C*-algebras associated to higher-rank graphs

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I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

(Signed) ____________________

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Contents

Abstract

Directed graphs are combinatorial objects used to model networks like fluid-flow systems in which the direction of movement through the network is important. In 1980, Enomoto and Watatani used finite directed graphs to provide an intuitive framework for the Cuntz-Krieger algebras introduced by Cuntz and Krieger earlier in the same year. The theory of the C^*-algebras of directed graphs has since been extended to include infinite graphs, and there is an elegant relationship between connectivity and loops in a graph and the structure theory of the associated C^*-algebra.

Higher-rank graphs are a higher-dimensional analogue of directed graphs introduced by Kumjian and Pask in 2000 as a model for the higher-rank Cuntz-Krieger algebras introduced by Robertson and Steger in 1999. The theory of the Cuntz-Krieger algebras of higher-rank graphs is relatively new, and a number of questions which have been answered for directed graphs remain open in the higher-rank setting. In particular, for a large class of higher-rank graphs, the gauge-invariant ideal structure of the associated C^*-algebra has not yet been identified.

This thesis addresses the question of the gauge-invariant ideal structure of the Cuntz-Krieger algebras of higher-rank graphs. To do so, we introduce and analyse the collections of relative Cuntz-Krieger algebras associated to higher-rank graphs.

The first two main results of the thesis are versions of the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem which apply to relative Cuntz-Krieger algebras. Using these theorems, we are able to achieve our main goal, producing a classification of the gauge-invariant ideals in the Cuntz-Krieger algebra of a higher-rank graph analogous to that developed for directed graphs by Bates, Hong, Raeburn and Szymański in 2002. We also demonstrate that relative Cuntz-Krieger algebras associated to higher-rank graphs are always nuclear, and produce conditions on a higher-rank graph under which the associated Cuntz-Krieger algebra is simple and purely infinite.

CHAPTER 1

Introduction

Cuntz-Krieger algebras associated to directed graphs, or graph algebras, have been of great interest in recent years due to the elegant relationship between loop-structure and connectivity in a graph and the structure of its Cuntz-Krieger algebra [**5**]. In particular this relationship can be exploited to produce a variety of examples of simple purely infinite nuclear C^*-algebras [**39**, Theorem 1.2]. In this thesis, we turn our attention to the higher-rank graphs introduced in [**18**]. We analyse the gauge-invariant ideal structure of their C^*-algebras, and provide conditions under which these C^*-algebras are simple, purely infinite and nuclear. To achieve our aims, we are forced to introduce and analyse a family of C^*-algebras which we call the *relative Cuntz-Krieger algebras* of a higher-rank graph. Our main results are a gauge-invariant uniqueness theorem (Theorem 4.3.12) and a Cuntz-Krieger uniqueness theorem (Theorem 4.5.2) for the relative Cuntz-Krieger algebras of higher-rank graphs, and a graph-theoretic description of the lattice of gauge-invariant ideals in the C^*-algebra of a higher-rank graph (Theorem 5.3.8).

In order to discuss the ideas in this thesis, it is helpful first to review briefly the development of the theory of graph C^*-algebras, especially with respect to the study of their ideal structure. We start in Section 1.1 by recalling the fundamental theory of row-finite graphs and their C^*-algebras, stating the theorems which we intend to generalise.

This thesis studies the C^*-algebras of higher-rank graphs, and our main task is to extend the theory of higher-rank graph C^*-algebras, which currently deals only with row-finite higher-rank graphs, to the non-row-finite case. To introduce the obstacles that our analysis must overcome, we go on to discuss in Section 1.2 the problems which arose when the theory of ordinary directed graphs was extended to include the non-row-finite case, and describe the solutions to these problems. In Section 1.3 we discuss k-graphs and indicate what is known in the row-finite case. In particular, we indicate how the theory of higher-rank graphs and their C^*-algebras parallels that of directed graphs, as well as indicating the key differences between the two theories. In Section 1.4, we discuss how the theory of

Cuntz-Krieger algebras of row-finite k-graphs has been extended to the non-row-finite setting. In Section 1.5, we give a brief summary of the main results and achievements of this thesis. In Section 1.6, we then give a more detailed overview of the thesis and of how we achieve our goals.

1.1. Row-finite graphs and their C^*-algebras

A directed graph E consists of a countable set E^0 of vertices, a countable set E^1 of edges and range and source maps $r, s : E^1 \to E^0$ which indicate the directions of the edges. Enomoto and Watatani showed in [**8**] that the Cuntz-Krieger algebras of finite $(0, 1)$-matrices [**6**] have a natural interpretation as C^*-algebras associated to finite directed graphs. Groupoids were used to generalise this theory to infinite graphs by Kumjian, Pask, Raeburn and Renault in [**20**]. To make use of the existing theory of locally compact groupoids and their C^*-algebras, it was necessary to restrict attention to the *locally finite* graphs in which $r^{-1}(v)$ and $s^{-1}(v)$ are both finite for all $v \in E^0$, and to assume that the graphs have *no sources*[†] in the sense that $r^{-1}(v)$ is nonempty for all $v \in E^0$.

For a locally finite graph E with no sources, the Cuntz-Krieger algebra $C^*(E)$ is the universal C^*-algebra generated by projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$ and

$$\sum_{e \in r^{-1}(v)} s_e s_e^* = p_v \quad \text{for all } v \in E^0; \tag{1.1.1}$$

such a family of projections and partial isometries is called a *Cuntz-Krieger E-family.*

In [**33, 34**], Renault described the ideal structure of the C^*-algebra associated to a locally compact groupoid $\mathcal{G}$, and in particular formulated a hypothesis under which the ideals of the C^*-algebra correspond to easily identifiable subsets of the unit space of the groupoid. In [**20**], Kumjian, Pask, Raeburn and Renault realised the C^*-algebra of a locally finite graph E as the C^*-algebra associated to a locally compact groupoid $\mathcal{G}_E$ of the sort studied in [**33, 34**]. They then showed that Renault's hypothesis for $\mathcal{G}_E$ was equivalent to a hypothesis on E which they called *Condition (K)*; namely that no vertex of E can lie on precisely one loop. When E satisfies Condition (K), Renault's theory of groupoid C^*-algebras [**33, 34**] can be used to describe the ideal structure of $C^*(E)$ in terms of the loop-structure of E. Building on the analysis of [**20**], Kumjian, Pask and Raeburn formulated

[†]Readers who are familiar with graph algebras should note that this thesis follows the edge-direction conventions of [**18, 29, 30**] where the partial isometry s_e has the same direction as the edge e; hence *no sources* here corresponds to *no sinks* in, for example, [**20, 3, 23**]

the weaker *Condition (L)*, which with our edge-direction convention says that every loop in E has an entrance. Theorem 3.7 of [**19**] says that when E satisfies Condition (L), every representation of $C^*(E)$ which is nonzero on all the vertex projections is faithful. This is the typical form of what is known as a *Cuntz-Krieger uniqueness theorem* — see Theorem 1.1.2 below for details.

In [**3**], Bates, Pask, Raeburn and Szymański employed direct methods rather than groupoid models to extend the structure theorems of [**20, 19**] to the *row-finite graphs* in which $r^{-1}(v)$ is always finite, but $s^{-1}(v)$ may not be, and to graphs which may contain sources. The generalisation from locally-finite to row-finite graphs required no modification of the relations; the definition of $C^*(E)$ in [**20, 19**] makes sense for row-finite graphs, and it was only the particular groupoid model used to prove the uniqueness theorems of [**20, 19**] that required the locally-finite hypothesis. On the other hand, allowing sources does require an adjustment of the definition of $C^*(E)$. To see this, notice that if v is a source in E, then $r^{-1}(v)$ is empty and hence (1.1.1) would force $p_v = 0$. The solution in [**3**] was to insist that (1.1.1) hold only when $r^{-1}(v)$ is nonempty, so that no relation is imposed at sources. To obtain results about graphs with sources, [**3**] introduced a construction called *adding tails*. Adding tails to an arbitrary row-finite directed graph E produces a row-finite graph F with no sources such that $C^*(E)$ is a full corner in $C^*(F)$ [**3**, Lemma 1.2]. The results of [**3**] were the first to apply to arbitrary row-finite directed graphs, and we shall now discuss them.

Given a directed graph E, the path-space E^* of E consists of all sequences $\mu = e_1e_2 \dots e_n$ of edges of E such that $s(e_{i-1}) = r(e_i)$ for $2 \le i \le n$. This definition may seem back-to-front at first. To make sense of it, it is helpful to think of edges $e \in E^1$ as maps $e : s(e) \to r(e)$, so that the path $e_1 \dots e_n$ can be thought of as the composition map $e_1 \circ \cdots \circ e_n : s(e_n) \to r(e_1)$. For $\mu = e_1 \dots e_n \in E^*$ we write s_μ for the partial isometry $s_{e_1}s_{e_2} \cdots s_{e_n}$, and we write $s(\mu)$ for $s(e_n)$. It is shown in [**3**, Equation (1.1)] that

$$C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}. \tag{1.1.2}$$

This spanning condition is crucial to the analysis of the Cuntz-Krieger algebra. The universal property of $C^*(E)$ shows that there is a strongly continuous action γ of $\mathbb{T}$ on $C^*(E)$ such that $\gamma_z(p_v) = p_v$ for all $v \in E^0$ and $\gamma_z(s_e) = zs_e$ for all $e \in E^1$ (see [**3**, page 309]); this action γ is called the *gauge action*.

We can now state the first of the results in [**3**] which we intend to generalise in this thesis: the gauge-invariant uniqueness theorem. The first version of the gauge-invariant uniqueness theorem applied to C^*-algebras generated by $(0, 1)$-matrices and was formulated and proved by an Huef and Raeburn in [**16**, Theorem 2.3]. The version given below is that found in [**3**].

THEOREM 1.1.1 ([**3**, Theorem 2.1]: The gauge-invariant uniqueness theorem). *Let E be a row-finite graph, let $\{S_e, P_v : e \in E^1, v \in E^0\}$ be a Cuntz-Krieger E-family in $\mathcal{B}(\mathcal{H})$, and let $\pi_{S,P}$ be the representation of $C^*(E)$ such that $\pi_{S,P}(s_e) = S_e$ and $\pi_{S,P}(p_v) = P_v$ for all $e \in E^1$ and $v \in E^0$. Suppose that each P_v is nonzero and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(\{S_e, P_v : e \in E^1, v \in E^0\})$ such that $\beta_z \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_z$ for all $z \in \mathbb{T}$. Then $\pi_{S,P}$ is faithful.*

The second theorem from [**3**] which we wish to generalise is the Cuntz-Krieger uniqueness theorem. In this result, the hypothesis that there exists an action β which implements the gauge action under $\pi_{S,P}$ is replaced by an extra structural condition on the graph. Theorems of this type for Cuntz-Krieger algebras associated to (0,1)-matrices date back to Cuntz and Krieger's original analysis in [**6**], and a version of this theorem for the C^*-algebras of locally finite graphs appeared in [**19**]. The version given below is once again from [**3**].

THEOREM 1.1.2 ([**3**, Theorem 3.1]: The Cuntz-Krieger uniqueness theorem). *Suppose that E is a row-finite graph which satisfies Condition* (L)*, and that $\{S_e, P_v : e \in E^1, v \in E^0\}$ and $\{T_e, Q_v : e \in E^1, v \in E^0\}$ are two Cuntz-Krieger E-families in which all the projections P_v and Q_v are nonzero. Then there is an isomorphism ϕ of $C^*(\{S_e, P_v : e \in E^1, v \in E^0\})$ onto $C^*(\{T_e, Q_v : e \in E^1, v \in E^0\})$ such that $\phi(S_e) = T_e$ and $\phi(P_v) = Q_v$ for all $e \in E^1$ and $v \in E^0$.*

In this thesis, we prove versions of Theorems 1.1.1 and 1.1.2 for a collection of C^*-algebras which we call *relative Cuntz-Krieger algebras* associated to non-row-finite higher-rank graphs. Our motivation for this was the study of the gauge-invariant ideals in Cuntz-Krieger algebras of non-row-finite higher-rank graphs. Specifically, the intention was to generalise Theorem 4.1 of [**3**] to the non-row-finite higher-rank setting. Theorem 4.1 of [**3**] completely describes the gauge-invariant ideals in $C^*(E)$ for a row-finite graph E. To state this theorem, we need some notation and terminology from [**3**, Section 4].

Given a set $H \subset E^0$ of vertices in E, we write I_H for the ideal in $C^*(E)$ generated by the collection of projections $\{p_v : v \in H\}$; since all these projections are fixed by the gauge action, I_H is always gauge-invariant. A set $H \subset E^0$ is *hereditary* if $e \in E^1$ and $r(e) \in H$ imply $s(e) \in H$. A set $H \subset E^0$ is *saturated* if whenever $v \in E^0$ has the property that $s(e) \in H$ for all $e \in r^{-1}(v)$, we have $v \in H$.

Theorem 1.1.3 ([**3**, Theorem 4.1]: The gauge-invariant ideal structure). *Let E be a row-finite graph.*

(a) *The map $H \mapsto I_H$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of closed gauge-invariant ideals of $C^*(E)$.*
(b) *Suppose that $H \subset E^0$ is saturated and hereditary. Let F be the graph with vertices $E^0 \setminus H$ and edges $\{e \in E^1 : s(e) \notin H\}$. Then $C^*(E)/I_H$ is canonically isomorphic to $C^*(F)$.*
(c) *If X is any hereditary subset of E^0 and G is the graph with vertices X and edges $\{e \in E^1 : r(e) \in X\}$, then $C^*(G)$ is canonically isomorphic to $C^*(\{s_e, p_v : e \in G^1, v \in X\})$, and this subalgebra is a full corner in the ideal I_X.*

1.2. C^*-algebras of non-row-finite graphs

The problem of generalising the results of [**3**] to deal with Cuntz-Krieger algebras associated to non-row-finite directed graphs is tricky because it is not immediately clear how to define $C^*(E)$ when E is not row-finite. If $v \in E^0$ has the property that $r^{-1}(v)$ is infinite, then relation (1.1.1) contains an infinite sum of projections that cannot converge in a C^*-algebra. However, if no relationship at all is imposed on the projections $s_e s_e^*$ for $e \in r^{-1}(v)$ when $r^{-1}(v)$ is infinite, then the spanning condition (1.1.2) will fail, which has serious ramifications for the structure of the resulting C^*-algebra.

The solution to this problem was suggested by Fowler and Raeburn's analysis of the Toeplitz algebra of a Hilbert bimodule in [**14**]. Example 1.2 of [**14**] shows how to construct a Hilbert bimodule $X(E)$ from a directed graph E with no sources in such a way that if E is row-finite then the representations and universal algebras of E and $X(E)$ coincide. Theorem 4.1 of [**14**] shows that the Toeplitz algebra $\mathcal{T}_{X(E)}$ of the Hilbert bimodule $X(E)$ is generated by a family of projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ such that $s_e^* s_e = p_{s(e)}$ and such that $p_v \geq \sum_{e \in F} s_e s_e^*$ for all $v \in E^0$ and all finite $F \subset r^{-1}(v)$. Such a family is called a Toeplitz-Cuntz-Krieger E-family. Theorem 4.1 of [**14**] shows that a Toeplitz-Cuntz-Krieger E-family $\{s_e, p_v : e \in E^1, v \in E^0\}$ generates an isomorphic copy of $\mathcal{T}_{X(E)}$ if and only if

(1) $p_v \neq 0$ for all $v \in E^0$; and
(2) if $v \in E^0$ and if $F \subset r^{-1}(v)$ is finite, then $p_v - \sum_{\lambda \in F} s_\lambda s_\lambda^* \neq 0$.

Most signficantly, [**14**, Theorem 4.1] holds whether E is row-finite or not.

The uniqueness theorems for the Cuntz-Krieger algebras of row-finite graphs are both based on [**3**, Corollary 2.4] which says that a Cuntz-Kreiger E-family $\{s_e, p_v : e \in E^1, v \in E^0\}$ corresponds to a homomorphism of $C^*(E)$ which is injective on the fixed point algebra $C^*(E)^\gamma$ for the gauge action if and only if it satisfies (1) above. Hence Theorem 4.1 of [**14**] suggests how to define $C^*(E)$ for a non-row-finite directed graph E as follows. Suppose that $v \in E^0$ and $F \subset r^{-1}(v)$ have the property that there exists a Toeplitz-Cuntz-Krieger E-family $\{t_e, q_v : e \in E^1, v \in E^0\}$ which satisfies (1) and in which $q_v - \sum_{e\in F} t_e t_e^* = 0$. Since $\{t_e, q_v\}$ satisfies (1), the corresponding homomorphism $\pi_{t,q}$ should be injective on $C^*(E)^\gamma$. But $q_v - \sum_{e\in F} t_e t_e^*$ will belong to $\pi_{t,q}(C^*(E)^\gamma)$, so $\pi_{t,q}$ can only be injective on $C^*(E)^\gamma$ if $p_v - \sum_{e\in F} s_e s_e^* = 0$ in every Cuntz-Krieger E-family $\{s_e, p_v\}$. It is easy to see that if $\{s_e, p_v\}$ satisfies (1), then $p_v - \sum_{e\in F} s_e s_e^*$ is nonzero whenever either $r^{-1}(v)$ is infinite or $r^{-1}(v) \setminus F$ is nonempty, but if $F = r^{-1}(v)$ is finite, then $p_v - \sum_{e\in F} s_e s_e^*$ may be equal to zero. Hence [**14**, Theorem 4.1] suggests that to define $C^*(E)$ for an arbitrary directed graph E, the appropriate analogue of (1.1.1) is

$$p_v \geq \textstyle\sum_{e\in F} s_e s_e^* \quad \text{for all } v \in E^0 \text{ and finite } F \subset r^{-1}(v)$$
$$\text{with equality when } F = r^{-1}(v) \text{ is finite.}$$

Fowler, Laca and Raeburn followed up on this suggestion in [**12**] using results of Exel and Laca [**9**] to prove a version of the Cuntz-Krieger uniqueness theorem for arbitrary directed graphs [**12**, Theorem 2].

The approach of [**12**] was simplified significantly by Raeburn and Szymański in [**31**] by viewing the C^*-algebras of arbitrary graphs as direct limits of C^*-algebras of finite subgraphs. Using this direct limit approach, Raeburn and Szymański were also able to compute the K-theory of $C^*(E)$. Theorem 2 of [**12**] together with information about the structure of the algebras $C^*(E)$ for arbitrary directed graphs were also obtained independently using groupoids and inverse semigroups by Paterson in [**25**, Theorem 4], and using category-theoretic methods by Spielberg in [**37**, Theorem 3.15], confirming that the relations formulated in [**12**] were the right ones.

