{3}(k + 2)$

$= RHS$

Therefore, Claim($k+1$) is also true when Claim($k$) is true for some $k \geq 1$ and, by the Principle of Mathematical Induction, Claim($n$) must be true for all natural numbers $n$.

Question 2.

From the definition of divisibility, $\forall n \in \mathbb{Z} \ (6|n(n^2 + 5) \Rightarrow \exists c \in \mathbb{Z}, n(n^2 + 5) = 6c)$

Claim($n$) is $\exists c \in \mathbb{Z}, n(n^2 + 5) = 6c$ for each natural number $n$

Let $n = 1$. Claim(1) is $\exists c \in \mathbb{Z}, 1(1^2 + 5) = 6c$.
$LHS = 6$, $RHS = 6c$, so for $LHS = RHS$ we need $c = 1$, and so Claim(1) is true.

Let $n = 2$. Claim(1) is $\exists c \in \mathbb{Z}, 2(2^2 + 5) = 6c$.
$LHS = 18$, $RHS = 6c$, so for $LHS = RHS$ we need $c = 3$, and so Claim(2) is true.

Let $n = 1$. Claim(1) is $\exists c \in \mathbb{Z}, 3(3^2 + 5) = 6c$
$LHS = 42$, $RHS = 6c$, so for $LHS = RHS$ we need $c = 7$, and so Claim(3) is true.

Assume that Claim($k$) is true for some $k \geq 1$, that is, $\exists c \in \mathbb{Z}, k(k^2 + 5) = 6c$

Now try to show that Claim($k+1$) is also true, that is, $\exists d \in \mathbb{Z}, (k+1)((k + 1)^2 + 5) = 6d$
LHS = (k + 1)((k + 1)^2 + 5)  
= (k + 1)^3 + 5(k + 1)  
= (k^3 + 3k^2 + 3k + 1) + (5k + 5)  
= (k^3 + 5k) + (3k^2 + 3k + 6)  
= 6c + (3k^2 + 3k + 6)  
= 6c + 3k(k + 1) + 6  

using Claim(k)

We now have a problem: does 6|3k(k + 1)?

If we let k be an even number, say k = 2m for some integer m, then 3k(k + 1) = 6m(2m + 1) = 6p where p = m(2m + 1), and so 6|3k(k + 1) is true.

Similarly, if we let k be an odd number, say k = 2m + 1 for some integer m, then 3k(k + 1) = 3(2m + 1)(2m + 2) = 6(2m + 1)(m + 1) = 6p where p = (m + 1)(2m + 1), and so 6|3k(k + 1) is again true.

Therefore, 6|3k(k + 1) is true for any integer k, and continuing our proof

LHS = ...
= 6c + 3k(k + 1) + 6  
= 6c + 6p + 6  
= 6(c + p + 1)  
= 6d  
= RHS

Therefore, Claim(k + 1) is also true when Claim(k) is true for some k \geq 1 and, by the Principle of Mathematical Induction, Claim(n) must be true for all natural numbers n.

Question 3.

Using the Generalised Principle of Mathematical Induction, prove

\[(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})... (1 - \frac{1}{n^2}) = \frac{n + 1}{2n}\]

Claim(n) is \[(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})... (1 - \frac{1}{n^2}) = \frac{n + 1}{2n}\] for each natural number n \geq 2

Let n = 2. Claim(2) is \[(1 - \frac{1}{2^2}) = \frac{2 + 1}{2(2)}\].

LHS = \[\frac{3}{4}\], RHS = \[\frac{3}{4}\], so Claim(2) is true.

Let n = 3. Claim(3) is \[(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) = \frac{3 + 1}{2(3)}\].

LHS = \[\frac{3}{4}, \frac{8}{9} = \frac{2}{3}\], RHS = \[\frac{4}{6} = \frac{2}{3}\], so Claim(3) is true.

Let n = 4. Claim(4) is \[(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) = \frac{4 + 1}{2(4)}\].

LHS = \[\frac{3}{4}, \frac{8}{9}, \frac{15}{16} = \frac{5}{8}\], RHS = \[\frac{4 + 1}{2(4)} = \frac{5}{8}\], so Claim(4) is true.

Assume that Claim(k) is true for some k \geq 2, that is, \[(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})... (1 - \frac{1}{k^2}) = \frac{k + 1}{2k}\]

Now try to show that Claim(k + 1) is also true, that is, \[(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})... (1 - \frac{1}{k^2})(1 - \frac{1}{(k+1)^2}) = \frac{k + 2}{2(k+1)}\]
\[ \text{LHS} = \left( 1 - \frac{1}{2^2} \right) \left( 1 - \frac{1}{3^2} \right) \cdots \left( 1 - \frac{1}{k^2} \right) \left( 1 - \frac{1}{(k+1)^2} \right) \]
\[ \stackrel{\text{k+1}}{=} \frac{1}{2k} \left( \frac{(k+1)^2 - 1}{(k+1)^2} \right) \]
\[ \stackrel{\text{k+1}}{=} \frac{k+1}{2k} \left( \frac{k^2 + 2k}{(k+1)^2} \right) \]
\[ \stackrel{\text{k+1}}{=} \frac{k+1}{2k} \left( \frac{k(k+2)}{(k+1)^2} \right) \]
\[ \stackrel{\text{k+1}}{=} \frac{k+2}{2(k+1)} \]
\[ = \text{RHS} \]

Therefore, Claim\((k+1)\) is also true when Claim\((k)\) is true for some \(k \geq 2\) and, by the General Principle of Mathematical Induction, Claim\((n)\) must be true for all natural numbers \(n \geq 2\).

**Question 4.**

Claim\((n)\) is \(n^3 > 2n + 1\) for each natural number \(n \geq 2\)

Let \(n = 2\). Claim\((2)\) is \(2^3 > 2(2) + 1\).

\[ \text{LHS} = 8, \ \text{RHS} = 5, \ \text{and} \ 8 > 5, \ \text{so Claim}(2) \text{ is true.} \]

Let \(n = 3\). Claim\((3)\) is \(3^3 > 2(3) + 1\).

\[ \text{LHS} = 27, \ \text{RHS} = 7, \ \text{and} \ 27 > 7, \ \text{so Claim}(3) \text{ is true.} \]

Let \(n = 4\). Claim\((4)\) is \(4^3 > 2(4) + 1\).

\[ \text{LHS} = 64, \ \text{RHS} = 9, \ \text{and} \ 64 > 9, \ \text{so Claim}(4) \text{ is true.} \]

Assume that Claim\((k)\) is true for some \(k \geq 2\), that is, \(k^3 > 2k + 1\)

Now try to show that Claim\((k+1)\) is also true, that is, \((k + 1)^3 > 2(k + 1) + 1\)

\[ \text{LHS} = (k + 1)^3 \]
\[ = (k^3 + 3k^2 + 3k + 1) \]
\[ > (k^3 + 12 + 6 + 1) \]
\[ = (k^3 + 19) \]
\[ > (2k + 1) + 19 \]
\[ > (2k + 1) + 2 \]
\[ = 2(k + 1) + 1 \]
\[ = \text{RHS} \]

Therefore, Claim\((k+1)\) is also true when Claim\((k)\) is true for some \(k \geq 2\) and, by the General Principle of Mathematical Induction, Claim\((n)\) must be true for all natural numbers \(n \geq 2\).

**Question 5.**

Claim\((n)\) is \(a_n \leq 1\) for each natural numbers \(n\)

Let \(n = 1\). Claim\((1)\) is \(a_1 \leq 1\).

We are given \(a_1 = \frac{9}{10}\), so \(LHS = \frac{9}{10}, \ \text{RHS} = 1, \ \text{LHS} \leq \text{RHS}, \ \text{so Claim}(1) \text{ is true.} \)

Let \(n = 2\). Claim\((2)\) is \(a_2 \leq 1\).

We are given \(a_2 = \frac{10}{11}\), so \(LHS = \frac{10}{11}, \ \text{RHS} = 1, \ \text{LHS} \leq \text{RHS}, \ \text{so Claim}(2) \text{ is true.} \)
Let $n = 3$. Claim(3) is $a_3 \leq 1$.

Now $a_3 = a_2 a_1 = \frac{10}{11} \cdot \frac{9}{10} = \frac{9}{11}$, so $LHS = \frac{9}{11}$, $RHS = 1$, $LHS \leq RHS$, so Claim(3) is true.

Assume that Claim($k$), Claim($k-1$), … Claim(1) are true for some $k \geq 1$, that is, $a_k \leq 1$, $a_{k-1} \leq 1$, … and $a_1 \leq 1$

Now try to show that Claim($k+1$) is also true, that is, $a_{k+1} \leq 1$

$LHS = a_{k+1}$

$= a_k \cdot a_{k-1}$

$\leq a_{k-1}$ using Claim($k$)

$\leq 1$ using Claim($k-1$)

$\leq RHS$

Therefore, Claim($k+1$) is also true when Claim($k$), Claim($k-1$), … Claim(1) are true for some $k \geq 1$ and, by the Strong Principle of Mathematical Induction, Claim($n$) must be true for all natural numbers $n$.

**Question 6.**

The three equations seem to indicate the following formulae

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{n}\right) = \frac{1}{n} \quad \text{where} \quad n \geq 2$$

or

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{(n+1)}\right) = \frac{1}{(n+1)} \quad \text{where} \quad n \geq 1$$

The second of these formulae can be proved using the Principle of Mathematical Induction, while the first of these formulae would require the General Principle of Mathematical Induction. These proofs are essentially the same so I will not be doing both of them.

Claim($n$) is

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{n}\right) = \frac{1}{n} \quad \text{for each natural number} \quad n \geq 2$$

Let $n = 2$. Claim(2) is

$LHS = \frac{1}{2}$, $RHS = \frac{1}{2}$, so Claim(2) is true.

Let $n = 3$. Claim(3) is

$LHS = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$, $RHS = \frac{1}{3}$, so Claim(3) is true.

Let $n = 4$. Claim(4) is

$LHS = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$, $RHS = \frac{1}{4}$, so Claim(4) is true.

Assume that Claim($k$) is true for some $k \geq 2$,

that is, 

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{k}\right) = \frac{1}{k}$$

Now try to show that Claim($k+1$) is also true,

that is,

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{k}\right) \left(1 - \frac{1}{(k+1)}\right) = \frac{1}{k+1}$$
\[ \text{LHS} = \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{k}\right) \left(1 - \frac{1}{(k+1)}\right) \]
\[= \frac{1}{k} \left(1 - \frac{1}{(k+1)}\right) \quad \text{using Claim}(k) \]
\[= \frac{1}{k} \left(\frac{(k+1)-1}{(k+1)}\right) \]
\[= \frac{1}{k} \left(\frac{k}{(k+1)}\right) \]
\[= \frac{1}{k+1} \]
\[= \text{RHS} \]

Therefore, Claim\((k+1)\) is also true when Claim\((k)\) is true for some \(k \geq 2\) and, by the General Principle of Mathematical Induction, Claim\((n)\) must be true for all natural numbers \(n \geq 2\).

**Question 7.**

a) **Modus Ponens:**

If \( p \) is true and \( p \Rightarrow q \) is true, then \( q \) is also true

\((p \land (p \Rightarrow q)) \Rightarrow q\)

**Modus Tolens:**

If \( \neg q \) is true and \( p \Rightarrow q \) is true, then \( \neg p \) is also true

\((\neg q \land (p \Rightarrow q)) \Rightarrow \neg p\)

**Law of Syllogism**

If \( p \Rightarrow q \) is true and \( q \Rightarrow r \) is true, then \( p \Rightarrow r \) is also true

\(((p \Rightarrow q) \land (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)\)

b) \( P(x) \quad x \) is a politician

\( Q(x) \quad x \) can be trusted

Julia Gillard and Tony Abbott are politicians

\( \equiv P(\text{Julia Gillard}) \land P(\text{Tony Abbott})\)

No politician can be trusted

\( \equiv \forall \text{politicians } x \ (x \text{ cannot be trusted}) \)

\( \equiv \forall x \ (x \text{ is a politician } \Rightarrow x \text{ cannot be trusted}) \)

\( \equiv \forall x \ (P(x) \Rightarrow \neg Q(x)) \)

Therefore, neither Julian Gillard nor Tony Abbott can be trusted

\( \equiv \neg( Q(\text{Julia Gillard}) \lor Q(\text{Tony Abbott}) ) \)

The argument is therefore of the form:

\( ( P(\text{Julia Gillard}) \land P(\text{Tony Abbott}) \land \forall x \ (P(x) \Rightarrow \neg Q(x)) ) \)

\( \Rightarrow \neg (Q(\text{Julia Gillard}) \lor Q(\text{Tony Abbott}) ) \)

The argument is a tautology. It uses the Rules of Substitution, Substitution of Equivalence, Universal Modus Ponens, and the De Morgan Laws, and so should be a tautology.

\( ( P(\text{Julia Gillard}) \land \forall x \ (P(x) \Rightarrow \neg Q(x)) ) \Rightarrow \neg Q(\text{Julia Gillard}) \quad \text{by Universal Modus Ponens} \)

\( ( P(\text{Tony Abbott}) \land \forall x \ (P(x) \Rightarrow \neg Q(x)) ) \Rightarrow \neg Q(\text{Tony Abbott}) \quad \text{by Universal Modus Ponens} \)
**Question 8.**

a) A Universal statement has the form

\[ \forall x \in A, P(x) \]

This can be shown to be false by showing that the truth value of the predicate \( P(x) \) is false for any element \( x \) in the domain. This is known as a counterexample.

b) An Existential statement has the form

\[ \exists x \in A, P(x) \]

This can be shown to be true by showing that the truth value of the predicate \( P(x) \) is true for any element \( x \) in the domain \( A \). This is known as an example.