Another approach to the study of the C^*-algebras of arbitrary directed graphs is presented by Drinen and Tomforde in [**7**]. Here, a construction called *desingularisation* is introduced. Desingularisation generalises the adding-a-tail construction for eliminating sources in a row-finite graph. Given an arbitrary directed graph

E, the desingularisation of E is a row-finite graph F with no sources such that $C^*(E)$ is a full corner in $C^*(F)$. The desingularised graph F satisfies conditions (K) and (L) respectively if and only if E does, so [**12**, Theorem 2] for E follows from [**3**, Theorem 3.1] for F [**7**, Corollary 2.15].

The gauge-invariant ideal structure of $C^*(E)$ when E is not row-finite is more complicated than when E is row-finite. Suppose that H is a saturated hereditary subset of E^0 in the sense appropriate for non-row-finite graphs, and let F be the subgraph of E described in Theorem 1.1.3(b). If $v \in F^0$ is such that $r^{-1}(v)$ is infinite in E but $r^{-1}(v) \cap F^1$ is finite and nonempty, then relation (1.1.1) will not hold at v in the family $\{s_e + I_H, p_v + I_H : e \in F^1, v \in F^0\}$. Hence $C^*(E)/I_H$ will not in general be isomorphic to $C^*(F)$.

The problem of identifying the gauge-invariant ideals of $C^*(E)$ for a non-row-finite graph E is solved by Bates, Hong, Raeburn and Szymański in [**2**], as follows. Let H be a saturated hereditary subset of E^0, and let H^{fin}_∞ be the collection of all vertices v in F^0 such that $r^{-1}(v)$ is infinite in E but finite and nonempty in F. From such data, Bates *et al.* construct a *quotient graph*, denoted E/H, by attaching a source to F for each v in H^{fin}_∞, and show that $C^*(E)/I_H$ is canonically isomorphic to $C^*(E/H)$. Using this construction, they give a complete listing of the gauge-invariant ideals in $C^*(E)$ for an arbitrary directed graph E [**2**, Theorem 3.6]. Each saturated hereditary subset H of E^0 gives rise to an ideal I_H in $C^*(E)$ as before, and also gives rise to an additional ideal $J_{H,B}$ for each subset B of H^{fin}_∞. In [**17**], Hong and Szymański built upon the results of [**2**] to give a complete graph-theoretic description of the primitive ideal space of $C^*(E)$ and its hull-kernel topology.

Recent work of Muhly and Tomforde [**23**] provides another way of viewing the quotients of $C^*(E)$ by gauge-invariant ideals. Muhly and Tomforde define the *relative graph algebra* $C^*(E, V)$ associated to a graph E and a subset V of E^0 to be the universal algebra obtained by imposing relation (1.1.1) only at vertices which belong to V [**23**, Definition 3.2]. By attaching a source to E for every vertex in V [**23**, Definition 3.3], Muhly and Tomforde produce a graph E_V such that the relative graph algebra $C^*(E, V)$ is canonically isomorphic to the graph algebra $C^*(E_V)$. Putting $V = H^{\mathrm{fin}}_\infty$, it is easy to see that the graph E_V of [**23**] is precisely the quotient graph E/H of [**2**]. Hence the quotient of $C^*(E)$ by the ideal I_H, which [**2**] shows to be isomorphic to $C^*(E/H)$, can be viewed instead as a relative graph algebra associated to the subgraph F of E described in Theorem 1.1.3(b).

1.3. Row-finite higher-rank graphs

In [**18**], Kumjian and Pask introduced a new class of combinatorial objects known as *higher-rank* or *rank-k* graphs, and affectionately known as k-graphs. The C^*-algebras associated to k-graphs in [**18**] generalise Robertson and Steger's higher-rank Cuntz-Krieger algebras [**35**] in the same way that the graph C^*-algebras of [**3**] generalise the original Cuntz-Krieger algebras of [**6**].

Kumjian and Pask defined a k-graph to consist not just of vertices and edges, but of all paths which can be obtained by concatenating edges. Each path λ has a range $r(\lambda)$ and a source $s(\lambda)$ in Λ^0, and a *degree* $d(\lambda)$ in $\mathbb{N}^k$. This degree is the higher-rank analogue of the length n of a path $e_1 \dots e_n$ in the path space E^* of a directed graph E. The higher-dimensional nature of paths in a k-graph is encoded by the *factorisation property*: if λ has degree $m + n$, then there are unique paths μ and ν with degrees m and n respectively such that $\lambda = \mu\nu$.

A 1-graph is therefore the path space E^* of a directed graph E, rather than just E itself. The factorisation property is automatic for 1-graphs; it says that for $e_1 \cdots e_n \in E^*$ and $0 < m < n$, the paths $e_1 \cdots e_m$ and $e_{m+1} \cdots e_n$ are the unique paths of lengths m and $n - m$ such that $e_1 \cdots e_n = (e_1 \cdots e_m)(e_{m+1} \cdots e_n)$.

The analysis of the Cuntz-Krieger algebras of k-graphs in [**18**] is based on groupoid methods as were the analyses of the Cuntz-Krieger algebras of directed graphs in [**20**] and [**19**]. Modifications to the groupoid construction allowed Kumjian and Pask to consider k-graphs which were row-finite, rather than locally finite. A k-graph Λ is row-finite if $v\Lambda^n := \{\lambda \in \Lambda : r(\lambda) = v, d(\lambda) = n\}$ is finite for all $n \in \mathbb{N}^k$ and $v \in \Lambda^0$. As in [**20, 19**] however, it is assumed that the k-graphs in [**18**] have no sources, so $v\Lambda^n$ is nonempty for all n and v.

For a k-graph Λ a Cuntz-Krieger Λ-family consists of a family of partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ satisfying four Cuntz-Krieger relations. The vertices in a k-graph Λ are regarded as paths of degree 0, so $\Lambda^0 \subset \Lambda$ and hence the vertex projections are denoted t_v rather than p_v. We describe the Cuntz-Krieger relations given in Definition 1.5 of [**18**] in the same order as they appear there, motivating each one with reference to the corresponding property for Cuntz-Krieger families associated to directed graphs.

(i) Just as for directed graphs, relation (i) of [**18**, Definition 1.5] insists that the partial isometries $\{t_v : v \in \Lambda^0\}$ are mutually orthogonal projections.

(ii) In a directed graph, the partial isometry associated to a path $e_1 \cdots e_n$ in E^* is equal to the composition of partial isometries $t_{e_1} \cdots t_{e_n}$ by definition.

In a Cuntz-Krieger family for a k-graph on the other hand, there already exists a partial isometry associated to each path in Λ. Relation (ii) of [**18**, Definition 1.5] insists that if the path $\lambda \in \Lambda$ decomposes as the concatenation of paths $\lambda = \mu\nu$, then $t_\lambda = t_\mu t_\nu$.

(iii) An easy induction on n shows that if E is a row-finite directed graph, μ a path of length n in E^*, and $\{p_v, s_e : v \in E^0, e \in E^1\}$ a Cuntz-Krieger E-family, then $s_\mu^* s_\mu = p_{s(\mu)}$. Relation (iii) of [**18**, Definition 1.5] insists that $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$.

(iv) Let E be a row-finite directed graph with no sources, let $\{p_v, s_e : v \in E^0, e \in E^1\}$ be a Cuntz-Krieger E-family, let $v \in E^0$ and let $n \in \mathbb{N}$. Write vE^n for the collection of paths in E^* with length n and range v. Then an easy induction on n shows that $p_v = \sum_{\mu \in vE^n} s_\mu s_\mu^*$. The analogue of (1.1.1) imposed by [**18**, Definition 1.5](iv) is that

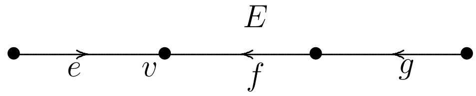
$$t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^* \quad \text{for each vertex } v \text{ of } \Lambda \text{ and each } n \in \mathbb{N}^k. \tag{1.3.1}$$

The Cuntz-Krieger algebra $C^*(\Lambda)$ of a row-finite k-graph Λ with no sources is by definition the C^*-algebra generated by a universal Cuntz-Krieger Λ-family. Kumjian and Pask showed that $C^*(\Lambda)$ always satisfies the analogue of (1.1.2), namely that $C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : d(\lambda) = d(\mu)\}$ [**18**, Lemma 3.1]. They also showed that $C^*(\Lambda)$ carries a strongly continuous gauge action γ of $\mathbb{T}^k$. Using this gauge action, Kumjian and Pask proved a version of the gauge-invariant uniqueness theorem for the C^*-algebras of row-finite k-graphs with no sources.

Kumjian and Pask also formulated an *aperiodicity condition* (A) for higher-rank graphs [**18**, Definition 4.3], and showed that if Λ satisfies Condition (A), then a Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ generates an isomorphic copy of $C^*(\Lambda)$ if and only if t_v is nonzero for each $v \in \Lambda^0$. They also produced graph-theoretic conditions which ensure that $C^*(\Lambda)$ is simple [**18**, Proposition 4.8] and purely infinite [**18**, Proposition 4.9]; $C^*(\Lambda)$ is always nuclear [**18**, Theorem 5.5].

In [**29**], Raeburn, Sims and Yeend generalised the results of [**18**] to cover some row-finite k-graphs containing sources. This is less straightforward than for directed graphs because vertices in a k-graph may receive edges of some degrees and not of others. Thus there can be many different types of sources in a k-graph. The results in [**29**] deal with k-graphs that are *locally convex*; that is, certain configurations of sources are not permitted. For row-finite k-graphs which are not locally convex, the definition of a Cuntz-Krieger family given in [**29**] is unsuitable because if Λ is row-finite but not locally convex then there exists a vertex $v \in \Lambda^0$ such that for any family $\{t_\lambda : \lambda \in \Lambda\}$ satisfying the relations given in [**29**], $t_v = 0$.

To see how vertex projections t_v might be forced to be equal to zero, consider the 1-graph E^* arising from the directed graph



Taking $n = 1$ in (1.3.1) gives $p_v = \sum_{\mu \in vE^1} s_\mu s_\mu^* = s_e s_e^* + s_f s_f^*$. However, since e cannot be extended to a path of length two, applying the same relation with $n = 2$ says that $p_v = \sum_{\mu \in vE^2} s_\mu s_\mu^* = s_{fg} s_{fg}^*$, and it follows that $s_e = 0$ and hence $p_{s(e)} = 0$. This is problematic for two reasons. Firstly, on a philosophical level, the idea of Cuntz-Krieger algebras is that they should encode the structure of a graph E in the C^*-algebra $C^*(E)$. If portions of the graph are associated to the zero element of $C^*(E)$, then the structure of these sections of the graph is not present in the Cuntz-Krieger algebra. Secondly, on a technical level, the typical uniqueness theorems for Cuntz-Krieger algebras apply to Cuntz-Krieger families in which all the vertex projections are nonzero. Consequently the statements of the uniqueness theorems would have to be more complicated to account for Cuntz-Krieger algebras in which some of the vertex projections are necessarily equal to zero.

The problem of associating Cuntz-Krieger families to k-graphs with sources in such a way that the vertex projections are all nonzero was addressed in [**29**] by regarding paths such as e which "run out of puff" before reaching length 2 as having a length of 2 anyway for the purposes of the Cuntz-Krieger relation. More formally, given a k-graph Λ and given $n \in \mathbb{N}^k$, Raeburn *et al.* write $\Lambda^{\leq n}$ for the collection of paths $\lambda \in \Lambda$ such that $d(\lambda) \leq n$ and there is no non-trivial extension $\lambda\mu$ of λ such that $d(\lambda\mu) \leq n$ [**29**, Definition 3.1]. They write $v\Lambda^{\leq n}$ for $\{\lambda \in \Lambda^{\leq n} : r(\lambda) = v\}$. Relation (iv) of [**29**, Definition 3.1] is then that

$$s_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^* \quad \text{for all } v \in \Lambda^0 \text{ and all } n \in \mathbb{N}^k. \tag{1.3.2}$$

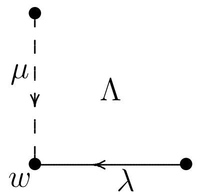
In the 1-graph drawn above, both e and fg belong to $E^{\leq 2}$, so that putting $n = 2$ in (1.3.2) gives $s_v = \sum_{\mu \in vE^{\leq 2}} s_\mu s_\mu^* = s_e s_e^* + s_{fg} s_{fg}^*$. Hence under (1.3.2), $p_{s(e)}$ need not be equal to zero.

In [**29**], Raeburn *et al.* use the $\Lambda^{\leq n}$ construction to analyse the C^*-algebras of locally convex row-finite k-graphs. They proved a version of the gauge-invariant uniqueness theorem [**29**, Theorem 4.1], and they sharpened the aperiodicity Condition (A) slightly to obtain their Condition (B) and a version of the Cuntz-Krieger uniqueness theorem for locally convex row-finite k-graphs [**29**, Theorem 4.3]. Using these uniqueness theorems and the ideas of [**3**], they also showed that the gauge-invariant ideals in $C^*(\Lambda)$ for locally convex row-finite k-graphs Λ correspond to the appropriate higher-rank analogue of saturated hereditary subsets of Λ^0 [**29**, Theorem 5.2].

For a 1-graph E, Condition (K) of [**20**] is equivalent to the statement that for every saturated hereditary $H \subset E^0$, the subgraph $E \backslash s^{-1}(H)$ satisfies Condition (L) of [**19**]. Hence the results of [**20**] can, in retrospect, be interpreted as stating that if a row-finite graph E is such that the removal of any saturated hereditary set $H \subset E^0$ yields a graph which satisfies Condition (L), then every ideal in $C^*(E)$ is gauge-invariant. The higher-rank analogue of [**2**, Corollary 4.8] is proved for row-finite locally convex k-graphs in [**29**, Theorem 5.3]. That is, if a row-finite locally convex k-graph Λ is such that the removal of any saturated hereditary set $H \subset \Lambda^0$ yields a k-graph satisfying [**29**, Condition (B)], then every ideal of $C^*(\Lambda)$ is gauge-invariant.

For some k-graphs, even the $\Lambda^{\leq n}$ construction and (1.3.2) do not guarantee that each s_v is nonzero. Consider the 2-graph

(1.3.3)



where the solid path λ has degree $(1, 0)$ and the dashed path μ has degree $(0, 1)$. Relation (1.3.2) with $n = (1, 0)$ says that $t_\lambda t_\lambda^* = t_w$, and with $n = (0, 1)$ it says that $t_\mu t_\mu^* = t_w$. On the other hand, both λ and μ belong to $\Lambda^{\leq(1,1)}$, so taking $n = (1, 1)$ in (1.3.2) gives $t_\lambda t_\lambda^* + t_\mu t_\mu^* = t_w$, and it follows that $C^*(\{t_\lambda : \lambda \in \Lambda\}) = \{0\}$. This 2-graph Λ is the prototypical example of a k-graph which is not locally convex, and it is k-graphs such as this which are ruled out in [**29**].

1.4. Non-row-finite higher-rank graphs

Fowler and Sims demonstrated in [**15**] that k-graphs can be viewed as product systems of directed graphs over the semigroup $\mathbb{N}^k$. Following the program of [**14**], Raeburn and Sims [**28**] studied the Toeplitz algebras of non-row-finite k-graphs using Fowler's work on the Toeplitz algebras of product systems of Hilbert bimodules [**11**]. Specifically, they showed how to extend to the higher-rank setting the construction of a Hilbert bimodule $X(E)$ from a graph E developed in [**14**]. Given a k-graph Λ, Raeburn and Sims show in [**28**] how to produce a product system $X(\Lambda)$ of Hilbert bimodules from Λ in such a way that when Λ is row-finite and has no sources the representation theory of $X(\Lambda)$ as studied in [**11**] matches up with the representation theory of Λ as studied in [**18**].

To discuss the outcomes of [**28**], we must briefly describe Fowler's results. Given a product system X of Hilbert bimodules, Fowler initially associates to it a Toeplitz algebra $\mathcal{T}_X$ and a Cuntz-Pimsner algebra $\mathcal{O}_X$. His primary interest is in the Toeplitz algebra, but to obtain a satisfactory uniqueness theorem, he turns his attention to the quotient $\mathcal{T}_{\text{cov}}(X)$ of $\mathcal{T}_X$ which is universal for what he calls *Nica covariant* Toeplitz representations of X. For representations of general product systems X of Hilbert bimodules, the definition of Nica covariance involves infinite sums of partial isometries which can only converge in the strong operator topology, so it makes no sense to talk about the universal C^*-algebra generated by such a representation. To avoid this problem, Fowler identifies the class of *compactly aligned* product systems for which the Nica covariance condition only involves finite sums.

To apply Fowler's results to the Toeplitz algebras of non-row-finite k-graphs in [**28**], Raeburn and Sims had to interpret the Nica covariance condition in terms of Cuntz-Krieger families, and decide for which k-graphs Λ the product system $X(\Lambda)$ is compactly aligned. For a k-graph Λ, Fowler's $\mathcal{T}_{\text{cov}}(X(\Lambda))$ is the universal algebra generated by a family of partial isometries $\{s_{\mathcal{T}}(\lambda) : \lambda \in \Lambda\}$ satisfying three relations. To state them, we need to introduce some handy notation from [**30**]. Given paths λ and μ in a k-graph Λ, Raeburn *et al.* define $\Lambda^{\min}(\lambda, \mu)$ to be the collection of pairs (α, β) which give rise to common extensions $\lambda\alpha = \mu\beta$ of λ and μ such that the degree of $\lambda\alpha$ is equal to the least upper bound $d(\lambda) \vee d(\mu)$ of $d(\lambda)$ and $d(\mu)$ in $\mathbb{N}^k$ [**30**, Definition 2.2]. The idea is that the notation $\Lambda^{\min}$ should suggest the notion of minimal common extensions in Λ. With this $\Lambda^{\min}$ notation in hand, we can state the Toeplitz-Cuntz-Krieger relations of [**28**]:

(TCK1) $\{s_{\mathcal{T}}(v) : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(TCK2) $s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu) = s_{\mathcal{T}}(\lambda\mu)$ whenever $s(\lambda) = r(\mu)$; and
(TCK3) $s_{\mathcal{T}}(\lambda)^* s_{\mathcal{T}}(\mu) = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$ for all $\lambda, \mu \in \Lambda$.

The first two of these relations are identical to relations (i) and (ii) from [**18**], but (TCK3) is new to [**28**], and is precisely the relation needed to ensure that the corresponding representation of $\mathcal{T}_{\text{cov}}(X(\Lambda))$ is Nica covariant. For row-finite k-graphs and for arbitrary directed graphs, (TCK3) is a consequence of the usual relations, but for non-row-finite k-graphs, it must be imposed separately. Moreover, for some non-row-finite k-graphs $\Lambda^{\min}(\lambda, \mu)$ is infinite for some pairs λ, μ.