It is worth noting that disproving a Universal statement is essentially the same process as proving its negation and vice-versa

\[ \sim \forall x \in A, P(x) \equiv \exists x \in A, \sim P(x) \]

\[ \sim \exists x \in A, P(x) \equiv \forall x \in A, \sim P(x) \]
Question 1.

a) A Universal statement has the form
\[ \forall x \in A, P(x) \]
This can be shown to be true by either (1) showing that the truth value of the predicate \( P(x) \) is true for all elements \( x \) in the domain \( A \) or (2) by a general proof.

b) An Existential statement has the form
\[ \exists x \in A, P(x) \]
This can be shown to be false by either (1) showing that the truth value of the predicate \( P(x) \) is false for all elements \( x \) in the domain \( A \) or (2) by a general proof.

It is worth noting that proving a Universal statement is essentially the same process as disproving its negation and vice-versa
\[ \sim \forall x \in A, P(x) \equiv \exists x \in A, \sim P(x) \]
\[ \sim \exists x \in A, P(x) \equiv \forall x \in A, \sim P(x) \]

Question 2.

a) \[ \forall a, b \in \mathbb{Z} \ (a \mid b \Rightarrow \exists c \in \mathbb{Z} \ (b = ac)) \]
This statement reads: For all integers \( a \) and \( b \), if \( a \) divides \( b \) then there exists an integer \( c \) such that \( b \) is equal to the product of \( a \) and \( c \). This is the definition of divisibility over the set of integers. It is therefore true.

b) \[ \exists a, b \in \mathbb{Z} \ (a \mid b \land b \mid a \land a \neq b) \]
This statement reads: There exist integers \( a \) and \( b \) such that \( a \) divides \( b \), \( b \) divides \( a \), and \( a \) is not equal to \( b \). This statement is true whenever \( a = -b \).

Question 3.

a) Proof by Contrapositive uses \( (p \Rightarrow q) \equiv (\sim q \Rightarrow \sim p) \)

b) Proof by Cases uses \( ((p \lor q) \Rightarrow r) \equiv ((p \Rightarrow r) \land (q \Rightarrow r)) \)

c) Proof by Contradiction uses \( (p \land (q \land \sim q)) \Rightarrow \sim p \)

Question 4.

a) This statement may be written as \( \forall n \in \mathbb{N} \ (n^3 \text{ is odd } \Rightarrow n \text{ is odd}) \).

This may be proven using Proof by Contrapositive where
\( (n^3 \text{ is odd } \Rightarrow n \text{ is odd}) \equiv (n \text{ is not odd } \Rightarrow n^3 \text{ is not odd}) \)

Since \( n \in \mathbb{N} \), \( n \) is not odd \( \equiv n \) is even, and \( n^3 \) is not odd \( \equiv n^3 \) is even, so we should try to prove
\[ \forall n \in \mathbb{N} \ (n \text{ is even } \Rightarrow n^3 \text{ is even}) \]

Our proof must begin and end with the definition of even:
\[ n \text{ is even } \Rightarrow \exists k \in \mathbb{N} \ (n = 2k) \]
\[ \ldots \]
\[ \exists p \in \mathbb{N} \ (n^3 = 2p) \Rightarrow n^3 \text{ is even} \]
n is even $\Rightarrow \exists k \in \mathbb{N} \ (n = 2k)$
\[ \Rightarrow n^3 = (2k)^3 \]
\[ \Rightarrow n^3 = 8k^3 \]
\[ \Rightarrow n^3 = 2(4k^3) \]
\[ \Rightarrow n^3 = 2p, \text{ where } p = 4k^3 \in \mathbb{N} \]
\[ \exists p \in \mathbb{N} \ (n^3 = 2p) \Rightarrow n^3 \text{ is even} \]

Therefore, $\forall n \in \mathbb{N} \ (n \text{ is even } \Rightarrow n^3 \text{ is even})$ is true, and so is $\forall n \in \mathbb{N} \ (n^3 \text{ is odd } \Rightarrow n \text{ is odd}).$

b) This statement may be written as $(n = 1 \lor n = 2 \lor n = 3) \Rightarrow (n-1)(n-2)(n-3) = 0$.

This statement may be proven using Proof by Cases:

Case #1: Let $n = 1$.
\[ LHS = (1-1)(1-2)(1-3) = (0)(-1)(-2) = 0 = RHS, \text{ so Case #1 is true.} \]

Case #2: Let $n = 2$.
\[ LHS = (2-1)(2-2)(3-2) = (1)(0)(-1) = 0 = RHS, \text{ so Case #2 is true.} \]

Case #3: Let $n = 3$.
\[ LHS = (3-1)(3-2)(3-2) = (2)(1)(0) = 0 = RHS, \text{ so Case #3 is true.} \]

\[ (n = 1 \Rightarrow (n-1)(n-2)(n-3) = 0) \land (n = 2 \Rightarrow (n-1)(n-2)(n-3) = 0) \land (n = 3 \Rightarrow (n-1)(n-2)(n-3) = 0) \]

is therefore true, and so is $(n = 1 \lor n = 2 \lor n = 3) \Rightarrow (n-1)(n-2)(n-3) = 0$.

c) This statement may be written as $\sim \exists x \in \mathbb{R}^+, \forall y \in \mathbb{R} \ (x \leq y)$.

[Note $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of positive real numbers and negative real numbers respectively.]

This statement may be disproven using Proof by Contradiction. We will start by assuming the negation of this statement is true, that is, $\exists x \in \mathbb{R}^+, \forall y \in \mathbb{R} \ (x \leq y)$, and hope to show that this assumption is actually false.

1) Let $x$ be the smallest positive real number.
2) Now consider $y = x/2$. Since $x \in \mathbb{R}^+$ is true, then $y \in \mathbb{R}^+$ must also be true by the closure of multiplication of real numbers. However, $x/2 < x$, so $y < x$, which contradicts the assumption in step (1).
3) Therefore, our assumption is false. That is, there is no smallest positive real number.

**Question 5.**

a) If $n$ and $d$ are both integers and $d > 0$, then there exist unique integers $q$ and $r$ such that $n = dq + r$, where $0 \leq r < d$.

b) Now let $d = 4$. By the Quotient-Remainder Theorem, there must exist unique integers $q$ and $r$ such that $n = 4q + r$, where $0 \leq r < 3$, that is, $n = 4q$, $n = 4q + 1$, $n = 4q + 2$, or $n = 4q + 3$.

Therefore, any integer can be written in the form $4k, 4k + 1, 4k + 2$, or $4k + 3$, for some integer $k$.

**Question 6.**

a) If $a \in \mathbb{N}$ and $a > 1$ then $a$ can be factorised in a unique way in the form
\[ a = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k} \]
where $p_1, p_2, \ldots p_k$ are each prime numbers and $\alpha_i \in \mathbb{N}$ for each $i = 1, 2, \ldots, k$
b) \[17493 = 3(5831)\]
\[= 3(7)(833)\]
\[= 3(7^2)(119)\]
\[= 3(7^3)(17)\]

The prime factors of 17493 are 3, 7, and 17.

**Question 7.**

a) The Sieve of Eratosthenes is a method of finding primes up to \(n\) as follows.

A. List all the primes up to \(\lfloor \sqrt{n} \rfloor\)
B. Write down all integers from 1 to \(n\), noting the listed primes
C. Delete all multiples of the listed primes
D. The remaining values are the prime numbers up to \(n\).

Note \(\lfloor x \rfloor\) is the floor function and is defined as:

\(\lfloor x \rfloor\) is the greatest integer that is less than or equal to \(x\), that is \(\lfloor x \rfloor = n\), where \(n \in \mathbb{N}\), \(n \leq x < n + 1\)

b) \(\lfloor \sqrt{399} \rfloor = 19\) so we need to consider the prime numbers 2, 3, 5, 7, 11, 13, 17, and 19. We must write down all of the numbers from 300 to 399, and eliminate all multiples of the listed primes

Eliminate all multiples of 2: 300, 302, 304, … 398
Eliminate all remaining multiples of 3: 303, 309, 315, … 399
Eliminate all remaining multiples of 5: 305, 325, 335, 355, 365, 385, 395
Eliminate all remaining multiples of 7: 301, 329, 343, 371
Eliminate all remaining multiples of 11: 319, 341
Eliminate all remaining multiples of 13: 377
Eliminate all remaining multiples of 17: 323, 391
Eliminate all remaining multiples of 19: 361

The remaining numbers 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, and 397, are prime numbers.

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[Note: I cheated here and used MS Excel to format my table and determine which numbers were divisible by the listed primes. How did you do it?] 😊

c) Twin primes are prime numbers that differ by 2. The twin primes between 300 and 399 are
(i) 311 & 313, and (2) 347 & 349.
Question 1.

a)  \( A \cup B = \{x \in \mathbb{Z} : 2 \leq x \leq 5\} = \{2, 3, 4, 5\} \)

b)  \( A \cap B = \{x \in \mathbb{Z} : 3 \leq x \leq 4\} = \{3, 4\} \)

c)  \( A - B = \{x \in \mathbb{Z} : 2 \leq x < 3\} = \{2\} \)

d)  \( B - A = \{x \in \mathbb{Z} : 4 < x \leq 5\} = \{5\} \)

e)  \( \bar{A} = \{x \in \mathbb{Z} : 1 \leq x < 2 \lor 4 < x \leq 6\} = \{1, 5, 6\} \)

f)  \( \bar{B} = \{x \in \mathbb{Z} : 1 \leq x < 3 \lor 5 < x \leq 6\} = \{1, 2, 6\} \)

Please notice the effect of changing the domain set from \( \mathbb{Z} \) (question 1) to \( \mathbb{R} \) (question 2). On the other hand, the property that defines corresponding sets in both questions is the same.

Question 2.

a)  \( A \cup B = \{x \in \mathbb{R} : 2 \leq x \leq 5\} = [2, 5] \)

b)  \( A \cap B = \{x \in \mathbb{R} : 3 \leq x \leq 4\} = [3, 4] \)

c)  \( A - B = \{x \in \mathbb{R} : 2 \leq x < 3\} = [2, 3) \)

d)  \( B - A = \{x \in \mathbb{R} : 4 < x \leq 5\} = (4, 5] \)

e)  \( \bar{A} = \{x \in \mathbb{R} : 1 \leq x < 2 \lor 4 < x \leq 6\} = [1, 2) \cup (4, 6] \)

f)  \( \bar{B} = \{x \in \mathbb{R} : 1 \leq x < 3 \lor 5 < x \leq 6\} = [1, 3) \cup (5, 6] \)

Question 3.

a)  The power set of \( A \) is the set of all subsets of \( A \), that is, \( \mathcal{P}(A) = \{X : X \subseteq A\} \)

b)  (i)  \( \mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{A\}\} \)

Note that \( A \) has 2 elements so \( \mathcal{P}(A) \) has \( 2^2 = 4 \) elements.
(ii) \( \mathcal{P}(\mathcal{P}(A)) = \{ \emptyset, \\
\{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{A\}\}, \\
\{\{\emptyset, \{1\}\}, \{\{\emptyset, \{2\}\}, \{\{\emptyset, \{A\}\}, \\
\{\{1, \{2\}\}, \{\{1, \{A\}\}, \\
\{\{2, \{A\}\}, \{\{\emptyset, \{1\}, \{2\}\}, \{\{\emptyset, \{1\}, \{A\}\}, \\
\{\{\emptyset, \{2\}, \{A\}\}, \{\{1\}, \{2\}, \{A\}\}}} \\
\mathcal{P}(A) \}
\)

Note that \( \mathcal{P}(A) \) has 4 elements so \( \mathcal{P}(\mathcal{P}(A)) \) has \( 2^4 = 16 \) elements. Start with the empty set \( \emptyset \), then list the subsets of \( \mathcal{P}(A) \) containing only 1 element, then the subsets of \( \mathcal{P}(A) \) containing 2 elements, then the subsets of \( \mathcal{P}(A) \) containing 3 elements, and finally \( \mathcal{P}(A) \) itself. It is probably simpler just to draw the Hasse diagram. 😊

c) \( \{1, 2\} \)
\[
\begin{array}{c}
\{1\} \\
\emptyset \\
\{2\}
\end{array}
\]

**Question 4.**

gcd(a, b) is defined as the greatest common divisor of a and b, by which we mean that all divisors of a and b must also divide gcd(a, b)

i.e. \( x = gcd(a, b) \Rightarrow (x|a \land x|b \land \forall c \in \mathbb{N} ((c|a \land c|b) \Rightarrow c|x) \)

a) From the fundamental theorem of arithmetic, 512 = \(2^9\) and 172 = \(2^2.43\)
\[
gcd(512, 172) = 2^2.43^0 = 4
\]
b) From the fundamental theorem of arithmetic, 612 = \(2^2.3^2.17\) and 272 = \(2^2.3.31\)
\[
gcd(612, 372) = 2^2.3^1.17^0.31^0 = 12
\]

**Question 5.**
lcm(a, b) is defined as the least common multiple of a and b, by which we mean that all multiples of a and b must also be multiples of lcm(a, b)

i.e. \( x = lcm(a, b) \Rightarrow (a|x \land b|x \land \forall c \in \mathbb{N} ((a|c \land b|c) \Rightarrow x|c) \)

a) From the fundamental theorem of arithmetic, 512 = \(2^9\) and 172 = \(2^2.43\)
\[
lcm(512, 172) = 2^9.43^1 = 22,016
\]
b) From the fundamental theorem of arithmetic, 612 = \(2^2.3^2.17\) and 272 = \(2^2.3.31\)
\[
lcm(612, 372) = 2^2.3^2.17^1.31^1 = 18,972
\]
Question 6.

a) The Euclidean Algorithm is used to find \( \gcd(a, b) \) and can be described as follows:

1. Let \( a, b \in \mathbb{N} \), with \( |a| > |b| \geq 0 \).
2. If \( b = 0 \), then
   set \( \gcd(a, b) = a \).
   If \( b \neq 0 \), then
   apply the quotient remainder theorem to get \( a = bq + r \), where \( 0 \leq r < b \),
   and then set \( \gcd(a, b) = \gcd(b, r) \).