For such k-graphs (TCK3) involves an infinite sum of partial isometries which cannot converge in the norm-topology. A k-graph Λ is said to be *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is always finite [**28**, Definition 5.3]. The finitely-aligned k-graphs are precisely those which correspond to compactly aligned product systems of Hilbert bimodules, and for these k-graphs (TCK3) is a C^*-algebraic condition. For a finitely aligned k-graph Λ, we call $\mathcal{T}_{\text{cov}}(X(\Lambda))$ the *Toeplitz algebra of* Λ, and denote it by $\mathcal{T}C^*(\Lambda)$.

The main result of [**28**] was a uniqueness theorem analogous to [**14**, Theorem 4.1] for the Toeplitz algebras of finitely aligned k-graphs. Theorem 8.1 of [**28**] says that given a finitely aligned k-graph Λ, a Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ generates an isomorphic copy of $\mathcal{T}C^*(\Lambda)$ if and only if

(1) t_v is nonzero for all $v \in \Lambda^0$; and
(2) $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*)$ is nonzero for all $v \in \Lambda^0$ and finite $E \subset r^{-1}(v) \setminus \{v\}$.

As did the analogous result for directed graphs, Theorem 8.1 of [**28**] indicates how to define the Cuntz-Krieger algebra of a finitely aligned — but not necessarily row-finite — k-graph Λ. The idea is to decide for which sets $E \subset r^{-1}(v)$ there exist Toeplitz-Cuntz-Krieger Λ-families $\{t_\lambda : \lambda \in \Lambda\}$ which satisfy (1), in which $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = 0$. The Cuntz-Krieger algebra $C^*(\Lambda)$ is then defined by introducing a relation which insists that $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = 0$ for all such E.

The idea suggested by [**28**, Theorem 8.1] was implemented by Raeburn, Sims and Yeend in [**30**]. The sets $E \subset r^{-1}(v)$ for which there exist Toeplitz-Cuntz-Krieger Λ-families satisfying (1) but in which $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = 0$ were identified in terms of the combinatorial structure of Λ, and were dubbed *finite exhaustive sets*. A subset E of $r^{-1}(v)$ is exhaustive if for every path μ with range v, there exists a path $\lambda \in E$ for which $\Lambda^{\min}(\lambda, \mu)$ is nonempty.

The appropriate version of (1.1.1) for finitely aligned k-graphs is then

(CK) $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = 0$ whenever $E \subset r^{-1}(v)$ is a finite exhaustive set.

Though this relation looks quite different to its predecessors, it was shown in [**30**, Appendix B] that the relations given in [**30**] are equivalent to those of [**29**] for locally convex row-finite k-graphs, and are equivalent to those of [**12**] for arbitrary directed graphs. Relation (CK) also has the great advantage that it associates Cuntz-Krieger families in which all the vertex projections are nonzero even to non-locally-convex k-graphs. To see why, notice that in the pathological non-locally-convex k-graph (1.3.3), the only finite exhaustive subset of $r^{-1}(w)$ is the set $\{\lambda, \mu\}$, so there is no relation forcing either $t_\lambda t_\lambda^* = t_w$ or $t_\mu t_\mu^* = w$. Hence the C^*-algebra associated to Λ in [**30**] does not collapse as did the one in [**29**].

Using relation (CK), Raeburn, Sims and Yeend proved versions of the gauge-invariant uniqueness theorem [**30**, Theorem 4.2] and the Cuntz-Krieger uniqueness theorem [**30**, Theorem 4.5] for finitely aligned k-graphs. However, the problem of determining the ideal structure of $C^*(\Lambda)$ when Λ is not row-finite presents further difficulties, and was not addressed in [**30**].

1.5. Main results of this thesis

The object of this thesis is to describe the gauge-invariant ideal structure of the C^*-algebras of finitely aligned k-graphs. Unfortunately it is not clear how to generalise to non-row-finite k-graphs constructions such as the adding a tail construction of [**3**], the desingularisation of [**7**], the quotient graph construction of [**2**], and the construction of E_V in [**23**]. The problem is that whereas adding an edge to a directed graph is a local operation which does not affect the structure of the graph at a distance, adding an edge to a k-graph can have a global effect because of the factorisation property.

The main innovation in this thesis is to study the quotients of a Cuntz-Krieger algebra by first studying relative Cuntz-Krieger algebras associated to higher-rank graphs. Given a k-graph Λ and a collection $\mathcal{E}$ of finite exhaustive sets in Λ, the relative Cuntz-Krieger algebra $C^*(\Lambda; \mathcal{E})$ is the universal C^*-algebra generated by a family $\{s_\mathcal{E}(\lambda) : \lambda \in \Lambda\}$ which satisfy (TCK1)–(TCK3), and which satisfy (CK) at all finite exhaustive sets which belong to $\mathcal{E}$.

The first of the main results of the thesis is Theorem 3.5.8, which gives elementary criteria for deciding, given any two families $\{t_\lambda : \lambda \in \Lambda\}$ and $\{t'_\lambda : \lambda \in \Lambda\}$ which satisfy (TCK1)–(TCK3), whether the images of the fixed point algebra for the gauge action in $\mathcal{T}C^*(\Lambda)$ under the associated homomorphisms π_t and $\pi_{t'}$ are isomorphic. This result is fairly straightforward: we prove it by modifying arguments developed in joint work with Raeburn [**28**] and with Raeburn and Yeend [**30**].

The next main achievement of this thesis is the development of versions of the gauge-invariant uniqueness theorem (Theorem 4.3.12) and the Cuntz-Krieger uniqueness theorem (Theorem 4.5.2) for relative Cuntz-Krieger algebras. Since relative Cuntz-Krieger algebras are entirely new to this thesis, we must develop new techniques to formulate and prove these uniqueness theorems. The general idea is to use Theorem 3.5.8, but the key question which must be answered first is:

at precisely which finite exhaustive sets E will relation (CK) hold in a particular relative Cuntz-Krieger algebra? The *satiation* $\overline{\mathcal{E}}$ of a collection $\mathcal{E}$ of finite exhaustive sets is the collection of all finite exhaustive sets at which (CK) holds in the relative Cuntz-Krieger algebra $C^*(\Lambda; \mathcal{E})$. One of our key technical achievements is a graph-theoretic characterisation of $\overline{\mathcal{E}}$ (see Section 4.2 and Corollary 4.3.13).

Our last two main results, Theorem 5.3.8 and Theorem 5.4.2, achieve the overall objective of the thesis. In Theorem 5.3.8, we use our gauge-invariant uniqueness theorem to give a complete graph-theoretic description of the lattice of gauge-invariant ideals in $C^*(\Lambda)$. In Theorem 5.4.2, we use our Cuntz-Krieger uniqueness theorem to produce a structural condition on Λ under which every ideal in $C^*(\Lambda)$ is gauge-invariant.

1.6. Overview of the thesis

Chapter 2. In this chapter, we provide a detailed discussion of k-graphs and their properties. Here we present the basic definitions and notation which we will use to discuss k-graphs later in the thesis. We indicate how to picture k-graphs in terms of their *1-skeletons*, and illustrate our key definitions using examples described in terms of these 1-skeletons.

Chapter 3. Here we prove a C^*-algebraic uniqueness theorem for the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ of a k-graph Λ (Theorem 3.1.6). We proved a version of Theorem 3.1.6 for product systems over general quasi-lattice ordered semigroups P in joint work with Raeburn in [**28**, Theorem 8.1]. Taking $P = \mathbb{N}^k$ gives an example of a quasi-lattice ordered semigroup, and product systems of graphs over $\mathbb{N}^k$ are k-graphs [**15**, Example 1.3(iv)]. When $P = \mathbb{N}^k$, Theorem 3.1.6 and [**28**, Theorem 8.1] are equivalent. We give a new proof of Theorem 3.1.6 here rather than appealing to [**28**, Theorem 8.1]. To explain why, it is necessary to describe first, in broad terms, the difference between the two proofs.

The proof of [**28**, Theorem 8.1] appeals to Fowler's spatial faithfulness theorem for representations of product systems of bimodules [**11**, Theorem 7.1] to obtain a faithful conditional expectation onto the *diagonal* subalgebra of the Toeplitz algebra. The diagonal is the closed linear span of the range projections $s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\lambda)^*$ where λ ranges over all paths in the k-graph. The proof of [**28**, Theorem 8.1] then proceeds by identifying finite-dimensional subalgebras of the diagonal, and using these to characterise the representations of the Toeplitz algebra which are faithful on the diagonal. A calculation-intensive technical argument shows that given any representation π which is faithful on the diagonal, the expectation onto the diagonal in $\mathcal{T}C^*(\Lambda)$ is implemented by a well-defined linear map in $\pi(\mathcal{T}C^*(\Lambda))$. A standard argument (see, for example, [**30**, Proposition 4.1]) combines these ingredients to prove [**28**, Theorem 8.1].

The proof of Theorem 3.1.6 given in this thesis, on the other hand, proceeds by analysing the structure of the *core*, rather than the diagonal, in representations of $\mathcal{T}C^*(\Lambda)$. The core is the closed linear span of the partial isometries $s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*$ such that $d(\lambda) = d(\mu)$. It is a standard consequence of the universal property of $\mathcal{T}C^*(\Lambda)$ that there is a *gauge action* of $\mathbb{T}^k$ on the universal algebra, and that the core is the fixed-point algebra for this action. The usual procedure for producing a uniqueness theorem is to exploit the faithful conditional expectation onto the core obtained by averaging over the gauge action. By showing that the core is AF, we obtain elementary conditions under which a homomorphism $\pi : \mathcal{T}C^*(\Lambda) \to A$ is injective on the core. Calculations like those for the diagonal in [**28**] then show that whenever π is faithful on the core, the expectation onto the core in $\mathcal{T}C^*(\Lambda)$ is implemented by a well-defined linear map in $\pi(\mathcal{T}C^*(\Lambda))$. The argument of [**30**, Proposition 4.1] then proves Theorem 3.1.6.

The reason for providing a new proof of Theorem 3.1.6 rather than appealing to [**28**, Theorem 8.1], then, is that our technique explicitly identifies finite-dimensional subalgebras of the core along with families of matrix units for them. We prove that the core is AF by giving formulas in terms of matrix units for the inclusion maps of nested finite-dimensional algebras. The existence of the gauge action, the existence of a faithful conditional expectation onto the core, and the AF structure of the core all descend from $\mathcal{T}C^*(\Lambda)$ to the relative Cuntz-Krieger algebras studied later, whereas the methods used to obtain a faithful conditional expectation onto the diagonal in $\mathcal{T}C^*(\Lambda)$ do not carry over easily to relative Cuntz-Krieger algebras. In particular our technique for proving Theorem 3.1.6 provides us with Theorem 3.5.8 which is one of the main tools we use to prove our uniqueness theorems for relative Cuntz-Krieger algebras later on. Theorem 3.5.8 gives us elementary criteria for deciding whether the images of the core of $\mathcal{T}C^*(\Lambda)$ under two homomorphisms π and π' are isomorphic. Thus, for the purposes of this thesis, the proof of Theorem 3.1.6 given here is superior to that of [**28**, Theorem 8.1].

Chapter 4. In Chapter 4, we investigate the relative Cuntz-Krieger algebras associated to a k-graph Λ, with the intention of paralleling the theory of relative Cuntz-Pimsner algebras for Hilbert bimodules (see [**22**]), and of the quotients of Cuntz-Pimsner algebras by gauge-invariant ideals (see [**13**]). The relative Cuntz-Krieger algebras associated to a k-graph Λ are each determined by a collection

$\mathcal{E} \subset \mathrm{FE}(\Lambda)$, where $\mathrm{FE}(\Lambda)$ denotes the collection of all finite exhaustive sets in Λ. Given such a collection $\mathcal{E}$, we call a collection of partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ which satisfy (TCK1)–(TCK3) and satisfy (CK) at all finite exhaustive sets E which belong to $\mathcal{E}$ a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. The relative Cuntz-Krieger algebra $C^*(\Lambda; \mathcal{E})$ is the universal C^*-algebra generated by a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family; we denote the universal generating Cuntz-Krieger $(\Lambda; \mathcal{E})$-family in $C^*(\Lambda; \mathcal{E})$ by $\{s_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$.

The universal property of $C^*(\Lambda; \mathcal{E})$ guarantees the existence of a gauge action of $\mathbb{T}^k$, and averaging over this action gives an expectation onto the core. To analyse the core, we apply Theorem 3.5.8 to compare the image of the core of $\mathcal{T}C^*(\Lambda)$ under the canonical homomorphism $\pi^{\mathcal{T}}_{s_{\mathcal{E}}}$ from $\mathcal{T}C^*(\Lambda)$ to $C^*(\Lambda; \mathcal{E})$ with the image of the core under the homomorphism $\pi^{\mathcal{T}}_t$ determined by any other relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. To do this, we need to: (1) show that all the vertex projections are nonzero in $C^*(\Lambda; \mathcal{E})$; and (2) decide which gap projections $\prod_{\lambda \in E} \big(s_{\mathcal{E}}(v) - s_{\mathcal{E}}(\lambda) s_{\mathcal{E}}(\lambda)^*\big)$ are nonzero in $C^*(\Lambda; \mathcal{E})$.

To achieve (1) and (2), we identify the *satiation* $\overline{\mathcal{E}}$ of $\mathcal{E}$ (see Definition 4.2.3 and Lemma 4.2.13). The satiation of $\mathcal{E}$ is a collection of finite exhaustive sets which contains $\mathcal{E}$ and has the property that if $\{t_\lambda : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family and $E \subset r^{-1}(v)$ belongs to $\overline{\mathcal{E}}$, then $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = 0$.

We use $\overline{\mathcal{E}}$ to adapt the infinite path space representation (see for example [**18**, Proposition 2.11] or [**30**, Proposition 2.12]) to give a concrete representation $\pi_{S_{\mathcal{E}}}$ of $C^*(\Lambda; \mathcal{E})$, called the *$\mathcal{E}$-relative boundary-path space representation*. The point is that we can check directly that for a finite exhaustive set $E \subset r^{-1}(v)$ which does not belong to $\overline{\mathcal{E}}$, we have $\pi_{S_{\mathcal{E}}}\big(\prod_{\lambda \in E}(s_{\mathcal{E}}(v) - s_{\mathcal{E}}(\lambda) s_{\mathcal{E}}(\lambda)^*)\big) \neq 0$ (see Lemma 4.3.9). We conclude from this that (CK) holds in $C^*(\Lambda; \mathcal{E})$ at a finite exhaustive set E if and only if E belongs to $\overline{\mathcal{E}}$, and obtain from this result a necessary and sufficient condition for a homomorphism of $C^*(\Lambda; \mathcal{E})$ to be injective on the core (Corollary 4.3.13).

Corollary 4.3.13 is the first ingredient needed to obtain a uniqueness theorem for $C^*(\Lambda; \mathcal{E})$. The other ingredient required is a condition under which, given a homomorphism $\pi : C^*(\Lambda; \mathcal{E}) \to A$, the expectation onto the core in $C^*(\Lambda; \mathcal{E})$ is implemented by a well-defined linear map on A. For a gauge-invariant uniqueness theorem, the condition imposed is that there exists an action θ of $\mathbb{T}^k$ on A such that π is equivariant in the gauge action and θ; averaging over θ gives the required linear map. Hence our gauge-invariant uniqueness theorem, Theorem 4.3.12 says that if π satisfies the hypotheses of Corollary 4.3.13, and if there is an action θ of $\mathbb{T}^k$ on A such that $\theta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}^k$, then π is faithful.

When $\mathcal{E}$ is empty, so is $\overline{\mathcal{E}}$, so (CK) is imposed nowhere. In this case we recover the Toeplitz algebra as $C^*(\mathcal{E}; \emptyset)$. When $\mathcal{E}$ is all of $\mathrm{FE}(\Lambda)$, we obtain the Cuntz-Krieger algebra $C^*(\Lambda)$ of [**30**] as $C^*(\Lambda; \mathrm{FE}(\Lambda))$. In this case, a representation satisfies the hypotheses of Corollary 4.3.13 if and only if it is nonzero on all the vertex projections. Hence we recover the gauge-invariant uniqueness theorem for $C^*(\Lambda)$ [**30**, Corollary 4.3] as a special case of Theorem 4.3.12.

In Section 4.5, we describe an aperiodicity-style condition which we denote Condition (C). Condition (C) is analogous to the aperiodicity Condition (B) of [**30**]. We prove a version of the Cuntz-Krieger uniqueness theorem, Theorem 4.5.2 which says that if the pair $(\Lambda, \mathcal{E})$ satisfies Condition (C), then a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family $\{t_\lambda : \lambda \in \Lambda\}$ generates an isomorphic copy of $C^*(\Lambda; \mathcal{E})$ if and only if it satisfies the hypotheses of Corollary 4.3.13. When $\mathcal{E} = \emptyset$ so that $C^*(\Lambda; \mathcal{E}) = C^*(\Lambda)$, Condition (C) is formally weaker than [**30**, Condition (B)] so that our Cuntz-Krieger uniqueness theorem is formally stronger than [**30**, Theorem 4.5]. However, we do not have any examples of k-graphs Λ for which Theorem 4.6.5 is applicable but [**30**, Theorem 4.5] is not, and it seems likely that the two theorems are in fact equivalent.

In Sections 4.7 and 4.8, we use the gauge-invariant uniqueness theorem to show how to obtain a homomorphism which is injective on all of $C^*(\Lambda; \mathcal{E})$ from any homomorphism which is injective on the core by *augmenting* the representation. In particular, augmenting the $\mathcal{E}$-relative boundary-path space representation produces a faithful representation of any $C^*(\Lambda; \mathcal{E})$ on Hilbert space. The effect of this augmentation is trivial on the core.

Chapter 5. In Chapter 5, we consider ideals and quotients of $C^*(\Lambda)$. We define a notion of a saturated hereditary set which ensures that if $H \subset \Lambda^0$ is saturated and hereditary and I_H is the ideal in $C^*(\Lambda)$ generated by $\{t_v : v \in H\}$, then the set of vertices $w \in \Lambda^0$ such that $t_w \in I_H$ is precisely H. Hence the ideals I_H associated to saturated hereditary sets $H \subset \Lambda^0$ are distinct ideals of $C^*(\Lambda)$. We show that the collection $r^{-1}(H) := \{\lambda \in \Lambda : r(\lambda) \in H\}$ is a sub-k-graph of Λ and that I_H contains $C^*(r^{-1}(H))$ as a full corner.

We then use our gauge-invariant uniqueness theorem for $C^*(\Lambda; \mathcal{E})$ to prove that the quotient algebra $C^*(\Lambda)/I_H$ is canonically isomorphic to the relative Cuntz-Krieger algebra $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H)$ where $\Lambda \setminus \Lambda H$ is the subgraph of Λ consisting of paths whose source does not belong to H, and $\mathcal{E}_H$ is an appropriate subset of

$\mathrm{FE}(\Lambda \setminus \Lambda H)$ (see Definition 5.2.2 for details). We show that collections B of finite exhaustive sets in $r^{-1}(\Lambda^0 \setminus H) \setminus \mathcal{E}_H$ for which $\mathcal{E}_H \cup B$ is its own satiation (that is, $\overline{\mathcal{E}_H \cup B} = \mathcal{E}_H \cup B$) are in bijective correspondence with the gauge-invariant ideals in $C^*(\Lambda)/I_H$ which contain no vertex projections. Such collections B therefore give rise to gauge-invariant ideals $J_{H,B}$ in $C^*(\Lambda)$ such that $I_H \subset J_{H,B}$. We describe a bijection between pairs (H, B) as above and gauge-invariant ideals of $C^*(\Lambda)$, and describe the partial order $\preceq$ on pairs (H, B) which corresponds to the inclusion relation on gauge-invariant ideals.

Using Condition (C), we give a condition under which every ideal in $C^*(\Lambda)$ is gauge-invariant. We say that a k-graph Λ satisfies Condition (D) if for any saturated hereditary set $H \subset \Lambda^0$ and any set $B \subset \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$ for which $\mathcal{E}_H \cup B$ is its own satiation, the pair $(\Lambda \setminus \Lambda H, \mathcal{E}_H)$ satisfies Condition (C). We show that if Λ satisfies Condition (D), then every ideal of $C^*(\Lambda)$ is gauge-invariant.

Chapter 6. In Chapter 6, we investigate conditions on a k-graph Λ under which $C^*(\Lambda)$ is simple, purely infinite and nuclear. We show that all the relative Cuntz-Krieger algebras studied in this thesis are nuclear C^*-algebras. We show that if Λ satisfies Condition (C) then $C^*(\Lambda)$ is simple if and only if Λ is *cofinal*. Finally, we show that if Λ satisfies Condition (C) and every vertex of Λ can be reached from a *loop with an entrance* in Λ, then every hereditary subalgebra of $C^*(\Lambda)$ contains an infinite projection; when $C^*(\Lambda)$ is simple, this is precisely the definition of pure infinity.

The nuclearity of $C^*(\Lambda; \mathcal{E})$ follows immediately from a general result of Quigg on coactions. To prove our simplicity and pure infinity results, however, we draw heavily upon the corresponding arguments from [**3**] for row-finite directed graphs. The technical difficulties involved in generalising these arguments to finitely-aligned k-graphs are substantial. However, the description of the core of $C^*(\Lambda)$ developed in [**30**] allows us to overcome these technical difficulties very neatly, and the proofs of [**3**] end up carrying over quite smoothly to our setting.

1.7. Connections with other work

The analysis of the algebras $M^t_{\Pi E}$ in Sections 3.4 and 3.5 is adapted directly from joint work with Raeburn and Yeend in [**30**, Section 3]; the formulation of the Cuntz-Krieger relation (CK) is a result of the same joint work, while Section 4.9 is taken more or less verbatim from [**30**, Appendix A]. Additionally, Lemma 4.2.1 and Lemma 4.2.7 both appear in Appendix C of the same paper ([**30**, Lemma C.6] and [**30**, Lemma C.4] respectively).

The definition of a Toeplitz-Cuntz-Krieger family (Definition 3.1.1), the analysis of the diagonal in Section 3.3, and the material in Section 3.7 on the path-space representation all come from joint work with Raeburn in [**28**]. Theorem 3.1.6 is a special case of [**28**, Theorem 8.1], though our proof is different from the one given there.

I would like to make special mention of Trent Yeend's contributions to the proof of the technical Lemma 4.3.9, which is crucial to the main results of Chapters 4 and 5. Firstly, I am very grateful to Trent for providing me with a preprint of [**10**]. The technique of employing a diagonal listing of $\{s(\lambda_i)\overline{\mathcal{E}} : i \in \mathbb{N}\}$ used in the constructions of $\mathcal{E}$-relative boundary paths for both parts of Lemma 4.3.9 is due to Trent and can be found in [**10**, Lemma 3.9]. Additionally, [**10**, Lemma 1.5] is a key ingredient in the proof of statement (2) of Lemma 4.3.9. Secondly, I thank Trent for his insightful input into many discussions along the road to a proof of statement (2) of Lemma 4.3.9.

CHAPTER 2

Basic definitions

In this chapter, we define k-graphs, establish notation for dealing with them, and spend some time investigating their structure and a variety of their combinatorial properties. We begin with a few preliminary concepts.

By a *graph*, we mean a quadruple (E^0, E^1, r, s), where E^0 is the countable vertex set and E^1 is the countable edge set, and where $r, s : E^1 \to E^0$ are the range and source maps which give the edges their direction. Pictorially, E^0 is a collection of dots or vertices, E^1 is a collection of arrows joining the dots, and r, s indicate the direction of these arrows, so that the arrow $e \in E^1$ points from $s(e)$ to $r(e)$.

We regard $\mathbb{N}^k$ as a monoid under addition, with additive identity denoted 0. We denote the standard generators of $\mathbb{N}^k$ by $e_1, e_2, \dots, e_k$ (the same symbols are sometimes used to denote sequences of edges in a directed graph, but it is always clear from context which meaning they have in any given situation). We write $\leq$ for the partial order on $\mathbb{N}^k$ given by $m \leq n$ if and only if $n - m \in \mathbb{N}^k$, and we write $m < n$ when $n - m \in \mathbb{N}^k \setminus \{0\}$. For $n \in \mathbb{N}^k$, we write n_i for the i^{th} coordinate of n; that is, $n = \sum_{i=1}^k n_i e_i$. For $m, n \in \mathbb{N}^k$, we write $m \vee n$ for their coordinate-wise maximum, and $m \wedge n$ for their coordinate-wise minimum; so $(m \vee n)_i = \max\{m_i, n_i\}$, and $(m \wedge n)_i = \min\{m_i, n_i\}$. For $n \in \mathbb{N}^k$, we write $|n|$ for the *length* $\sum_{i=1}^k n_i$ of n.

2.1. Basics of categories

The notion of a k-graph is best formulated in terms of category theory, so we establish the basic notation before proceeding. The following definitions are taken from Chapter 1 of [**21**].

DEFINITION 2.1.1. A *category* $\mathcal{C}$ is a sextuplet $(\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}), \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ consisting of

- the *object* and *morphism* sets $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$,
- the *domain* and *codomain* functions $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$,
- the *identity* function $\mathrm{id} : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$, and

- the *composition* function $\circ : \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$, where $\mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) = \{(g, f) \in \mathrm{Mor}(\mathcal{C})^2 : \mathrm{dom}(g) = \mathrm{cod}(f)\}$ is the collection of all *composable pairs* in $\mathrm{Mor}(\mathcal{C})$,

which satisfies

- $\mathrm{dom}(\mathrm{id}(a)) = a = \mathrm{cod}(\mathrm{id}(a))$ for all $a \in \mathrm{Obj}(\mathcal{C})$,
- $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ and $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$ for all pairs $(f, g) \in \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C})$,
- the *unit law*: $\mathrm{id}(\mathrm{cod}(f)) \circ f = f = f \circ \mathrm{id}(\mathrm{dom}(f))$ for all $f \in \mathrm{Mor}(\mathcal{C})$,
- *associativity*: $(f \circ g) \circ h = f \circ (g \circ h)$.

We say that $\mathcal{C}$ is *countable* if $\mathrm{Mor}(\mathcal{C})$ is countable.

One regards the morphisms $\mathrm{Mor}(\mathcal{C})$ as arrows connecting objects, and composition as concatenation of arrows, read right to left. In practise, we will drop the $\circ$ in compositions, and write fg for $f \circ g$.

Example 2.1.2. We can think of a monoid $(S, e, \cdot)$ as (the morphisms of) a category $\mathcal{C}$ with a single object o; that is $\mathrm{Obj}(\mathcal{C}) = \{o\}$, $\mathrm{Mor}(\mathcal{C}) = S$, $\mathrm{dom}(s) = \mathrm{cod}(s) = o$ for all $s \in S$, $\mathrm{id}(o) = e$, and $s \circ t = s \cdot t$.

Definition 2.1.3. A *covariant functor* T from a category $\mathcal{C}$ to a category $\mathcal{B}$ is a pair of functions (both denoted T): an *object function* $T : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{B})$ and a morphism function $T : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{B})$ satisfying

- $\mathrm{dom}(T(f)) = T(\mathrm{dom}(f))$ and $\mathrm{cod}(T(f)) = T(\mathrm{cod}(f))$ for all $f \in \mathrm{Mor}(\mathcal{C})$,
- $T(\mathrm{id}(o)) = \mathrm{id}(T(o))$ for all $o \in \mathrm{Obj}(\mathcal{C})$,
- $T(f) \circ T(g) = T(f \circ g)$ for all $(f, g) \in \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C})$.

Remark 2.1.4. One can also define *contravariant functors* between categories which reverse the direction of composition (see [**21**, §II.2]). However for the purposes of this thesis it will suffice to consider only covariant functors, and from this point forward, we will refer to covariant functors simply as *functors*.

Informally, a functor between categories is a map from the arrows in one category to the arrows in the other which preserves connectivity. The object map is redundant since it is determined by the morphism map restricted to the identity morphisms.

EXAMPLE 2.1.5. Let $E = (E^0, E^1, r, s)$ be a directed graph, and let E^* denote the collection of all *paths* in E; that is

$$E^* = E^0 \cup \{e_1 e_2 \cdots e_n : n \in \mathbb{N}, e_i \in E^1, r(e_{i+1}) = s(e_i) \text{ for all } i \le n-1\}.$$

Define $\mathrm{dom}, \mathrm{cod} : E^* \to E^0$ as follows. For $e_1 \cdots e_n \in E^* \setminus E^0$, let $\mathrm{dom}(e_1 \cdots e_n) := s(e_n)$ and let $\mathrm{cod}(e_1 \cdots e_n) := r(e_1)$. For $v \in E^0 \subset E^*$, let $\mathrm{cod}(v) = \mathrm{dom}(v) = v$. Define $\mathrm{id} : E^0 \to E^*$ to be the inclusion map $v \mapsto v$. Finally, define $\circ : E^* \times_{E^0} E^* \to E^*$ as follows: for $(e_1 \cdots e_n, f_1 \cdots f_m) \in E^* \times_{E^0} E^*$, and $v = s(e_n), w = r(e_1) \in E^0 \subset E^*$, define

$$e_1 \cdots e_n \circ f_1 \cdots f_m := e_1 \cdots e_n f_1 \cdots f_m, \quad \text{and}$$
$$w \circ e_1 \cdots e_n := e_1 \cdots e_n =: e_1 \cdots e_n \circ v.$$

Then $E^* := (E^0, E^*, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ is a category, called the *free category generated by* E. There is a functor l, called the *length functor*, from E^* to the semigroup $(\mathbb{N}, +)$ regarded as a category as in Example 2.1.2 which satisfies $l(v) = 0$ for $v \in E^0 \subset E^*$, and $l(e_1 \cdots e_n) = n$ for $e_1 \cdots e_n \in E^*$.

2.2. Higher-rank graphs: basic notation

DEFINITION 2.2.1. Let $k \in \mathbb{N} \setminus \{0\}$. A k-graph is a pair (Λ, d) where Λ is a countable category and d is a functor from Λ to $\mathbb{N}^k$ (regarding $\mathbb{N}^k$ as a category as in Example 2.1.2) which satisfies the *factorisation property*:

> For all $\lambda \in \mathrm{Mor}(\Lambda)$ and all $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exist unique morphisms μ and ν in $\mathrm{Mor}(\Lambda)$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

See Section 2.3 for more details about the structure of k-graphs. If $\lambda \in \mathrm{Mor}(\Lambda)$ with $d(\lambda) = l$, and $0 \le m \le n \le l$, then two applications of the factorisation property ensure that there exist unique elements $\lambda', \lambda'', \lambda''' \in \mathrm{Mor}(\Lambda)$ such that $d(\lambda') = m$, $d(\lambda'') = n - m$ and $d(\lambda''') = l - n$ and such that $\lambda = \lambda'\lambda''\lambda'''$. We denote λ'' by $\lambda(m, n)$ (hence λ' is denoted $\lambda(0, m)$ and λ''' is denoted $\lambda(n, l)$).

If $m, n \in \mathbb{N}^k$ are not less than or equal to $d(\lambda)$, then $\lambda(m, n) := \lambda(m \wedge d(\lambda), n \wedge d(\lambda))$. Since we are regarding k-graphs as generalised graphs, we refer to elements of $\mathrm{Mor}(\Lambda)$ as *paths* and we write r and s for the codomain and domain maps.

EXAMPLE 2.2.2. A 1-graph (Λ, d) (that is, $k = 1$) is just the free category E^* generated by the directed graph

$$(E^0 := \mathrm{Obj}(\Lambda), E^1 := d^{-1}(1), r := \mathrm{cod}|_{E^1}, s := \mathrm{dom}|_{E^1}).$$

In this context the factorisation property is automatic: it boils down to the statement that given $\lambda = e_1 \dots e_n \in E^*$, the sequence $(e_1, e_2, \dots, e_n)$ is the unique sequence of edges in E^1 such that $\lambda = e_1 \cdots e_n$. Under the notation just established, we have $e_i = \lambda(i-1, i)$ for all $1 \le i \le n$.

EXAMPLE 2.2.3. Let $k \in \mathbb{N} \setminus \{0\}$, and $m \in (\mathbb{N} \cup \{\infty\})^k$. Define

$$\begin{gathered}\operatorname{Obj}(\Omega_{k,m}) := \{n \in \mathbb{N}^k : n_i \le m_i \text{ for all } i\} \\ \operatorname{Mor}(\Omega_{k,m}) := \{(n_1, n_2) \in \operatorname{Obj}(\Omega_{k,m}) \times \operatorname{Obj}(\Omega_{k,m}) : n_1 \le n_2\} \\ \operatorname{cod}((n_1, n_2)) := n_1 \qquad \operatorname{dom}((n_1, n_2)) := n_2.\end{gathered}$$

For $n_1 \le n_2 \le n_3 \in \operatorname{Obj}(\Omega_{k,m})$, we define $\operatorname{id}(n_1) := (n_1, n_1)$ and $(n_1, n_2) \circ (n_2, n_3) := (n_1, n_3)$. Then $\Omega_{k,m}$ is a countable category. If we further define $d((n_1, n_2)) := (n_2 - n_1)$, then $(\Omega_{k,m}, d)$ becomes a k-graph.

Let (Λ, d) be a k-graph. Since the degree map d is a functor, and since $\mathbb{N}^k$ has just one object o, we have $d(\operatorname{id}(v)) = \operatorname{id}(d(v)) = \operatorname{id}(o) = 0$ for every object $v \in \operatorname{Obj}(\Lambda)$. Fix $v \in \operatorname{Obj}(\Lambda)$, and suppose that $\lambda \ne \operatorname{id}(v)$ is a morphism of degree 0 with range v. Since $0 = 0 + 0$, the factorisation property then insists that there exist unique morphisms $\mu, \nu \in \operatorname{Mor}(\Lambda)$ such that $d(\mu) = d(\nu) = 0$ and $\mu\nu = \lambda$. Both $\mu = \operatorname{id}(v), \nu = \lambda$ and $\mu = \lambda, \nu = \operatorname{id}(\operatorname{dom}(\lambda))$ provide such factorisations, so the uniqueness of factorisations ensures that $\lambda = \operatorname{id}(v)$; that is $\{\lambda \in \operatorname{Mor}(\Lambda) : d(\lambda) = 0\} = \{\operatorname{id}(v) : v \in \operatorname{Obj}(\Lambda)\}$. Consequently, we will identify $\operatorname{Obj}(\Lambda)$ with $\{\lambda \in \operatorname{Mor}(\Lambda) : d(\lambda) = 0\}$. Since we are then able to regard $\operatorname{Obj}(\Lambda)$ as a subset of $\operatorname{Mor}(\Lambda)$, we will think of Λ as consisting entirely of its morphisms, and will henceforth write $\lambda \in \Lambda$ in place of $\lambda \in \operatorname{Mor}(\Lambda)$.

If $\lambda \in \Lambda$ and $n \le d(\lambda)$ then $\lambda(n, n) = s(\lambda(0, n)) = (\lambda(n, d(\lambda))$. We will abbreviate this to $\lambda(n) := \lambda(n, n)$.

In analogy with the path-space notation for 1-graphs, we will write

$$\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\};$$

hence $\Lambda^0 = \{\lambda \in \Lambda : d(\lambda) = 0\}$, and we regard this as the set of vertices of Λ under the identification of $\operatorname{Obj}(\Lambda)$ with Λ^0 described above. Abusing notation slightly, given $\lambda \in \Lambda$ and $E \subset \Lambda$, we write λE for the set $\{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}$ and $E\lambda$ for the set $\{\mu\lambda : \mu \in E, s(\mu) = r(\lambda)\}$. In particular if $v \in \Lambda^0$, then $vE = \{\lambda \in E : r(\lambda) = v\}$ and $Ev = \{\lambda \in E : s(\lambda) = v\}$.

We will generally use lower-case greek letters $(\lambda, \mu, \nu, \sigma, \tau, \dots)$ for paths in $\Lambda \setminus \Lambda^0$, although we reserve δ for the Kronecker delta, and we reserve γ for the gauge action. We use english letters $(u, v, w, \dots)$ for elements of Λ^0.

2.3. Higher-rank graphs from 1-skeletons

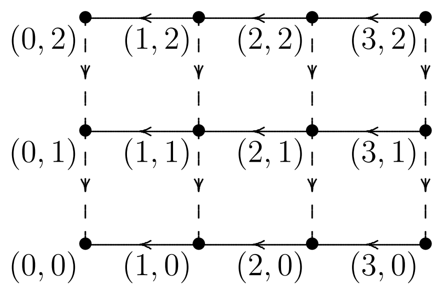
When studying k-graphs, and particularly when describing examples, it is helpful to think in terms of 1-skeletons as in [**30**, Section 2]. Let (Λ, d) be a k-graph, and for each pair of distinct generators e_i, e_j of $\mathbb{N}^k$, define a bijection $T_{i,j} : \Lambda^{e_i} \times_{\Lambda^0} \Lambda^{e_j} \to \Lambda^{e_j} \times_{\Lambda^0} \Lambda^{e_i}$ as follows: for $(\lambda, \mu) \in \Lambda^{e_i} \times_{\Lambda^0} \Lambda^{e_j}$, we define $T_{i,j}(\lambda, \mu)$ to be the unique pair $(\sigma, \tau) \in \Lambda^{e_j} \times_{\Lambda^0} \Lambda^{e_i}$ provided by the factorisation property such that $\lambda\mu = \sigma\tau$. Using the notation established in the previous section, we can rewrite this as

$$T_{i,j}(\lambda, \mu) := ((\lambda\mu)(0, e_j), (\lambda\mu)(e_j, e_i + e_j)).$$

Example 1.5(iv), Theorem 2.1 and Theorem 2.2 of [**15**] combine to show that every k-graph Λ is completely determined by its *coordinate graphs* $\{(\Lambda^0, \Lambda^{e_i}, r, s) : 1 \leq i \leq k\}$, and the bijections $T_{i,j} : \Lambda^{e_i} \times_{\Lambda^0} \Lambda^{e_j} \to \Lambda^{e_j} \times_{\Lambda^0} \Lambda^{e_i}$ described above.