Repeat the process in step 2, to find \( \gcd(b, r) \)

Note that the process is guaranteed to terminate eventually with \( r = 0 \) because each new
remainder is less than the preceding one and all are nonnegative.

b) \[
\begin{align*}
512 &= 2(172) + 168 \\
272 &= 1(168) + 4 \\
168 &= 42(4) + 0 \\
\therefore \gcd(512, 172) &= 4
\end{align*}
\]

c) \[
\begin{align*}
612 &= 1(372) + 240 \\
372 &= 1(240) + 132 \\
240 &= 1(132) + 108 \\
132 &= 1(108) + 24 \\
108 &= 4(24) + 12 \\
24 &= 2(12) + 0 \\
\therefore \gcd(612, 372) &= 12
\end{align*}
\]
Question 1.

a) \[512 = 2(172) + 168 \quad \#1 \]
\[172 = 1(168) + 4 \quad \#2 \]
\[168 = 42(4) + 0 \]
\[\therefore \gcd(512, 172) = 4 \]

Now rewriting these equations with the remainder as subject
\[4 = 172 - 1(168) \quad \#3, \text{ from } \#2 \]
\[168 = 512 - 2(172) \quad \#4, \text{ from } \#1 \]
\[4 = 172 - 1(512 - 2(172)) \quad \text{using } \#3 \text{ and } \#4 \]
\[= 3(172) - 1(512) \]
\[= 512m + 172n \]
\[\therefore m = -1, \quad n = 3 \]

You should check these calculations. Does 512(-1) + 172(3) = 4?

b) \[612 = 1(372) + 240 \quad \#1 \]
\[372 = 1(240) + 132 \quad \#2 \]
\[240 = 1(132) + 108 \quad \#3 \]
\[132 = 1(108) + 24 \quad \#4 \]
\[108 = 4(24) + 12 \quad \#5 \]
\[24 = 2(12) + 0 \]
\[\therefore \gcd(612, 372) = 12 \]

Now rewriting these equations with the remainder as subject
\[12 = 108 - 4(24) \quad \#6, \text{ from } \#5 \]
\[24 = 132 - 1(108) \quad \#7, \text{ from } \#4 \]
\[108 = 240 - 1(132) \quad \#8, \text{ from } \#3 \]
\[132 = 372 - 1(240) \quad \#9, \text{ from } \#2 \]
\[240 = 612 - 1(372) \quad \#10, \text{ from } \#1 \]
\[12 = 108 - 4(132 - 1(108)) \quad \text{using } \#6 \text{ and } \#7 \]
\[= 5(108) - 4(132) \]
\[= 5(240 - 1(132)) - 4(132) \quad \text{using } \#8 \]
\[= 5(240) - 9(132) \]
\[= 5(240) - 9(372 - 1(240)) \quad \text{using } \#9 \]
\[= 14(240) - 9(372) \]
\[= 14(612 - 1(372)) - 9(372) \quad \text{using } \#10 \]
\[= 14(612) - 23(372) \]
\[= 612m + 372n \]
\[\therefore m = 14, \quad n = -23 \]

You should check these calculations. Does 612(14) + 372(-23) = 24?

Question 2.

Using the pigeonhole principle, we treat the students as our \( n \) pigeons, and the 366 days of the year as our \( k \) pigeonholes. Note that there are 366 days in a year – we must always count February 29. I assume some of you will fall into this trap. ☺️
So we want \( \lceil n/k \rceil = 2 \) and so must choose \( n = k + 1 = 367 \). Therefore, we must have 367 students to guarantee that at least two students share the same birthday.

**Question 3.**

Using the generalized pigeonhole principle, we should treat the students as our \( n \) pigeons and the number of classes as our \( k \) pigeonholes. However, we have a complication – the number of students that be assigned to each class is limited by both a minimum of 12 and a maximum of 16. This requires a little more thought than the previous question.

Let us assume that there are only 4 classes with at least 13 students each. This leaves us with the remaining 11 classes with 12 students each.

Now completely fill the first 4 classes with 16 students each. The number of students we have assigned to classes is equal to \( 4(16) + 11(12) = 196 \). However, we have 200 students, so must assign the remaining 4 students amongst the 11 classes that are not yet filled. No matter how we achieve this, at least one of these 11 classes must have at least 13 students, which contradicts our original assumption.

Therefore, there must be at least 5 classes with at least 13 students in them.

**Question 4.**

a) Consider the Quotient-Remainder theorem:

If \( n \) and \( d \) are integers and \( d > 0 \), then there exist unique integers \( q \) and \( r \) such that \( n = qd + r \), where \( 0 \leq r < d \).

Div returns the integer quotient of \( n \) divided by \( d \), that is, \( n \text{ div } d = q \), and mod returns the integer remainder of \( n \) divided by \( d \), that is, \( n \text{ mod } d = r \)

b) \( 1632 = 32(51) + 0 \)
\[ \therefore \ 1632 \text{ div } 51 = 32 \text{ and } 1632 \text{ mod } 51 = 0 \]

c) \( 2783 = 95(29) + 28 \)
\[ \therefore \ 2783 \text{ div } 29 = 95 \text{ and } 2783 \text{ mod } 29 = 28 \]

**Question 5.**

a) The equivalence class of \( x \) modulo \( n \) is the set of integers congruent to \( x \) modulo \( n \), that is \( [x] = \{ a \in \mathbb{Z} : a \equiv x \mod n \} \)

b) The set of residues modulo \( n \) is the complete set of equivalence classes modulo \( n \), that is \( S_n = \{[0], [1], [2], \ldots, [n-1] \} \)

c) Addition table:

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Multiplication table:

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</tbody>
</table>

5d) The additive identity is [0] since \([x] + [0] = [x]\) for all classes \([x]\).

5e) The multiplicative identity is [1] since \([x] \cdot [1] = [x]\) for all classes \([x]\).

5f) The additive inverses are as follows:


5g) The multiplicative inverses are as follows:


There is no inverse for [0] as \([0] \cdot [x] \neq [1]\) for all classes \([x]\).

**Question 6.**

\((A - B) - C = (A - B) \cap \overline{C}\)

from set difference

\[= (A \cap \overline{B}) \cap \overline{C}\]

from set difference

\[= A \cap (B \cap \overline{C})\]

from associativity

\[= A \cap (B \cup \overline{C})\]

from De Morgan’s laws

**Question 7.**

If \((A - B) - C = A \cap (B \cup \overline{C})\) is a true statement, we require \(\forall x ( x \in ((A - B) - C) \implies x \in (A \cap (B \cup \overline{C})) )\) to also be a true statement, and so we require \(x \in ((A - B) - C) \implies x \in (A \cap (B \cup \overline{C}))\) to be a true statement for some typical element \(x\).

**LHS**

\(x \in ((A - B) - C)\)

\(\equiv x \in (A - B) \land \neg x \in C\)

from definition of set difference

\(\equiv (x \in A \land \neg x \in B) \land \neg x \in C\)

from definition of set difference

\(\equiv x \in A \land (\neg x \in B \land \neg x \in C)\)

from associativity

\(\equiv x \in A \land \neg (x \in B \lor x \in C)\)

from De Morgan’s laws

\(\equiv x \in A \land \neg (x \in B \cup C)\)

from definition of set union

\(\equiv x \in A \land x \in B \cup \overline{C}\)

from definition of set complement

\(\equiv x \in A \land (B \cup \overline{C})\)

from definition of set complement

\(\equiv x \in A \land (B \cup \overline{C})\)

from definition of set intersection

**RHS**

Therefore, \(x \in ((A - B) - C) \implies x \in (A \cap (B \cup \overline{C}))\) is true for some typical element \(x\).