We picture this by drawing the 1-*skeleton* of Λ, which is the graph with vertex set Λ^0 and edges $\bigcup_{i=1}^{k} \Lambda^{e_i}$, with edges of different degrees distinguished using k different colours. In the pictures here, solid edges have degree e_1, dashed edges have degree e_2, and dotted edges have degree e_3. We will not need any examples with $k > 3$ at this stage; as we will soon discover, the 1-skeletons of 3-graphs are basically as complicated as the situation ever gets.

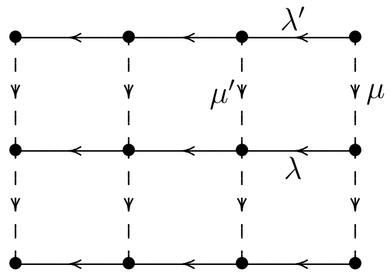
EXAMPLE 2.3.1. The 1-skeleton of the 2-graph $\Omega_{2,(3,2)}$ is



Furthermore, $\Omega_{2,(3,2)}$ is the unique 2-graph with this 1-skeleton since there is a unique bijection $T_{1,2} : \Lambda^{e_1} \times_{E^0} \Lambda^{e_2} \to \Lambda^{e_2} \times_{E^0} \Lambda^{e_1}$ that preserves ranges and sources.

It is not generally the case that a k-graph is completely determined by its 1-skeleton. Some 1-skeletons do not correspond to any k-graph, while other 1-skeletons correspond to many different k-graphs. The missing information is the factorisation property which is specified in terms of *bi-coloured paths* and *bi-coloured squares*.

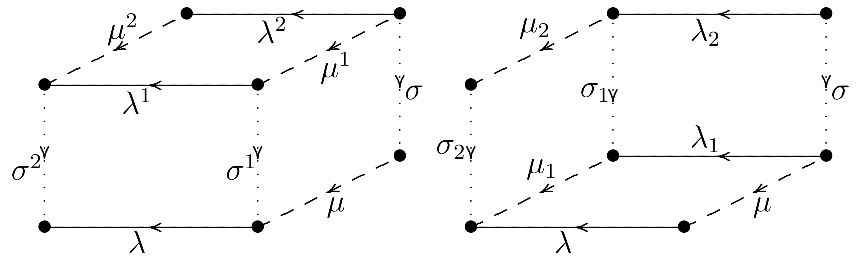
The bi-coloured paths in a 1-skeleton are pairs (λ, μ) of edges of different colours such that $s(\lambda) = r(\mu)$. If $d(\lambda) = e_i$ and $d(\mu) = e_j$, then the pair (λ, μ) corresponds to the path $\lambda\mu$ of degree $e_i + e_j$, and the factorisation property requires that there exist a unique pair $(\mu', \lambda') \in \Lambda^{e_j} \times_{\Lambda^0} \Lambda^{e_i}$ such that $\mu'\lambda' = \lambda\mu$. We write (λ, μ) for the bi-coloured path rather than writing $\lambda\mu$ because (λ, μ) and (μ', λ') are distinct bi-coloured paths in the 1-skeleton of Λ even though they correspond to the same path $\lambda\mu = \mu'\lambda' \in \Lambda$. Pictorially, the pairing of (λ, μ) with (μ', λ') corresponds to a commuting diagram in Λ called a *bi-coloured square* like the square $\{(e, g), (f, h)\}$ labelled in the 1-skeleton of $\Omega_{2,(3,2)}$ shown below:



The factorisation property says that the collection S of bi-coloured squares thus obtained is such that every bi-coloured path occurs in exactly one bi-coloured square.

It is proved in Theorems 2.1 and 2.2 of [**15**] that the 1-skeleton E of Λ together with the collection S of bi-coloured squares of E completely determine Λ. Hence, given a 1-skeleton E and a collection S of bi-coloured squares such that every bi-coloured path appears in exactly one bi-coloured square, there is at most one k-graph with 1-skeleton E and bi-coloured squares S. For $k = 2$, no additional conditions are required [**18**, Section 6], and there is always exactly one 2-graph with 1-skeleton E and bi-coloured squares S. For $k \geq 3$, an associativity condition is required of the collection S of bi-coloured squares. Suppose that λ, μ, σ are edges in E with $d(\lambda) = e_i$, $d(\mu) = e_j$ and $d(\sigma) = e_l$ where i, j, l are distinct elements of $\{1, \ldots, k\}$, and suppose that $s(\lambda) = r(\mu)$ and $s(\mu) = r(\sigma)$. There are two ways in which one can produce a factorisation of the path $\lambda\mu\sigma$ in $\Lambda^{e_l} \times_{\Lambda^0} \Lambda^{e_j} \times_{\Lambda^0} \Lambda^{e_i}$:

(2.3.1)



(1) We could proceed as in the picture on the left of (2.3.1). That is, use the bi-coloured square containing μ and σ on the right-hand face of the cube to replace $\mu\sigma$ with $\sigma^1\mu^1$, then use the bi-coloured square containing λ and

σ^1 on the front face of the cube to replace $\lambda\sigma^1$ with $\sigma^2\lambda^1$, and finally use the bi-coloured square containing λ^1 and μ^1 on the top face of the cube to replace $\lambda^1\mu^1$ with $\mu^2\lambda^2$.

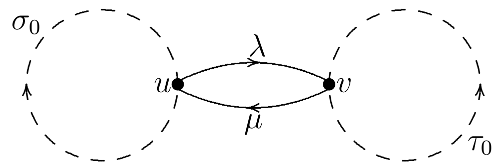
(2) We could proceed as in the picture on the right of (2.3.1). That is, first replace $\lambda\mu$ with $\mu_1\lambda_1$ using the bottom bi-coloured square, then $\lambda_1\sigma$ with $\sigma_1\lambda_2$ in the back bi-coloured square, and finally use the left bi-coloured square to replace $\mu_1\sigma_1$ with $\sigma_2\mu_2$.

These two procedures give us two factorisations in $\Lambda^{e_l} \times_{\Lambda^0} \Lambda^{e_j} \times_{\Lambda^0} \Lambda^{e_i}$ of the path $\lambda\mu\sigma$, namely

$$\sigma^2\mu^2\lambda^2 = \lambda\mu\sigma = \sigma_2\mu_2\lambda_2.$$

Theorems 2.1 and 2.2 of [**15**] together with [**15**, Example 1.5(iv)] show that for $k \geq 3$, a 1-skeleton E and a collection S of bi-coloured squares in E determine a k-graph if and only if the collection S is such that in all diagrams like (2.3.1), we have $\lambda^2 = \lambda_2$, $\mu^2 = \mu_2$, and $\sigma^2 = \sigma_2$.

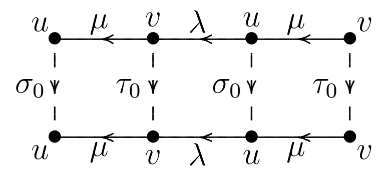
Example 2.3.2. There is precisely one 2-graph Λ corresponding to the following 1-skeleton.



This is because the only bi-coloured paths are (λ, σ_0), (σ_0, μ), (μ, τ_0) and (τ_0, λ), so the only possible choice of bi-coloured squares available is

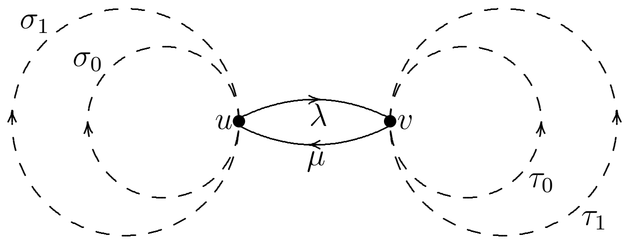
$$S = \{\{(\lambda, \sigma_0), (\tau_0, \lambda)\}, \{(\sigma_0, \mu), (\mu, \tau_0)\}\}.$$

Since $k = 2$, this choice automatically corresponds to a unique 2-graph with 1-skeleton E and bi-coloured squares S. With this information, we can see that the unique element $\rho \in u\Lambda^{(3,1)}v$ corresponds to the commuting diagram



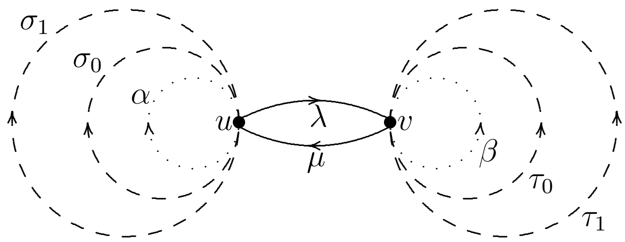
and we can read off its factorisations from this diagram: $\rho = \mu\lambda\mu\tau_0 = \mu\lambda\sigma_0\mu = \mu\tau_0\lambda\mu = \sigma_0\mu\lambda\mu$.

EXAMPLE 2.3.3. If we add extra edges to the 1-skeleton in Example 2.3.2, we may have to make a choice. For example, in the 1-skeleton



there are four possible bi-coloured paths from u to v, and we have to decide how to pair these off into bi-coloured squares: either $\{(\lambda, \sigma_i), (\tau_i, \lambda)\}$ or $\{(\lambda, \sigma_i), (\tau_{1-i}, \lambda)\}$. Likewise, the bi-coloured paths from v to u can be paired off in two ways: either $\{(\mu, \tau_i), (\sigma_i, \mu)\}$ or $\{(\mu, \tau_i), (\sigma_{1-i}, \mu)\}$. Consequently, there are four possible 2-graphs with 1-skeleton E, one corresponding to each of the choices of bi-coloured squares above.

EXAMPLE 2.3.4. If we add a third dimension to our example, then we need to worry about the tri-coloured cubes as well. Consider the 1-skeleton



The choices of bi-coloured squares containing (λ, σ_i), (τ_i, λ), (μ, τ_i) and (σ_i, μ) are as in Example 2.3.3, and there is only one way to pair the bi-coloured pairs containing one dotted and one solid edge, namely $\{(\lambda, \alpha), (\beta, \lambda)\}$ and $\{(\mu, \beta), (\alpha, \mu)\}$. However, the bi-coloured loops based at each of u and v can be paired in two possible ways: we have either

$$\{(\sigma_i, \alpha), (\alpha, \sigma_i)\} \quad \text{or} \quad \{(\sigma_i, \alpha), (\alpha, \sigma_{1-i})\}$$

based at u and either

$$\{(\tau_i, \beta), (\beta, \tau_i)\} \quad \text{or} \quad \{(\tau_i, \beta), (\beta, \tau_{1-i})\}$$

based at v. Consequently, it would appear at first sight that there are $2^4 = 16$ possible 3-graphs with the above 1-skeleton. However, closer inspection reveals that if, for example, we take $\{(\sigma_i, \alpha), (\alpha, \sigma_i)\}$ at u, $\{(\tau_i, \beta), (\beta, \tau_{1-i})\}$ at v and take

$\{(\lambda,\sigma_i),(\tau_i,\lambda)\}$ in between, the factorisations of $\lambda\alpha\sigma_0$ as in (2.3.1) are

$$\lambda\alpha\sigma_0 \overset{\alpha\sigma_0\leftrightarrow\sigma_0\alpha}{\longrightarrow} \lambda\sigma_0\alpha \overset{\lambda\sigma_0\leftrightarrow\tau_0\lambda}{\longrightarrow} \tau_0\lambda\alpha \overset{\lambda\alpha\leftrightarrow\beta\lambda}{\longrightarrow} \tau_0\beta\lambda \text{ and}$$
$$\lambda\alpha\sigma_0 \overset{\lambda\alpha\leftrightarrow\beta\lambda}{\longrightarrow} \beta\lambda\sigma_0 \overset{\lambda\sigma_0\leftrightarrow\tau_0\lambda}{\longrightarrow} \beta\tau_0\lambda \overset{\beta\tau_0\leftrightarrow\tau_1\beta}{\longrightarrow} \tau_1\beta\lambda.$$

Since these are not consistent, this is not a valid choice of S. Similar calculations for other choices of S show that in order for the associativity condition to hold, we must have $\{(\sigma_i,\alpha),(\alpha,\sigma_i)\}$ in S if and only if $\{(\tau_i,\beta),(\beta,\tau_i)\}$ is in S, and that this suffices to ensure that the associativity condition holds for all tri-coloured cubes. It follows that precisely eight of the sixteen possible choices of S give rise to a valid 3-graph with the given 1-skeleton.

2.4. Further definitions

In this section we define some important properties of k-graphs. These definitions are fundamental to the rest of the thesis, so we take the time to illustrate them using 1-skeletons.

Definition 2.4.1. Let (Λ, d) be a k-graph, and suppose that $\mu,\nu\in\Lambda$. We say that $\lambda\in\Lambda$ is a *common extension* of μ and ν if $\lambda(0,d(\mu))=\mu$ and $\lambda(0,d(\nu))=\nu$ (it necessarily follows that $d(\lambda)\geq d(\mu)\vee d(\nu)$). We call λ a *minimal common extension* of μ and ν if it is a common extension of μ and ν and also satisfies $d(\lambda)=d(\mu)\vee d(\nu)$. We denote the collection of all minimal common extensions of μ and ν by $\mathrm{MCE}(\mu,\nu)$, and we use the notation $\Lambda^{\min}(\mu,\nu)$ for the collection

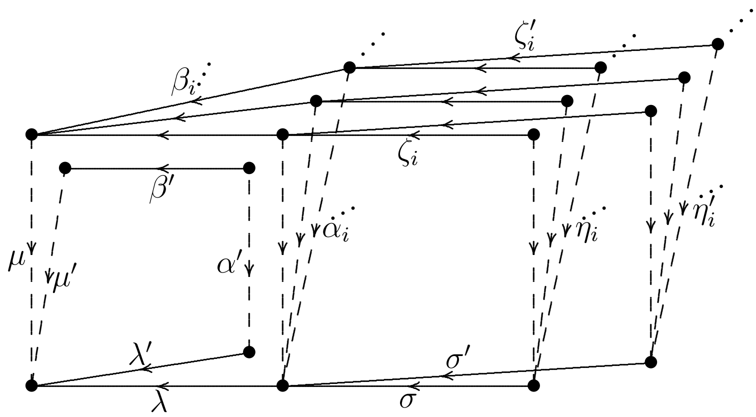
$$\Lambda^{\min}(\mu,\nu) := \{(\alpha,\beta)\in\Lambda\times\Lambda : \mu\alpha=\nu\beta\in\mathrm{MCE}(\mu,\nu)\}.$$

If $E\subset\Lambda$ and $\mu\in\Lambda$, then $\mathrm{Ext}_\Lambda(\mu;E)$ denotes the set

$$\textstyle\bigcup_{\lambda\in E}\{\beta\in s(\mu)\Lambda : \text{ there exists } \alpha\in s(\lambda)\Lambda \text{ such that } (\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)\}$$

of minimal common extendors of μ with respect to elements of E. Usually the k-graph Λ is clear from context, in which case we just write $\mathrm{Ext}(\lambda;E)$ rather than $\mathrm{Ext}_\Lambda(\lambda;E)$.

EXAMPLE 2.4.2. Consider the unique 2-graph (Λ, d) with 1-skeleton



We have that $\lambda\sigma\eta_i = \mu\beta_i\zeta_i$ for all i, that $\lambda\sigma'\eta_i' = \mu\beta_i\zeta_i'$ for all i, and that $\lambda\alpha_i = \mu\beta_i$ for all i; hence each $\lambda\sigma\eta_i$, each $\lambda\sigma'\eta_i'$, and each $\lambda\alpha_i$ is a common extension of λ and μ. We have $d(\lambda) \vee d(\mu) = (1,1) = d(\lambda\alpha_i)$ for all i, so

$$\mathrm{MCE}(\lambda, \mu) = \{\lambda\alpha_i : i \in \mathbb{N}\} \quad \text{and} \quad \Lambda^{\min}(\lambda, \mu) = \{(\alpha_i, \beta_i) : i \in \mathbb{N}\}.$$

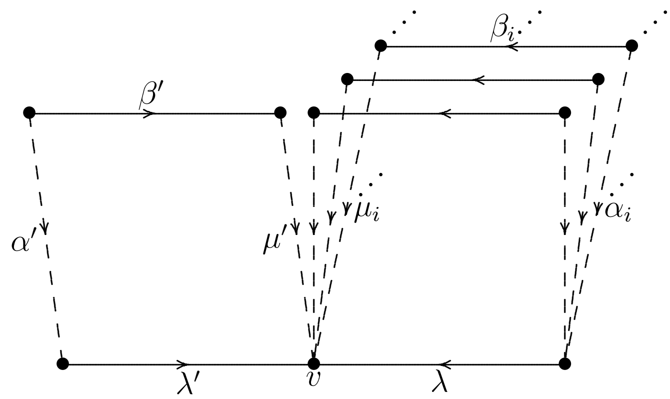
It follows that $\mathrm{Ext}(\mu; \{\lambda\}) = \{\beta_i : i \in \mathbb{N}\}$, and $\mathrm{Ext}(\sigma; \{\alpha_i : i \in I\}) = \{\eta_i : i \in I\}$ for all $I \subset \mathbb{N}$. Not all pairs of paths with the same range have any common extensions: $\Lambda^{\min}(\lambda, \mu') = \Lambda^{\min}(\lambda', \mu) = \emptyset$.

DEFINITION 2.4.3. Let (Λ, d) be a k-graph and let $E \subset \Lambda$. We say that E is *exhaustive* if $E \subset v\Lambda$ for some $v \in \Lambda^0$ and for each $\lambda \in v\Lambda$ there exists $\mu \in E$ such that $\mathrm{MCE}(\lambda, \mu) \neq \emptyset$. If E is also finite, we say E is *finite exhaustive*. By a slight abuse of notation, for each $E \subset \Lambda$ such that E is exhaustive, we $r(E)$ for the unique $v \in \Lambda^0$ such that $E \subset v\Lambda$. We write $\mathrm{FE}(\Lambda)$ for the collection of all finite exhaustive subsets of $\Lambda \setminus \Lambda^0$. If $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, and $v \in \Lambda^0$, we write $v\mathcal{E}$ for the collection $\{E \in \mathcal{E} : r(E) = v\}$.

REMARK 2.4.4. It is a key point in the above definition that if $E \in \mathrm{FE}(\Lambda)$, then we have $E \cap \Lambda^0 = \emptyset$. Any finite subset of $v\Lambda$ which contains v is trivially exhaustive, but such sets are ruled out of $\mathrm{FE}(\Lambda)$ to simplify the statements of later results (see for example Corollary 3.8.3).

If E is a 1-graph and $v \in E^0$, then $r^{-1}(v)$ is exhaustive whenever v is not a source in E. The set $r^{-1}(v)$ is finite exhaustive when $0 < |r^{-1}(v)| < \infty$. However, finite exhaustive sets are typically somewhat more complicated:

EXAMPLE 2.4.5. Consider the unique 2-graph Λ with 1-skeleton



Since every path in $v\Lambda$ is a subpath of a bi-coloured square with range v, and since each such bi-coloured square has either λ or λ' as an initial segment, the set $\{\lambda, \lambda'\}$ is finite exhaustive as are $\{\lambda, \mu'\}$ and $\{\lambda, \lambda'\alpha'\}$. The set $\{\mu', \mu_1, \mu_2, \dots\}$ is exhaustive, but is not finite exhaustive. So $v\,\mathrm{FE}(\Lambda)$ consists of all finite subsets of $v\Lambda \setminus \{v\}$ which contain as a subset any one of $\{\lambda, \lambda'\}$, $\{\lambda, \mu'\}$ or $\{\lambda, \lambda'\alpha\}$, and $\mathrm{FE}(\Lambda)$ consists of the sets in $v\,\mathrm{FE}(\Lambda)$ together with the sets $\{\alpha'\}$, $\{\alpha_1\}$, $\{\alpha_2\}$,..., and $\{\beta'\}$, $\{\beta_1\}$, $\{\beta_2\}$,...

It is crucial to the study of C^*-algebras associated to both graphs and k-graphs that said C^*-algebras are spanned by a collection of partial isometries of the form $s_\lambda s_\mu^*$ where λ and μ are paths with common source. This is always the case for 1-graphs and for row-finite k-graphs, but for more general k-graphs it is non-trivial. To ensure that such a spanning set of partial isometries exists, we restrict our attention to the finitely aligned k-graphs of [**28, 30**]:

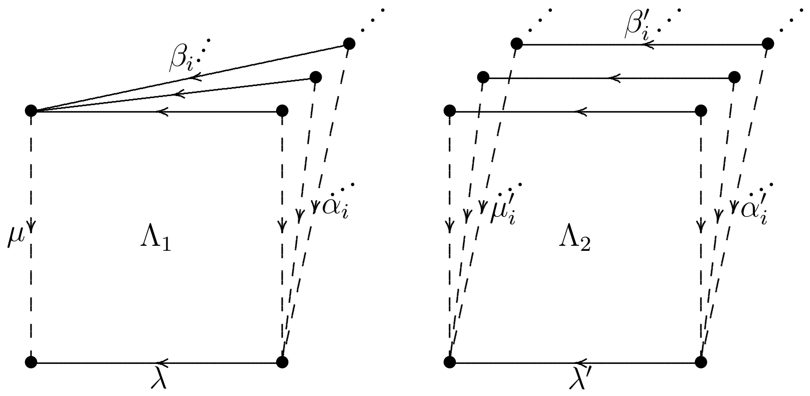
DEFINITION 2.4.6. Let (Λ, d) be a k-graph. We say that Λ is finitely aligned if and only if $|\Lambda^{\min}(\mu, \nu)| < \infty$ for all $\mu, \nu \in \Lambda$.

REMARK 2.4.7. The following three conditions are equivalent:

(1) Λ is finitely aligned.
(2) $\mathrm{MCE}(\lambda, \mu)$ is finite for all $\lambda, \mu \in \Lambda$.
(3) $\mathrm{Ext}(\lambda; E)$ is finite for all $\lambda \in \Lambda$ and all finite $E \subset \Lambda$.

The finitely aligned condition arose in [**28**] from Fowler's study of C^*-algebras associated to product systems of Hilbert bimodules (see [**28**, Theorem 5.4] and [**11**, Definition 5.7]). The most blatant indication of the importance of finite alignedness is relation (TCK3) of Definition 3.1.1 at the beginning of Chapter 3.

EXAMPLE 2.4.8. Consider the 2-graphs Λ_1 and Λ_2 with 1-skeletons



Neither of these 2-graphs is row-finite; each contains vertices which receive infinitely many paths of a single degree. The 2-graph Λ_1 on the left is not finitely aligned because we have

$$|\operatorname{MCE}(\lambda, \mu)| = |\Lambda^{\min}(\lambda, \mu)| = |\{(\alpha_i, \beta_i) : i \in \mathbb{N}\}| = \infty.$$

However, Λ_2 is finitely aligned because $\Lambda^{\min}(\lambda', \mu_i') = \{(\alpha_i', \beta_i')\}$ is a singleton set for all i.

CHAPTER 3

The Toeplitz algebra

In this chapter, we associate collections of partial isometries called Toeplitz-Cuntz-Krieger Λ-families to finitely aligned k-graphs Λ. The Toeplitz algebra of a finitely aligned k-graph Λ is the C^*-algebra $\mathcal{T}C^*(\Lambda)$ generated by a universal Toeplitz-Cuntz-Krieger Λ-family $\{s_{\mathcal{T}}(\lambda) : \lambda \in \Lambda\}$. We investigate the structure of the C^*-algebra generated by a Toeplitz-Cuntz-Krieger Λ-family, and decide when such a family determines an injective homomorphism of $\mathcal{T}C^*(\Lambda)$.

3.1. Toeplitz-Cuntz-Krieger families

DEFINITION 3.1.1. Let (Λ, d) be a finitely aligned k-graph. A family $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra is a *Toeplitz-Cuntz-Krieger Λ-family* if

(TCK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(TCK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$; and
(TCK3) $t_\lambda^* t_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$ for all $\lambda, \mu \in \Lambda$.

The following lemma, which expands upon Lemma 2.7 of [**30**], lists some elementary but useful consequences of Definition 3.1.1.

LEMMA 3.1.2. *Let (Λ, d) be a finitely aligned k-graph, and suppose that $\{t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family.*

1. $t_\lambda t_\lambda^* t_\mu t_\mu^* = \sum_{\sigma\in\mathrm{MCE}(\lambda,\mu)} t_\sigma t_\sigma^*$ *for all $\lambda, \mu \in \Lambda$.*
2. *The range projections $\{t_\lambda t_\lambda^* : \lambda \in \Lambda\}$ pairwise commute.*
3. *If $\Lambda^{\min}(\lambda, \mu) = \emptyset$, then $t_\lambda^* t_\mu = 0$; in particular, if $d(\lambda) = d(\mu)$, then $t_\lambda^* t_\mu = \delta_{\lambda,\mu} t_{s(\lambda)}$.*
4. *If $v \in \Lambda^0$ and $E \subset v\Lambda^n$ is finite, then $t_v \geq \sum_{\lambda\in E} t_\lambda t_\lambda^*$.*
5. $C^*(\{t_\lambda : \lambda \in \Lambda\}) = \overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}$.
6. *The partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ are all nonzero if and only if the vertex projections $\{t_v : v \in \Lambda^0\}$ are all nonzero.*

PROOF. Statement (1) of the lemma is obtained by multiplying (TCK3) on the left by t_λ and on the right by t_μ^*. Statement (2) is then immediate from statement (1) since $\mathrm{MCE}(\lambda, \mu) = \mathrm{MCE}(\mu, \lambda)$ for all λ and μ. The first part of statement (3) is a special case of (TCK3). For the second part of statement (3), note that if $d(\lambda) = d(\mu)$ then $\sigma \in \mathrm{MCE}(\lambda, \mu)$ implies in particular that $\lambda = \sigma(0, d(\lambda)) = \sigma(0, d(\mu)) = \mu$. For statement (4), notice that $t_v \geq t_\lambda t_\lambda^*$ for $\lambda \in v\Lambda$ by (TCK3), and that since $E \subset v\Lambda^n$, statement (3) shows that the range projections $\{t_\lambda t_\lambda^* : \lambda \in E\}$ are mutually orthogonal. For statement (5) note that $\mathrm{span}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\}$ contains $\{t_\lambda : \lambda \in \Lambda\}$, is closed under adjoints by definition, and is closed under multiplication by (TCK3). Finally, for statement (6), note that the "if" direction is trivial, so we need only establish the "only if" direction. For this, suppose that each t_v is nonzero, and suppose that $\lambda \in \Lambda$ satisfies $t_\lambda = 0$. Then $t_{s(\lambda)} = t_\lambda^* t_\lambda$ by statement (3), and it follows that $t_{s(\lambda)}$ is equal to zero, contradicting the assumption that each t_v is nonzero. □

REMARK 3.1.3. It is worth checking that (TCK1)–(TCK3) are equivalent to conditions (1) to (5) of [**28**, Definition 7.1]. To see this, notice that (1) and (2) of [**28**, Definition 7.1] are precisely (TCK1) and (TCK2), and conditions (3) and (5) of [**28**, Definition 7.1] put together boil down to (TCK3). Since (4) of [**28**, Definition 7.1] is precisely (4) of Lemma 3.1.2, it follows that the two definitions are equivalent.

REMARK 3.1.4. Notice the significance of the finitely aligned condition in Definition 3.1.1. If Λ is not finitely aligned, then (TCK3) may involve an infinite sum nonzero of partial isometries with mutually orthogonal range and source projections which cannot be norm-convergent.

THEOREM 3.1.5. *Let (Λ, d) be a finitely aligned k-graph. There exists a C^*-algebra $\mathcal{T}C^*(\Lambda)$ generated by a Toeplitz-Cuntz-Krieger Λ-family $\{s_\mathcal{T}(\lambda) : \lambda \in \Lambda\}$ which is universal in the sense that if $\{t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family in a C^*-algebra B, then there exists a unique homomorphism $\pi_t^\mathcal{T} : \mathcal{T}C^*(\Lambda) \to B$ such that $\pi_t^\mathcal{T}(s_\mathcal{T}(\lambda)) = t_\lambda$ for all $\lambda \in \Lambda$. We call $\mathcal{T}C^*(\Lambda)$ the* Toeplitz algebra *of Λ.*

PROOF. Let E_Λ be the product system over $\mathbb{N}^k$ of graphs associated to Λ as in [**15**, Example 1.5(iv)] and [**28**, Example 3.5]. Let $X(E_\Lambda)$ be the product system of Hilbert bimodules associated to E_Λ as in [**28**, Proposition 3.8]. Let $\mathcal{T}_{\mathrm{cov}}(X(E_\Lambda))$ be the C^*-algebra of [**28**, Definition 6.2] which is universal for Nica covariant Toeplitz-representations of $X(E_\Lambda)$, and let i_X be the universal generating Nica covariant Toeplitz representation of $X(E_\Lambda)$ in $\mathcal{T}_{\mathrm{cov}}(X(E_\Lambda))$. Define $\mathcal{T}C^*(\Lambda) := \mathcal{T}_{\mathrm{cov}}(X(E_\Lambda))$. By definition of $X(E_\Lambda)$, we have that for each $n \in \mathbb{N}^k$, the vector space $C_0(\Lambda^n)$ is dense in the Hilbert bimodule $X(E_\Lambda)^n$. Let $\chi_\lambda \in X(E_\Lambda)^{d(\lambda)}$ denote the characteristic function of λ. Define $s_{\mathcal{T}}(\lambda) := i_X(\chi_\lambda)$ for all $\lambda \in \Lambda$. Then [**28**, Corollary 6.7] ensures that $\mathcal{T}C^*(\Lambda)$ and $\{s_{\mathcal{T}}(\lambda) : \lambda \in \Lambda\}$ have the required properties. □

The remainder of this chapter is aimed at proving the following theorem which characterises injective homomorphisms of $\mathcal{T}C^*(\Lambda)$.

THEOREM 3.1.6. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. The homomorphism $\pi_t^{\mathcal{T}}$ which takes $s_{\mathcal{T}}(\lambda)$ to t_λ for all $\lambda \in \Lambda$ is injective if and only if*

(1) *$t_v \neq 0$ for all $v \in \Lambda^0$; and*
(2) *$\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) \neq 0$ for all $v \in \Lambda^0$ and $E \in v\,\mathrm{FE}(\Lambda)$.*

3.2. The gauge action and the core

In this section we establish the existence of a gauge action γ of $\mathbb{T}^k$ on $\mathcal{T}C^*(\Lambda)$. We show that averaging over γ provides a faithful conditional expectation Φ^γ from $\mathcal{T}C^*(\Lambda)$ to the *core* $\mathcal{T}C^*(\Lambda)^\gamma$, which is the fixed point algebra for γ. We also show that $\mathcal{T}C^*(\Lambda)^\gamma$ is equal to the closed linear span of the partial isometries $s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*$ such that λ and μ have the same degree.

For $n \in \mathbb{N}^k$ and $z \in \mathbb{T}^k$, we write z^n for the product $\prod_{i=1}^k z_i^{n_i} \in \mathbb{T}$.

PROPOSITION 3.2.1. *Let (Λ, d) be a finitely aligned k-graph. There is a strongly continuous action $\gamma : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda))$, called the* gauge action *determined by $\gamma_z(s_{\mathcal{T}}(\lambda)) := z^{d(\lambda)}s_{\mathcal{T}}(\lambda)$. Let $\mathcal{T}C^*(\Lambda)^\gamma$ denote the set $\{a \in \mathcal{T}C^*(\Lambda) : \gamma_z(a) = a$ for all $z \in \mathbb{T}^k\}$. Then*

$$\mathcal{T}C^*(\Lambda)^\gamma = \overline{\mathrm{span}}\{s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^* : \lambda, \mu \in \Lambda, d(\lambda) = d(\mu)\}.$$

PROOF. Fix $z \in \mathbb{T}^k$. We claim that the collection $\{z^{d(\lambda)}s_{\mathcal{T}}(\lambda) : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family in $\mathcal{T}C^*(\Lambda)$. To see this, first notice that if S is a partial isometry and $z \in \mathbb{T}$, then zS is also a partial isometry with the same range and source projections as S; hence $\{z^{d(\lambda)}s_{\mathcal{T}}(\lambda) : \lambda \in \Lambda\}$ is a collection of partial isometries in $\mathcal{T}C^*(\Lambda)$. To check (TCK1), notice that for $v \in \Lambda^0$, $z^{d(v)}s_{\mathcal{T}}(v) = z^0 s_{\mathcal{T}}(v) = s_{\mathcal{T}}(v)$, so $\{z^{d(v)}s_{\mathcal{T}}(v) : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections. For (TCK2), notice that if $s(\lambda) = r(\mu)$, then

$$\begin{aligned} z^{d(\lambda)}s_{\mathcal{T}}(\lambda)z^{d(\mu)}s_{\mathcal{T}}(\mu) &= z^{d(\lambda)}z^{d(\mu)}s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu) \\ &= z^{d(\lambda)+d(\mu)}s_{\mathcal{T}}(\lambda\mu) \\ &= z^{d(\lambda\mu)}s_{\mathcal{T}}(\lambda\mu). \end{aligned}$$

Finally, to check (TCK3), we calculate

$$\begin{aligned}(3.2.1)\qquad \big(z^{d(\lambda)}s_{\mathcal{T}}(\lambda)\big)^* z^{d(\mu)}s_{\mathcal{T}}(\mu) &= \overline{z^{d(\lambda)}}z^{d(\mu)}s_{\mathcal{T}}(\lambda)^*s_{\mathcal{T}}(\mu)\\ &= z^{d(\mu)-d(\lambda)}\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} s_{\mathcal{T}}(\alpha)s_{\mathcal{T}}(\beta)^*.\end{aligned}$$

On the other hand, we have

$$(3.2.2)\qquad \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} z^{d(\alpha)}s_{\mathcal{T}}(\alpha)\big(z^{d(\beta)}s_{\mathcal{T}}(\beta)\big)^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} z^{d(\alpha)-d(\beta)}s_{\mathcal{T}}(\alpha)s_{\mathcal{T}}(\beta)^*.$$

If $(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)$, then $d(\lambda)+d(\alpha)=d(\mu)+d(\beta)$, so $d(\alpha)-d(\beta)=d(\mu)-d(\lambda)$. Hence the expressions on the right-hand sides of (3.2.1) and (3.2.2) are equal, establishing the claim.

It follows from Theorem 3.1.5 that there is a homomorphism $\gamma_z : \mathcal{T}C^*(\Lambda)\to\mathcal{T}C^*(\Lambda)$ determined by $\gamma_z(s_{\mathcal{T}}(\lambda)) = z^{d(\lambda)}s_{\mathcal{T}}(\lambda)$. Furthermore $\gamma_{\bar z}\circ\gamma_z(s_{\mathcal{T}}(\lambda)) = \overline{z^{d(\lambda)}}z^{d(\lambda)}s_{\mathcal{T}}(\lambda) = s_{\mathcal{T}}(\lambda)$ for all $\lambda\in\Lambda$. By Theorem 3.1.5, $\mathrm{id}_{\mathcal{T}C^*(\Lambda)}$ is the unique homomorphism of $\mathcal{T}C^*(\Lambda)$ with this property, so $\gamma_{\bar z}\circ\gamma_z = \mathrm{id}_{\mathcal{T}C^*(\Lambda)}$. Hence $\gamma_z\in\mathrm{Aut}(\mathcal{T}C^*(\Lambda))$ with $(\gamma_z)^{-1}=\gamma_{\bar z}$. Writing 1_k for the identity element in $\mathbb{T}^k$, we have $\gamma_{1_k}(s_{\mathcal{T}}(\lambda)) = 1_k^{d(\lambda)}s_{\mathcal{T}}(\lambda) = s_{\mathcal{T}}(\lambda)$ for all λ, so $\gamma_{1_k}=\mathrm{id}_{\mathcal{T}C^*(\Lambda)}$. Now let $z,z'\in\mathbb{T}^k$ then

$$\gamma_z\circ\gamma_{z'}(s_{\mathcal{T}}(\lambda)) = (z')^{d(\lambda)}\gamma_z(s_{\mathcal{T}}(\lambda)) = (z'z)^{d(\lambda)}s_{\mathcal{T}}(\lambda) = \gamma_{z'z}s_{\mathcal{T}}(\lambda)$$

for all $\lambda\in\Lambda$. Another application of Theorem 3.1.5 now shows that $\gamma_{z'}\circ\gamma_z=\gamma_{z'z}$, so γ is an action as required.