Therefore, \(\forall x ( x \in ((A - B) - C) \implies x \in (A \cap (B \cup \overline{C})) )\) is a true statement, and so \((A - B) - C = A \cap (B \cup \overline{C})\) is a true statement.
Question 8.

a) Partition the set universe into eight disjoint subsets as shown below and assign to each subset a unique element, say \( A = \{1, 2, 3, 4\} \), \( B = \{2, 3, 5, 6\} \), \( C = \{3, 4, 5, 7\} \), and \( U = \{1, 2, 3, 4, 5, 6, 7, 8\} \).

\[
\begin{array}{c}
1 & A \\
2 & \\
3 & B \\
4 & \\
5 & C \\
6 & \\
7 & \\
8 & \\
\end{array}
\]

Now \( B - C = \{2, 6\} \) and so \( A - (B - C) = \{1, 3, 4\} \).

Similarly, \( A - B = \{1, 4\} \) and so \( (A - B) - C = \{1\} \).

Clearly, \( A - (B - C) \neq (A - B) - C \) and the sets \( A, B, \) and \( C \) above are a suitable counterexample.

Care must be exercised when trying to prove or disprove a statement involving sets – you can wind up in a horrible mess if the statement is actually false.

For example, if you attempted to prove or disprove this statement using set theorems:

LHS
\[
\begin{align*}
&= A - (B - C) \\
&= A - (B \cap \overline{C}) \\
&= A \cap (B \cap \overline{C}) \\
&= A \cap (\overline{B} \cup \overline{C}) \\
&= A \cap (\overline{B} \cap \overline{C}) \\
&= \text{using De Morgan’s laws} \\
&= \text{using double complement}
\end{align*}
\]

RHS
\[
\begin{align*}
&= (A - B) - C \\
&= (A \cap \overline{B}) - C \\
&= (A \cap \overline{B}) \cap \overline{C} \\
&= A \cap (\overline{B} \cap \overline{C}) \\
&= \text{using set difference} \\
&= \text{using associativity}
\end{align*}
\]

We end up asking the question is \( A \cap (\overline{B} \cup \overline{C}) = A \cap (\overline{B} \cap \overline{C}) \) ? So be careful. ☺

b) \( (C \subseteq A \land C \subseteq B) \Rightarrow C \subseteq (A \cap B) \)

LHS
\[
\begin{align*}
&= (C \subseteq A \land C \subseteq B) \\
&\equiv \forall x \in C (x \in A) \land C \subseteq B \\
&\equiv \forall x \in C (x \in A) \land \forall x \in C (x \in B) \\
&\Rightarrow \forall x \in C (x \in A \land x \in B) \\
&\Rightarrow C \subseteq (A \cap B) \\
&\Rightarrow RHS
\end{align*}
\]

A formal proof using the typical element argument is somewhat harder. ☺

Alternately, we can attempt an informal proof using Venn Diagrams similar to part (a) above.

Partition the set universe into eight disjoint subsets and assign to each subset a unique element as previously, so we have \( A = \{1, 2, 3, 4\} \), \( B = \{2, 3, 5, 6\} \), \( C = \{3, 4, 5, 7\} \), and \( U = \{1, 2, 3, 4, 5, 6, 7, 8\} \).
To prove this statement, we would need to show we have \((T \land T) \Rightarrow T\).

For \(C \subseteq A\) to be true, we require that the subsets containing elements 5 and 7 be empty, that is, \(B = \{2, 3, 6\}, C = \{3, 4\},\) and \(U = \{1, 2, 3, 4, 6, 8\}\).

For \(C \subseteq B\) to be true, we also require that the subset containing elements 4 be empty, that is, \(A = \{1, 2, 3\}, B = \{2, 3, 6\}, C = \{3\},\) and \(U = \{1, 2, 3, 6, 8\}\).

Now \(A \cap B = \{2, 3\}\) and so \(C \subseteq (A \cap B)\) is true. Therefore, we have shown that we have \((T \land T) \Rightarrow T\), and so the statement is proven.

Note that if we had \(C \subseteq (A \cap B)\) is false, even after the assumptions we were required to make to ensure that \(C \subseteq A\) and \(C \subseteq B\) were true, then we would have disproven the statement.
**Question 1.**

a) An ordered pair is a set \((a, b)\) such that \((a, b) \neq (b, a)\) unless \(a = b\), that is, the order of the components is important. Further, two ordered pairs \((a, b)\) and \((c, d)\) are equal if and only if \(a = c\) and \(b = d\).

Note that for most sets \([a, b] = [b, a]\) as the order of components is not important.

b) The Cartesian product \(A \times B\) is the set of all ordered pairs \((x, y)\) such that \(x\) is an element of \(A\) and \(y\) is an element of \(B\), that is, \(A \times B = \{(x, y): x \in A \land y \in B\}\).

c) The binary relation \(T\) from \(A\) to \(B\) defined \(T = \{(x, y): P(x, y)\}\) is a subset of the Cartesian product \(A \times B\) such that the predicate \(P(x, y)\) is true.

**Question 2.**

a) \[ A \times B = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\} \]

\(A\) and \(B\) both have three elements so \(A \times B\) has nine elements.

It is sometimes useful to think of \(A \times B\) as a “table” where the elements of \(A\) and \(B\) correspond to “rows” and “columns” of the table.

b) Let \(T = \{(x, y) \in A \times B: x + y\ \text{is a prime number}\}\)

\[ = \{(1, 4), (1, 6), (2, 5), (3, 4)\}\]

**Question 3.**

Let \(T = \{(x, y): x \in A \land y \in B \land P(x, y)\}\) be a binary relation from \(A\) to \(B\).

a) The domain of \(T\) is the set of all \(x \in A\) such that there exists a matching \(y \in B\), that is, \(\text{Dom } T = \{x \in A: \exists y \in B (P(x, y))\}\).

The range of \(T\) is the set of all \(y \in B\) such that there exists a matching \(x \in A\), that is, \(\text{Ran } T = \{y \in B: \exists x \in A (P(x, y))\}\)

3b) \(T^{-1} = \{(x, y): x \in B \land y \in A \land P(y, x)\}\)

The inverse of the relation \(T\) may be obtained by swapping the components of the ordered pairs \((x, y)\). The domain of \(T^{-1}\) is the range of \(T\). The range of \(T^{-1}\) is the domain of \(T\).

**Question 4.**

All four relations involve the circle \(x^2 + y^2 = 4\), whose centre is the origin \((0, 0)\) and has radius 2.

The region defined by \(x^2 + y^2 = 4\) is the circle itself, the region defined by \(x^2 + y^2 > 4\) is the outside of the circle, and the region defined by \(x^2 + y^2 < 4\) is the inside of the circle.