We must now check that γ is strongly continuous. Since the γ_z are linear, Lemma 3.1.2 ensures that we need only demonstrate that for $\lambda,\mu\in\Lambda$ with $s(\lambda)=s(\mu)$, and for $(z_i)_{i=1}^\infty$ in $\mathbb{T}$ with $\lim_{i\to\infty} z_i = z$, we have $\lim_{i\to\infty}\gamma_{z_i}(s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*) = \gamma_z(s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*)$; that is, we must show that

$$z_i\to z \text{ in } \mathbb{T} \quad\text{implies}\quad z_i^{d(\lambda)-d(\mu)}s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*\to z^{d(\lambda)-d(\mu)}s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^* \text{ in } \mathbb{T}C^*(\Lambda).$$

But this is immediate because all polynomials, and in particular the polynomial $z\mapsto z^{d(\lambda)-d(\mu)}$, are continuous on $\mathbb{T}^k$.

Lastly notice that (TCK3) ensures that $\mathcal{T}C^*(\Lambda) = \overline{\mathrm{span}}\{s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^* : \lambda,\mu\in\Lambda\}$, so to prove the last statement of the Proposition it suffices to show that $s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*\in\mathcal{T}C^*(\Lambda)^\gamma$ if and only if $d(\lambda)=d(\mu)$. If $d(\lambda)=d(\mu)$, then

$$\gamma_z(s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*) = z^{d(\lambda)}\overline{z^{d(\mu)}}s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^* = z^{d(\lambda)-d(\mu)}s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^* = s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*.$$

On the other hand, if $d(\lambda)\neq d(\mu)$ then $d(\lambda)_i\neq d(\mu)_i$ for some i. Define $z\in\mathbb{T}^k$ by $z_j=1$ for $j\neq i$ and $z_i=\omega$ where $\omega\in\mathbb{T}$ is not an integer root of unity. Then

$$\gamma_z(s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*) = \omega^{(d(\lambda)-d(\mu))_i}s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*\neq s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^*$$

by choice of ω. □

DEFINITION 3.2.2. Let (Λ, d) be a finitely aligned k-graph. We call $\mathcal{T}C^*(\Lambda)^\gamma$ the *core* of $\mathcal{T}C^*(\Lambda)$.

Recall that if A is a C^*-algebra and A' is a subalgebra of A, then a map $\Phi : A \to A'$ is called a *faithful conditional expectation* if

(1) Φ is linear;
(2) Φ is bounded with $\|\Phi\| = 1$;
(3) $\Phi^2 = \Phi$; and
(4) if $a \in A$ is positive and nonzero, then $\Phi(a)$ is also positive and nonzero.

The idea is to use the gauge action to obtain a faithful conditional expectation onto the core, so that we can reduce the problem of showing that a homomorphism π of $\mathcal{T}C^*(\Lambda)$ is injective to the problem of showing that π is injective on the core.

PROPOSITION 3.2.3. *Let (Λ, d) be a finitely aligned k-graph. The core of $\mathcal{T}C^*(\Lambda)$ is a C^*-algebra, and there exists a faithful conditional expectation $\Phi^\gamma : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda)^\gamma$ which satisfies*

$$\Phi^\gamma(s_\mathcal{T}(\lambda)s_\mathcal{T}(\mu)^*) = \delta_{d(\lambda),d(\mu)} s_\mathcal{T}(\lambda)s_\mathcal{T}(\mu)^*.$$

The proof of Proposition 3.2.3 makes use of the following standard result regarding group actions on C^*-algebras. The result is folklore in the context of Cuntz-Krieger algebras (for example, the result can easily be obtained from [**32**, Appendix C], and the ideas of the proof given here can be gleaned from the much more general arguments of [**33**, Section II.5]). We present a proof here for the purposes of completeness. The author thanks Trent Yeend for access to his honours thesis, from which the proof given below was adapted.

PROPOSITION 3.2.4. *Let A be a C^*-algebra, G a compact group, μ normalised Haar measure on G and $\alpha : G \to \operatorname{Aut}(A)$ a strongly continuous action. Let A^α denote the set $\{a \in A : \alpha_g(a) = a$ for all $g \in G\}$. Then A^α is a C^*-algebra called the* fixed point algebra for α*, and there exists a faithful conditional expectation $\Phi^\alpha : A \to A^\alpha$ determined by*

$$\Phi^\alpha(a) := \int_{g\in G} \alpha_g(a)\, d\mu(g). \tag{3.2.3}$$

PROOF. Since the α_g are automorphisms, A^α is closed under multiplication and adjoints, and if $(a_n)_{n=1}^\infty \subset A^\alpha$ with $a_n \to a$ then

$$\alpha_g(a) = \alpha_g(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} \alpha_g(a_n) = \lim_{n\to\infty} a_n = a$$

for all $g \in G$, so A^α is closed, and hence a C^*-algebra.

Since α is strongly continuous, the map $\alpha_{(\,\cdot\,)}(a)$ which sends $g \in G$ to $\alpha_g(a) \in A$ belongs to $C(G, A) = C_c(G, A)$ for each $a \in A$. Hence [**32**, Lemma C.3] ensures that Equation 3.2.3 makes sense, and determines a linear *-preserving map from A to A.

Let $a \in A$, and $g \in G$. Then

$$\alpha_g(\Phi^\alpha(a)) = \alpha_g\left(\int_{h\in G} \alpha_h(a)\, d\mu(h)\right) = \int_{h\in G} \alpha_{gh}(a)\, d\mu(h)$$

by [**32**, Lemma C.3]. But left-invariance of Haar measure gives $\int_{h\in G} \alpha_{gh}(a)\, d\mu(h) = \int_{h\in G} \alpha_h(a)\, d\mu(h) = \Phi^\alpha(a)$. Since $a \in A$ and $g \in G$ were arbitrary, it follows that $\Phi^\alpha(A) \subset A^\alpha$.

For $a \in A^\alpha$, we have $\Phi^\alpha(a) = \int_{g\in G} a\, d\mu(g)$ because $\alpha_g(a) = a$ for all g, and we have $\int_{g\in G} a\, d\mu(g) = a$ because μ is normalised. Hence Φ^α fixes the elements of A^α and $\|\Phi^\alpha\| \geq 1$. For $a \in A$, we have

$$\|\Phi^\alpha(a)\| = \left\|\int_{g\in G} \alpha_g(a)\, d\mu(g)\right\| \leq \int_{g\in G} \|\alpha_g(a)\|\, d\mu(g) \tag{3.2.4}$$

by Equation (C.7) of [**32**]. But each α_g is a C^*-automorphism, and hence isometric, so $\int_{g\in G} \|\alpha_g(a)\|\, d\mu(g) = \int_{g\in G} \|a\|\, d\mu(g)$ and this is equal to $\|a\|$ because μ is normalised. Combining this with (3.2.4) shows that $\|\Phi^\alpha\| \leq 1$, so we conclude that $\|\Phi^\alpha\| = 1$. To see that Φ^α is positive and faithful on positive elements, let π be a faithful nondegenerate representation of A on $\mathcal{H}$, and calculate

$$\big(\pi(\Phi^\alpha(a^*a))h|h\big) = \int_{g\in G} \big(\pi(\alpha_g(a^*a))h|h\big)\, d\mu(g) = \int_{g\in G} \big\|\pi(\alpha_g(a))h\big\|^2\, d\mu(g) \tag{3.2.5}$$

for all $a \in A$ and $h \in \mathcal{H}$. It is immediate that $\pi(\Phi^\alpha(a^*a))$ is positive, and hence $\Phi^\alpha(a^*a)$ is positive. Suppose that $\Phi^\alpha(a^*a) = 0$. Then $\pi(\Phi^\alpha(a^*a)) = 0$ and so Equation 3.2.5 shows that $\int_{g\in G} \big\|\pi(\alpha_g(a))h\big\|^2 = 0$. It follows that $\pi(\alpha_g(a)) = 0$ for μ-almost all $g \in G$, and hence for all $g \in G$ since α is strongly continuous. It follows that $a = 0$. Hence if $a > 0$, then $\Phi^\alpha(a) > 0$. □

Proof of Proposition 3.2.3. Let Φ^γ be the faithful conditional expectation of Proposition 3.2.4, where γ is the gauge action of Proposition 3.2.1. Then Φ^γ is a faithful conditional expectation whose range is $\mathcal{T}C^*(\Lambda)^\gamma$ by Proposition 3.2.4. For the last claim, note that if $d(\lambda) = d(\mu)$ then $s_\mathcal{T}(\lambda)s_\mathcal{T}(\mu)^* \in \mathcal{T}C^*(\Lambda)^\gamma$ by Proposition 3.2.1, and so is fixed by Φ^γ. On the other hand, if $d(\lambda) \neq d(\mu)$ then

$$\Phi(s_\mathcal{T}(\lambda)s_\mathcal{T}(\mu)^*) = \left(\int_{z\in\mathbb{T}^k} z^{d(\lambda)-d(\mu)}\, d\mu(z)\right) s_\mathcal{T}(\lambda)s_\mathcal{T}(\mu)^*,$$

which is equal to zero since $z \mapsto z^n$ is entire for $n \in \mathbb{Z}^k \setminus \{0\}$. □

3.3. Orthogonalising range projections

The bulk of the material in this section, namely Definition 3.3.1 and the statement and proof of Lemma 3.3.4 — including the technical lemmas 3.3.7, 3.3.9 and 3.3.10 — is taken from joint work with Raeburn in [**28**, Section 8]. Indeed the proof of Lemma 3.3.4 differs from that of [**28**, Proposition 8.6] only in the minor technical changes needed to account for the change of context from product systems to k-graphs. The statements and proofs of both Proposition 3.3.3, and Corollary 3.3.11 are taken from joint work with Raeburn and Yeend in [**30**], and are more or less identical to those of Proposition 3.5 and Corollary 3.7 of [**30**] respectively. In fact, the details of the reduction of Proposition 3.3.3 to Lemma 3.3.4 in the proof of Proposition 3.3.3, and our subsequent proof of Lemma 3.3.4 represent the details of the final paragraph of the proof of [**30**, Proposition 3.5].

Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. As has become standard in the context of graph algebras, we study the C^*-algebra generated by $\{t_\lambda : \lambda \in \Lambda\}$ by studying the structure of the subalgebra $C^*(\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\})$. The objective is to demonstrate this algebra as the closure of an increasing union of finite-dimensional subalgebras $C^*(\{t_\lambda t_\mu^* : \lambda, \mu \in E, d(\lambda) = d(\mu)\})$ where $E \subset \Lambda$ is finite. To find matrix units for these algebras, we need to be able to "orthogonalise" their generators. The first step is to orthogonalise the range projections $t_\lambda t_\lambda^*$ associated to the paths in E. We do this using the procedure developed in joint work with Raeburn in [**28**, Section 8].

Definition 3.3.1. Let (Λ, d) be a finitely aligned k-graph, and let E be a finite subset of Λ. For $\lambda \in E$ define

$$Q(t)^E_\lambda := t_\lambda t_\lambda^* \prod_{\substack{\lambda\nu \in E \\ d(\nu) > 0}} (t_\lambda t_\lambda^* - t_{\lambda\nu} t_{\lambda\nu}^*).$$

Notice that $Q(t)^E_\lambda \leq t_\lambda t_\lambda^* \leq t_{r(\lambda)}$ for all $\lambda \in E$.

REMARK 3.3.2. (1) Lemma 3.1.2(2) shows that $Q(t)^E_\lambda$ is well-defined for $\lambda \in E$, and also that each $Q(t)^E_\lambda$ is a projection.

(2) By convention, when a formal product in A is indexed by the empty set, it is taken to be equal to the unit of the multiplier algebra of A. In particular, if $\lambda \in E$ is such that there exists no $\nu \in \Lambda \setminus \Lambda^0$ with $\lambda\nu \in E$, then $Q(t)^E_\lambda = t_\lambda t^*_\lambda \cdot 1_{\mathcal{M}(\mathcal{T}C^*(\Lambda))} = t_\lambda t^*_\lambda$.

The object of this section is to prove the following Proposition:

PROPOSITION 3.3.3. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Suppose that $E \subset \Lambda$ is finite and nonempty, and that E is closed under taking minimal common extensions in the sense that $\lambda, \mu \in E$ implies* $\mathrm{MCE}(\lambda, \mu) \subset E$. *Then $\{Q(t)^E_\lambda : \lambda \in E\}$ is a collection of mutually orthogonal (possibly zero) projections such that*

$$\Big(t_v \prod_{\lambda \in vE} (t_v - t_\lambda t^*_\lambda)\Big) + \sum_{\lambda \in vE} Q(t)^E_\lambda = t_v \tag{3.3.1}$$

for all $v \in r(E)$.

To prove Proposition 3.3.3, we first simplify to the situation where: (1) the set E consists of paths with a common range $v \in \Lambda^0$, and (2) this vertex v belongs to E. That is:

LEMMA 3.3.4. *Let (Λ, d) be a finitely aligned k-graph. Let $v \in \Lambda^0$, and let E be a finite subset of $v\Lambda$ such that $v \in E$. Suppose also that E is closed under taking minimal common extensions in the sense that $\lambda, \mu \in E$ implies* $\mathrm{MCE}(\lambda, \mu) \subset E$. *Let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Then $\{Q(t)^E_\lambda : \lambda \in E\}$ is a collection of mutually orthogonal (possibly zero) projections such that*

$$\sum_{\lambda \in E} Q(t)^E_\lambda = t_v. \tag{3.3.2}$$

REMARK 3.3.5. Lemma 3.3.4 differs from Proposition 8.6 of [**28**] only in that

(1) here we assume that E is a subset of $v\Lambda$ for some v whereas in [**28**, Proposition 8.6] E is taken to be an arbitrary finite set which is closed under taking minimal common extensions;
(2) here we do not assume that $\prod_{\lambda \in E}(t_v - t_\lambda t^*_\lambda) \neq 0$ for all finite $E \subset v\Lambda$ and we do not claim that all the projections $Q(t)^E_\lambda$ are nonzero, whereas in [**28**, Proposition 8.6] all the $\prod_{\lambda \in E}(t_v - t_\lambda t^*_\lambda)$ are assumed to be nonzero and all the $Q(t)^{\vee F}_\lambda$ are shown to be nonzero; and

(3) here we are dealing with k-graphs whereas [**28**, Proposition 8.6] applies to the more general setting of finitely aligned product systems of graphs over quasi-lattice ordered semigroups.

The proof changes only minimally to account for these differences. Point (1) requires no change at all because it represents a restriction of [**28**, Proposition 8.6] to a special case. Point (2) is addressed simply by removing the first sentence of the proof of [**28**, Proposition 8.6]. Point (3) amounts to a change in notation.

Once we have established Lemma 3.3.4, we can deduce Proposition 3.3.3 as follows:

Proof of Proposition 3.3.3. The following argument is taken from joint work with Raeburn and Yeend in [**30**]. We proceed by reducing Proposition 3.3.3 to Lemma 3.3.4. We do this in two steps.

Before starting, notice that we have already established that the $Q(t)^E_\lambda$ are projections in Remark 3.3.2, so we need only show that the $Q(t)^E_\lambda$ are mutually orthogonal projections and satisfy (3.3.1). In fact, since t_v is a projection, (3.3.1) shows that the $Q(t)^E_\lambda$ are mutually orthogonal, so it suffices to establish (3.3.1).

First, suppose that (3.3.1) holds whenever $E \subset v\Lambda$, and suppose that $F \subset \Lambda$ is finite and that $\lambda, \mu \in F$ implies $\mathrm{MCE}(\lambda, \mu) \subset F$. Fix $v \in r(F)$. Notice first that for all $\lambda \in F$ with $r(\lambda) = v$,

$$Q(t)^F_\lambda = t_\lambda t^*_\lambda \prod_{\substack{\lambda\nu \in F\\ d(\nu)>0}} (t_\lambda t^*_\lambda - t_{\lambda\nu} t^*_{\lambda\nu}) = t_\lambda t^*_\lambda \prod_{\substack{\lambda\nu \in vF\\ d(\nu)>0}} (t_\lambda t^*_\lambda - t_{\lambda\nu} t^*_{\lambda\nu}) = Q(t)^{vF}_\lambda. \tag{3.3.3}$$

Let $E := vF$. Then $E \subset v\Lambda$ by definition. Furthermore, if $\lambda, \mu \in E$, then $\mathrm{MCE}(\lambda, \mu) \subset vF = E$, so E is closed under taking minimal common extensions. It follows from (3.3.3) and from our assumption that (3.3.1) holds whenever $E \subset v\Lambda$ that

$$t_v = \Big(t_v \prod_{\lambda \in E} (t_v - t_\lambda t^*_\lambda) + \sum_{\lambda \in E} Q(t)^E_\lambda\Big) = \Big(t_v \prod_{\lambda \in vF} (t_v - t_\lambda t^*_\lambda) + \sum_{\lambda \in vF} Q(t)^F_\lambda\Big).$$

Now suppose that $F \subset v\Lambda$, $\lambda, \mu \in F$ implies $\mathrm{MCE}(\lambda, \mu) \subset F$, and that $v \notin F$. Let $E := F \cup \{v\}$. Since $\mathrm{MCE}(v, \lambda) = \{\lambda\}$ for all $\lambda \in F$, we know that E is closed under taking minimal common extensions, and it is clearly a subset of $v\Lambda$. Hence Lemma 3.3.4 gives

$$t_v = \sum_{\lambda \in E} Q(t)^E_\lambda = Q(t)^E_v + \sum_{\lambda \in E\setminus\{v\}} Q(t)^E_\lambda = t_v \prod_{\lambda \in F} (t_v - t_\lambda t^*_\lambda) + \sum_{\lambda \in F} Q(t)^F_\lambda,$$

completing the proof. □

We are left with the task of proving Lemma 3.3.4. The general idea is to proceed by induction on the size of E, but this is not as straightforward as it sounds. The problem is that when we remove a path λ from a set E which is closed under taking minimal common extensions, we do not in general obtain a new set which is closed under taking minimal common extensions. Instead, given a finite set $E \subset \Lambda$ which need not be closed under taking minimal common extensions, we show how to generate a set $\vee E$ which is closed under taking minimal common extensions. The construction is such that if E is closed under taking minimal common extensions to begin with, then $\vee E$ is equal to E. We then prove Lemma 3.3.4 for $\vee E$ by induction on $|E|$.

Definition 3.3.6. Recall from [**28**, Definition 8.3] that if (Λ, d) is a finitely aligned k-graph and $E \subset \Lambda$ is finite, then $\operatorname{MCE}(E)$ and $\vee E$ are defined as follows:

$$\operatorname{MCE}(E) = \{\lambda \in \Lambda : d(\lambda) = \bigvee_{\alpha \in E} d(\alpha) \text{ and } \lambda(0, d(\alpha)) = \alpha \text{ for all } \alpha \in E\},$$

and $\vee E = \bigcup_{G \subset E} \operatorname{MCE}(G)$.