Assignment 8 Spring 2010 Solutions
a) In $T_1$, we have both $x \in \mathbb{R}$ and $y \in \mathbb{R}$, so the relation is the region inside the circle.

b) In $T_2$, we have $x \in \mathbb{R}$ and $y \in \mathbb{Z}$, so the relation is those parts of the horizontals lines $y = 1$, $y = 0$, and $y = -1$, which lie inside the circle.

c) In $T_3$, we have $x \in \mathbb{Z}$ and $y \in \mathbb{R}$, so the relation is those parts of the vertical lines $x = 1$, $x = 0$, and $x = -1$, which lie inside the circle.

d) In $T_4$, we have both $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$, so the relation is the set of points formed by the intersection of the horizontal lines described in (b) above with the vertical lines described in (c) above which lie inside the circle, that is, $T_4 = \{(1, 1), (1, 0), (1, -1), (0, 1), (0, 0), (0, -1), (-1, 1), (-1, 0), (-1, -1)\}$.

The important point here is that the relation is defined by both the Cartesian product that it is a subset of as well as the predicate relating $x$ and $y$.

**Question 5.**

a) A simple graph is a graph with no loops or parallel edges.

b) A non-connected graph is a graph which consists of two or more components.

c) Adjacent vertices $v_1$ and $v_2$ are connected via a common edge $e_1 = (v_1, v_2)$.

d) Adjacent edges $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_3)$ are connected via a common vertex $v_2$.

e) The vertices $v_1$ and $v_2$ in part (c) above are said to be incident on edge $e_1$. The edges $e_1$ and $e_2$ in part (d) above are said to be incident on vertex $v_1$.

**Question 6.**

a) Solution does not exist – the sum of degrees of the vertices is not even.

b) Solution exists.
c) Solution does not exist – many graphs with four vertices of degree 1, 1, 1, and 5 exists, but none are simple graphs.

Question 7.

a) 

b) $G$ is not a simple graph – it contains parallel edges $e_3$ and $e_4$. If we were to remove one of these edges, the resulting sub-graph would be a simple graph.

c) $G$ is not a connected graph – it contains an isolated vertex $v_5$ and so consists of two components. If we were to add another edge from one of the vertices $v_1, v_2, v_3, v_4,$ or $v_6$, the resulting graph would be a connected graph.
**Question 1.**

Let $T$ be the relation from $A$ to $B$ defined as $T = \{(x, y) \in A \times B : P(x, y)\}$

a) $T$ is reflexive if and only if $\forall x ((x, y) \in T \Rightarrow (x, x) \in T)$
   
   or $\forall x (P(x, y) \Rightarrow P(x, x))$

b) $T$ is symmetric if and only if $\forall x \forall y ((x, y) \in T \Rightarrow (y, x) \in T)$
   
   or $\forall x \forall y (P(x, y) \Rightarrow P(y, x))$

c) $T$ is transitive if and only if $\forall x \forall y \forall z ((x, y) \in T \land (y, z) \in T \Rightarrow (x, z) \in T)$
   
   or $\forall x \forall y \forall z (P(x, y) \land P(y, z) \Rightarrow P(x, z))$

**Question 2.**

a) A binary function $f$ from $A$ to $B$ is a binary relation from $A$ to $B$ such that two additional properties are satisfied:

i) Domain $f = A$ (The “Existence” property), that is, $\forall x \in A, \exists y \in B, (x, y) \in f$

ii) $f$ is not one-to-many or many-to-many, (The “Uniqueness” property), that is, $\forall x \in A, \forall y_1, y_2 \in B ( (x, y_1) \in f \land (x, y_2) \in f \Rightarrow y_1 = y_2)$.

2b) Domain $f = A$ is always required.

2c) Range $f = A$ is only required if $f$ is onto $B$.

**Question 3.**

Let $H$ be the set of all human beings.

Let $T = \{(x, y) : x$ is the brother or sister of $y\}$ be a relation on $H$

a) Dom $T =$ set of human beings who are the brother or sister of another human being.

Ran $T =$ set of human beings who have brothers or sisters.

It should be clear that Dom $T =$ Ran $T$ and exclude all human beings who have no brothers or sisters.

b) $T^{-1} = \{(x, y) : y$ is the brother or sister of $x\}$

It should be clear that $T = T^{-1}$.

c) $T$ is not reflexive – $x$ cannot be the brother or sister of themself.

d) $T$ is symmetric – if $x$ is the brother or sister of $y$ then $y$ must also be the brother or sister of $x$.

e) $T$ is not transitive – this relation would seem to be transitive if we have three people $x, y, \text{and } z$: if $x$ is the brother or sister of $y$ and $y$ is the brother or sister of $z$, then $x$ must also the brother or sister of $z$. However, this property actually fails if we have only two people $x$ and $y$: if $x$ is the brother or sister of $y$ and $y$ is the brother or sister of $x$, we would require that $x$ be the brother or sister of themself, which is impossible.
f) \( T \) is not an equivalence relation because it is neither reflexive nor transitive.

g) \( T \) is not one-to-one because some people have more than one brother or sister. The relation is actually many-to-many.

h) \( T \) is not a function because it fails both the existence property (some people have no brothers or sisters) and the uniqueness property (some people have more than one brother or sister).

i) \( T \) is not onto \( H \) because \( \text{Ran} \ T \neq H \).

**Question 4.**

A permutation \( f \) on \( A \) is a binary function from \( A \) onto \( A \) which is both one-to-one and onto \( A \).

**Question 5.**

Let \( f = (1 \ 4 \ 3) \ (2 \ 5) \) and \( g = (3 \ 4 \ 5) \)

We interpret these permutations as follows:

\[
\begin{array}{cccccc}
  x & 1 & 2 & 3 & 4 & 5 \\
  f(x) & 4 & 5 & 1 & 3 & 2 \\
\end{array}
\]

i.e. \( f(1) = 4, f(2) = 5, f(3) = 1, f(4) = 3, \) and \( f(5) = 2 \)

\[
\begin{array}{cccccc}
  x & 1 & 2 & 3 & 4 & 5 \\
  g(x) & 1 & 2 & 4 & 5 & 3 \\
\end{array}
\]

i.e. \( g(1) = 1, g(2) = 2, g(3) = 4, g(4) = 5, \) and \( g(5) = 3 \)

a) \( f \cdot g = (1 \ 5 \ 2 \ 3) \)

We interpret the composition \( f \cdot g \) as \( g(f(x)) \) – we perform the permutation \( f \) and then perform the permutation \( g \) on the results:

\[
\begin{array}{cccccc}
  x & 1 & 2 & 3 & 4 & 5 \\
  f(x) & 4 & 5 & 1 & 3 & 2 \\
  g(f(x)) & 5 & 3 & 1 & 4 & 2 \\
\end{array}
\]

first permutation

second permutation

Note that this permutation does not change the value of 4 so we can omit it from the answer. We could also write it in as a separate cycle if we really wanted to, \( f \cdot g = (1 \ 5 \ 2 \ 3) \ (4) \)

b) \( f^{-1} = (3 \ 4 \ 1) \ (5 \ 2) \)

c) \( g^{-1} = (5 \ 4 \ 3) \)

d) \( g^{-1} \cdot f^{-1} = (1 \ 3 \ 2 \ 5) \)

We interpret the composition \( g^{-1} \cdot f^{-1} \) as \( f^{-1}(g^{-1}(x)) \) – we perform the permutation \( g^{-1} \) and then perform the permutation \( f^{-1} \) on the results:

\[
\begin{array}{cccccc}
  x & 1 & 2 & 3 & 4 & 5 \\
  g^{-1}(x) & 1 & 2 & 5 & 3 & 4 \\
  f^{-1}(g^{-1}(x)) & 3 & 5 & 2 & 4 & 1 \\
\end{array}
\]

first permutation

second permutation

It is worth noting that solutions to (a) and (d) are inverses of each other, that is, \( (f \cdot g)^{-1} = g^{-1} \cdot f^{-1} \)

This result is true for all permutations.
Question 6.

a) Let $G = \{V, E\}$ and $G' = \{V', E'\}$ be graphs. $G$ and $G'$ are said to be isomorphic if there exist a pair of functions $f : V \rightarrow V'$ and $g : E \rightarrow E'$ such that $f$ associates each element in $V$ with exactly one element in $V'$ and vice versa; $g$ associates each element in $E$ with exactly one element in $E'$ and vice versa, and for each $v \in V$, and each $e \in E$, if $v$ is an endpoint of the edge $e$, then $f(v)$ is an endpoint of the edge $g(e)$.

Informally, if two graphs are isomorphic, they must have:
- the same number of vertices
- the same number of edges
- the same degrees for corresponding vertices
- the same number of connected components
- the same number of loops
- the same number of parallel edges.

Further,
- both graphs are connected or both graphs are not connected, and
- pairs of connected vertices must have the corresponding pair of vertices connected.

b) There are many graphs with three vertices and three edges. Most are clearly non-isomorphic, for example,

- graph $G_1$ with vertices of degree 2, 2, 2
- graph $G_2$ with vertices of degree 1, 2, 3
- graph $G_3$ with vertices of degree 1, 1, 4

Isomorphic graphs typically exist which are “mirror images” of the above graphs, for example, $G_2$ and $G_2'$ below are isomorphic.

Question 7.

a) A simple graph $G = \{V, E\}$ is said to be bipartite if there exists sets $U \subseteq V$ and $W \subseteq V$, such that
1. $U \cup W = V$ and $U \cap W = \emptyset$
2. Every edge of $G$ connects a vertex in $U$ with a vertex in $W$.

b) A simple graph $G = \{V, E\}$, is said to be complete bipartite if
1. $G$ is bipartite and
2. Every vertex in $U$ is connected to every vertex in $W$. 
c) Any graph which contains five of the following six edges \((u_1, w_1), (u_1, w_2), (u_1, w_3), (u_2, w_1), (u_2, w_2),\) and \((u_2, w_3)\) would be a solution. Therefore, there are six possible solutions.

d) The graph which contains all six of the edges in part (c) above would be complete bipartite.

e) The solutions in part (c) above are clearly sub-graphs of the solution in part (d).

Question 8.

a) Let \(G = \{V, E\}\) be a graph and let \(v \in V\). A circuit in \(G\) is a path from \(v\) to \(v\) in which no edge is repeated.

b) Let \(G = \{V, E\}\) be a graph. A circuit in \(G\) is an Eulerian circuit if every edge of \(G\) is included exactly once in the circuit.

c) Let \(G = \{V, E\}\) be a graph. \(G\) is an Eulerian graph if \(G\) has an Eulerian circuit.

d) \[\begin{array}{c}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
\end{array}\]

This is not an Eulerian graph as the degree of each vertex is not even: \(\delta(v_3) = \delta(v_6) = 3\)

Question 9.

a) A tree is a connected graph which has no non-trivial circuits.

b) A spanning tree for a graph \(G\) is a sub-graph of \(G\) which is a tree that includes every vertex of \(G\).

c) There are seven vertices but we only need six edges to connect them all together. We have eight edges so we must remove two of them to form a tree – we should remove any one of \((v_3, v_4), (v_3, v_5), (v_4, v_6),\) and \((v_5, v_6),\) and then any one of the remaining seven edges, so there are more than twenty different spanning trees.

Question 10.

The minimum spanning tree is as follows:

The minimum spanning tree is as follows:
Step 1: create a table of edges and their weights

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a, b)</td>
<td>6</td>
</tr>
<tr>
<td>(a, e)</td>
<td>9</td>
</tr>
<tr>
<td>(b, c)</td>
<td>4</td>
</tr>
<tr>
<td>(b, g)</td>
<td>5</td>
</tr>
<tr>
<td>(c, d)</td>
<td>3</td>
</tr>
<tr>
<td>(c, g)</td>
<td>4</td>
</tr>
<tr>
<td>(d, e)</td>
<td>1</td>
</tr>
<tr>
<td>(d, f)</td>
<td>5</td>
</tr>
<tr>
<td>(e, f)</td>
<td>7</td>
</tr>
<tr>
<td>(f, g)</td>
<td>8</td>
</tr>
</tbody>
</table>

Step 2: sort the table according to the weights and apply the algorithm

<table>
<thead>
<tr>
<th>Edge</th>
<th>Weight</th>
<th>Will adding edge make circuit?</th>
<th>Action taken</th>
<th>Cumulative weight of sub-graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d, e)</td>
<td>1</td>
<td>No</td>
<td>Added</td>
<td>1</td>
</tr>
<tr>
<td>(c, d)</td>
<td>3</td>
<td>No</td>
<td>Added</td>
<td>4</td>
</tr>
<tr>
<td>(b, c)</td>
<td>4</td>
<td>No</td>
<td>Added</td>
<td>8</td>
</tr>
<tr>
<td>(c, g)</td>
<td>4</td>
<td>No</td>
<td>Added</td>
<td>12</td>
</tr>
<tr>
<td>(b, g)</td>
<td>5</td>
<td>Yes</td>
<td>Not Added</td>
<td>12</td>
</tr>
<tr>
<td>(d, f)</td>
<td>5</td>
<td>No</td>
<td>Added</td>
<td>17</td>
</tr>
<tr>
<td>(a, b)</td>
<td>6</td>
<td>No</td>
<td>Added</td>
<td>23</td>
</tr>
<tr>
<td>(e, f)</td>
<td>7</td>
<td>Yes</td>
<td>Not Added</td>
<td>23</td>
</tr>
<tr>
<td>(f, g)</td>
<td>8</td>
<td>Yes</td>
<td>Not Added</td>
<td>23</td>
</tr>
<tr>
<td>(a, e)</td>
<td>9</td>
<td>Yes</td>
<td>Not Added</td>
<td>23</td>
</tr>
</tbody>
</table>

The algorithm has a unique solution in this example, but this is not always the case – where several edges have the same cost and any of them can be added without forming a circuit, the order in which the edges are added may result in different solutions. All solutions, however, must have the same minimum cost.