Lemma 3.3.7 ([**28**, Lemma 8.4]). *Let $v \in \Lambda^0$ and suppose that $v \in E \subset v\Lambda$ where E is finite. Then*

(1) *$E \subset \vee E$;*
(2) *$\vee E$ is finite;*
(3) *$G \subset \vee E$ implies $\operatorname{MCE}(G) \subset \vee E$; and*
(4) *$\lambda \in \vee E$ implies $d(\lambda) \leq \bigvee_{\mu \in E} d(\mu)$.*

Proof. (1) For $\lambda \in E$, we have $\{\lambda\} \subset E$ and $\lambda \in \operatorname{MCE}(\{\lambda\})$, giving $\lambda \in \vee E$.

(2) It suffices to show that if $E \subset \Lambda$ is finite, then $\operatorname{MCE}(F)$ is finite. When $|E| = 1$, this assertion is trivial. Suppose as an inductive hypothesis that $\operatorname{MCE}(E)$ is finite whenever $|E| \leq n$ for some $n \geq 1$, and suppose that $|E| = n + 1$. Let $\lambda \in E$, and let $E' := E \backslash \{\lambda\}$. Suppose that $\gamma \in \operatorname{MCE}(E)$. Since $\gamma\big(0, \bigvee_{\alpha \in E'} d(\alpha)\big) \in \operatorname{MCE}(E')$, we have $\gamma \in \operatorname{MCE}(\lambda, \mu)$ for some $\mu \in \operatorname{MCE}(E')$. Hence $|\operatorname{MCE}(E)| \leq \sum_{\mu \in \operatorname{MCE}(E')} |\operatorname{MCE}(\lambda, \mu)|$. Each term in this sum is finite because (Λ, d) is finitely aligned, and the sum has only finitely many terms by the inductive hypothesis. Hence $\operatorname{MCE}(E)$ is finite.

(3) Let $G \subset \vee E$ and for $\alpha \in G$ choose $G_\alpha \subset E$ such that $\alpha \in \operatorname{MCE}(G_\alpha)$. Let $H := \bigcup_{\alpha \in G} G_\alpha$. We will show that $\operatorname{MCE}(G) \subset \operatorname{MCE}(H) \subset \vee E$. Suppose $\lambda \in \operatorname{MCE}(G)$. Then

$$d(\lambda) = \bigvee_{\alpha \in G} d(\alpha) = \bigvee_{\alpha \in G} \big(\bigvee_{\beta \in G_\alpha} d(\beta)\big) = \bigvee_{\beta \in H} d(\beta).$$

For $\beta \in H$, choose $\alpha \in G$ such that $\beta \in G_\alpha$. Then $\lambda(0, d(\beta)) = \alpha(0, d(\beta)) = \beta$. Thus $\lambda \in \mathrm{MCE}(H)$.

(4) Let $\lambda \in \vee E$. Then $\lambda \in \mathrm{MCE}(G_\lambda)$ for some $G_\lambda \subset E$ by definition, so $d(\lambda) = \bigvee_{\mu \in G_\lambda} d(\mu) \leq \bigvee_{\mu \in E} d(\mu)$. □

REMARK 3.3.8. Suppose that $E \subset \Lambda$ is already closed under taking minimal common extensions. A straightforward induction on $|G|$ shows that $G \subset E$ implies $\mathrm{MCE}(G) \subset E$, and it follows that $\vee E \subset E$; this, combined with (1) of Lemma 3.3.7 shows that $\vee E = E$.

We now prove two technical lemmas that will allow us to relate the $Q(t)^{\vee E}_\lambda$ to the $Q(t)^{\vee G}_\mu$ when $G \subset E$.

LEMMA 3.3.9 ([**28**, Lemma 8.7]). *Let (Λ, d) be a finitely aligned k-graph and let $v \in \Lambda^0$. Suppose $v \in E \subset v\Lambda$ with E finite, and that $\lambda \in E \setminus \{v\}$; let $G := E \setminus \{\lambda\}$. Then for each $\mu \in \vee E \setminus \vee G$, there exists a unique $\xi_\mu \in \vee G$ such that*

(1) *$d(\mu) \geq d(\xi_\mu)$ and $\mu(0, d(\xi_\mu)) = \xi_\mu$; and*
(2) *$\xi \in \vee G$ and $\mu(0, d(\xi)) = \xi$ imply $d(\xi) \leq d(\xi_\mu)$.*

Furthermore, for all $\mu \in \vee G$, we have $\mu \in \mathrm{MCE}(\xi_\mu, \lambda)$.

PROOF. Fix $\mu \in \vee E \setminus \vee G$. Then

$$(\vee G)_\mu := \{\sigma \in \vee G : \mu(0, d(\sigma)) = \sigma\}$$

is nonempty because it contains v. Hence we can define

$$n := \bigvee\nolimits_{\sigma \in (\vee G)_\mu} d(\sigma) \qquad \text{and} \qquad \xi_\mu := \mu(0, n).$$

We have $\xi_\mu \in \mathrm{MCE}((\vee G)_\mu)$ by definition, and it then follows from Lemma 3.3.7(3) that ξ_μ belongs to $\vee G$. We have that ξ_μ satisfies (1) and (2) by definition. To see that ξ_μ is unique, suppose that $\xi \in \vee G$ also satisfies (1) and (2). Since ξ and ξ_μ both satisfy (2), we have $d(\xi_\mu) = d(\xi)$, and then since they both satisfy (1), we have $\xi = \mu(0, d(\xi)) = \mu(0, d(\xi_\mu)) = \xi_\mu$.

It remains to show that $\mu \in \mathrm{MCE}(\lambda, \xi_\mu)$. By definition of $\vee E$, we have $\mu \in \mathrm{MCE}(H)$ for some $H \subset E$, and since $\mu \notin \vee G$ we have $\lambda \in H$, so $\mu(0, d(\lambda)) = \lambda$. Let $H' = H \setminus \{\lambda\}$. Let $m := \bigvee_{\tau \in H'} d(\tau) \leq d(\mu)$. Then we have

$$d(\mu) = \bigvee\nolimits_{\tau \in H} d(\tau) = d(\lambda) \vee \big(\bigvee\nolimits_{\tau \in H'} d(\tau)\big) = d(\lambda) \vee m$$

so $\mu \in \mathrm{MCE}(\lambda, \mu(0, m))$, where $\mu(0, m) \in \mathrm{MCE}(H') \subset \vee G$. By part (2) above, we have $m \leq d(\xi_\mu)$, so that $d(\mu) = d(\lambda) \vee m \leq d(\lambda) \vee d(\xi_\mu) \leq d(\mu)$ where the last inequality follows from the fact that both λ and ξ_μ are initial segments of μ. It follows that $d(\mu) = d(\lambda) \vee d(\xi_\mu)$, and hence $\mu \in \mathrm{MCE}(\lambda, \xi_\mu)$ by definition. □

LEMMA 3.3.10 ([**28**, Lemma 8.8]). *Let (Λ, d) be a finitely aligned k-graph and let $v \in \Lambda^0$. Suppose $v \in E \subset v\Lambda$ with E finite, and that $\lambda \in E \setminus \{v\}$; let $G := E \setminus \{\lambda\}$. Let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Suppose that $\mu \in \vee E \setminus \vee G$. Let ξ_μ be the maximal subpath of μ in $\vee G$ provided by Lemma 3.3.9. Then*

$$Q(t)^{\vee E}_{\mu} = Q(t)^{\vee G}_{\xi_\mu} t_\mu t_\mu^*. \tag{3.3.4}$$

PROOF. The result is proved by way of two preliminary claims.

Claim 1: $Q(t)^{\vee E}_{\mu} \leq Q(t)^{\vee G}_{\xi_\mu}$.

Proof of Claim 1. Since $\mu(0, d(\xi_\mu)) = \xi_\mu$, we have $t_\mu t_\mu^* \leq t_{\xi_\mu} t_{\xi_\mu}^*$, and hence Lemma 3.1.2 ensures that

$$Q(t)^{\vee G}_{\xi_\mu} Q(t)^{\vee E}_{\mu} = t_\mu t_\mu^* \Big(\prod_{\substack{\xi_\mu \nu \in \vee G \\ d(\nu) \neq 0}} (t_{\xi_\mu} t_{\xi_\mu}^* - t_{\xi_\mu \nu} t_{\xi_\mu \nu}^*) \Big) Q(t)^{\vee E}_{\mu}.$$

Hence we need only show that $t_\mu t_\mu^* (t_{\xi_\mu} t_{\xi_\mu}^* - t_{\xi_\mu \nu} t_{\xi_\mu \nu}^*) Q(t)^{\vee E}_{\mu} = Q(t)^{\vee E}_{\mu}$ whenever $\xi_\mu \nu \in \vee G$ with $d(\nu) > 0$. But if $\xi_\mu \nu \in \vee G$ with $d(\nu) > 0$, then

$$\begin{aligned} t_\mu t_\mu^* (t_{\xi_\mu} t_{\xi_\mu}^* - t_{\xi_\mu \nu} t_{\xi_\mu \nu}^*) &= t_\mu t_\mu^* - \sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu, \xi_\mu \nu)} t_{\mu\alpha} t_{\mu\alpha}^* \\ &= \prod_{(\alpha,\beta) \in \Lambda^{\min}(\mu, \xi_\mu \nu)} (t_\mu t_\mu^* - t_{\mu\alpha} t_{\mu\alpha}^*). \end{aligned} \tag{3.3.5}$$

We claim that $(\alpha, \beta) \in \Lambda^{\min}(\mu, \xi_\mu \nu)$ implies $\mu\alpha \in \vee E$ and $d(\alpha) > 0$. To see this, let $(\alpha, \beta) \in \Lambda^{\min}(\mu, \xi_\mu \nu)$. We have $(\mu\alpha)(0, d(\xi_\mu \nu)) = \xi_\mu \nu \in \vee G$, and hence part (2) of Lemma 3.3.9 shows that $d(\xi_{\mu\alpha}) \geq d(\xi_\mu \nu) > d(\xi_\mu)$. It follows that $\mu\alpha \neq \mu$, so $d(\alpha) > 0$. Furthermore, $\xi_\mu \nu \in \vee G \subset \vee E$ and $\mu \in \vee E$ by assumption, so we have $\mu\alpha \in \mathrm{MCE}(\mu, \xi_\mu \nu) \subset \vee E$ by Lemma 3.3.7(4) as required.

It follows that each factor in (3.3.5) is already a factor in $Q(t)^{\vee E}_{\mu}$, giving $t_\mu t_\mu^* (t_{\xi_\mu} t_{\xi_\mu}^* - t_{\xi_\mu \nu} t_{\xi_\mu \nu}^*) Q(t)^{\vee E}_{\mu} = Q(t)^{\vee E}_{\mu}$ as required. □ Claim 1

Claim 2: If $\mu\nu \in \vee E$ with $d(\nu) \neq 0$, then $Q(t)^{\vee G}_{\xi_\mu} t_{\mu\nu} t_{\mu\nu}^* = 0$.

Proof of Claim 2. Fix $\mu\nu \in \vee E$ with $d(\nu) \neq 0$. Suppose for contradiction that $\xi_{\mu\nu} = \xi_\mu$. Then $d(\mu\nu) = d(\lambda) \vee d(\xi_{\mu\nu}) = d(\lambda) \vee d(\xi_\mu) = d(\mu)$ by two applications of the last statement of Lemma 3.3.7, contradicting $d(\nu) > 0$. Since $(\mu\nu)(0, d(\xi_\mu)) = \mu(0, d(\xi_\mu)) = \xi_\mu \in \vee G$, Lemma 3.3.7(2) now shows that $d(\xi_{\mu\nu}) > d(\xi_\mu)$, so that

$\xi_{\mu\nu} = \xi_\mu\tau$ for some τ with $d(\tau) > 0$. But $\xi_{\mu\nu} \in \vee G$, and so we have

$$\begin{aligned}
&Q(t)^{\vee G}_{\xi_\mu} t_{\mu\nu}t^*_{\mu\nu} \\
&\quad = t_{\xi_\mu}t^*_{\xi_\mu} \prod_{\substack{\xi_\mu\rho\in\vee G\\ d(\rho)>0}} (t_{\xi_\mu}t^*_{\xi_\mu} - t_{\xi_\mu\rho}t^*_{\xi_\mu\rho})t_{\mu\nu}t^*_{\mu\nu} \\
&\quad = t_{\xi_\mu}t^*_{\xi_\mu}\Big(\prod_{\substack{\xi_\mu\rho\in\vee G\setminus\{\xi_\mu\tau\}\\ d(\rho)>0}} (t_{\xi_\mu}t^*_{\xi_\mu} - t_{\xi_\mu\rho}t^*_{\xi_\mu\rho})\Big)(t_{\xi_\mu}t^*_{\xi_\mu} - t_{\xi_\mu\tau}t^*_{\xi_\mu\tau})t_{\mu\nu}t^*_{\mu\nu} \\
&\quad = t_{\xi_\mu}t^*_{\xi_\mu}\Big(\prod_{\substack{\xi_\mu\rho\in\vee G\setminus\{\xi_\mu\tau\}\\ d(\rho)>0}} (t_{\xi_\mu}t^*_{\xi_\mu} - t_{\xi_\mu\rho}t^*_{\xi_\mu\rho})\Big)(t_{\xi_\mu}t^*_{\xi_\mu} - t_{\xi_{\mu\nu}}t^*_{\xi_{\mu\nu}})t_{\mu\nu}t^*_{\mu\nu}
\end{aligned}$$

which vanishes because both $t_{\xi_\mu}t^*_{\xi_\mu}$ and $t_{\xi_{\mu\nu}}t^*_{\xi_{\mu\nu}}$ dominate $t_{\mu\nu}t^*_{\mu\nu}$. □ Claim 2

We can now use Claim 1, the definition of $Q(t)^{\vee E}_\mu$, and Claim 2 in that order to calculate

$$Q(t)^{\vee E}_\mu = Q(t)^{\vee G}_{\xi_\mu}Q(t)^{\vee E}_\mu = Q(t)^{\vee G}_{\xi_\mu} \prod_{\mu\nu\in\vee F, d(\nu)>0} (t_\mu t^*_\mu - t_{\mu\nu}t^*_{\mu\nu}) = Q(t)^{\vee G}_{\xi_\mu} t_\mu t^*_\mu,$$

completing the proof of Lemma 3.3.10. □

We can now prove Lemma 3.3.4, and thereby complete the proof of Proposition 3.3.3

Proof of Lemma 3.3.4. By Remark 3.3.8, it suffices to show that if $E \subset v\Lambda$ is finite with $v \in E$, then $Q(t)^{\vee E}_\lambda Q(t)^{\vee E}_\mu = \delta_{\lambda,\mu}Q(t)^{\vee E}_\lambda$ for all $\lambda, \mu \in \vee E$ and $t_v = \sum_{\lambda\in\vee E} Q(t)^{\vee E}_\lambda$.

First let $\lambda, \mu \in \vee E$ with $\lambda \neq \mu$. Suppose that $d(\lambda) = d(\mu)$. We have $Q(t)^{\vee E}_\lambda \leq t_\lambda t^*_\lambda$ and $Q(t)^{\vee E}_\lambda \leq t_\lambda t^*_\lambda$ by definition. Lemma 3.1.2(2) therefore shows that $Q(t)^{\vee E}_\lambda Q(t)^{\vee E}_\mu \leq t_\lambda t^*_\lambda t_\mu t^*_\mu$, and Lemma 3.1.2(3) shows that $t_\lambda t^*_\lambda t_\mu t^*_\mu = 0$ giving $Q(t)^{\vee E}_\lambda Q(t)^{\vee E}_\mu = 0$.

Now suppose that $d(\lambda) \neq d(\mu)$. Then we have $d(\lambda) \vee d(\mu)$ strictly larger than one of $d(\lambda)$ and $d(\mu)$; say $d(\lambda) \vee d(\mu) > d(\lambda)$. Then $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ implies

$d(\alpha) > 0$ and $\lambda\alpha \in \vee E$. But then

$$\begin{aligned}
&Q(t)^{\vee E}_\lambda Q(t)^{\vee E}_\mu \\
&\quad = t_\lambda t^*_\lambda t_\mu t^*_\mu Q(t)^{\vee E}_\lambda Q(t)^{\vee E}_\mu \qquad \text{by Lemma 3.1.2(2)} \\
&\quad = \Big(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_{\lambda\alpha} t^*_{\lambda\alpha}\Big)\Big(\prod_{\substack{\lambda\nu\in\vee E\\ d(\nu)>0}} (t_\lambda t^*_\lambda - t_{\lambda\nu} t^*_{\lambda\nu})\Big) Q(t)^{\vee E}_\mu \\
&\quad = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} \Big(t_{\lambda\alpha} t^*_{\lambda\alpha}\Big((t_\lambda t^*_\lambda - t_{\lambda\alpha} t^*_{\lambda\alpha}) \prod_{\substack{\lambda\nu\in\vee E\setminus\{\lambda\alpha\}\\ d(\nu)>0}} (t_\lambda t^*_\lambda - t_{\lambda\nu} t^*_{\lambda\nu})\Big)\Big) Q(t)^{\vee E}_\mu \\
&\quad = 0.
\end{aligned}$$

It remains to show that $\sum_{\lambda\in\vee E} Q(t)^{\vee E}_\lambda = t_v$. We proceed by induction on $|E|$. If $|E| = 1$, then since $v \in E$ by assumption, we have $\vee E = E = \{v\}$, and $\sum_{\lambda\in\vee E} Q(t)^{\vee E}_\lambda = Q(t)^{\{v\}}_v = t_v$, giving a basis case.

Now suppose that $|E| = n \geq 2$ and that (3.3.2) holds whenever $|E| \leq n - 1$. Since $|E| > 1$, there exists $\lambda \in E \setminus \{v\}$. Write G for $E \setminus \{\lambda\}$. Fix $\mu \in \vee G$, and rewrite

$$\begin{aligned}
Q(t)^{\vee E}_\mu &= t_\mu t^*_\mu\Big(\prod_{\substack{\mu\nu\in\vee E\\ d(\nu)>0}} (t_\mu t^*_\mu - t_{\mu\nu} t^*_{\mu\nu})\Big) \\
&= t_\mu t^*_\mu\Big(\prod_{\substack{\mu\nu\in\vee G\\ d(\nu)>0}} (t_\mu t^*_\mu - t_{\mu\nu} t^*_{\mu\nu})\Big)\Big(\prod_{\mu\sigma\in\vee F\setminus\vee G} (t_\mu t^*_\mu - t_{\mu\sigma} t^*_{\mu\sigma})\Big). \qquad (3.3.6)
\end{aligned}$$

We claim that if $\mu\sigma \in \vee E \setminus \vee G$ and $\xi_{\mu\sigma} \neq \mu$, then we can delete the factor $(t_\mu t^*_\mu - t_{\mu\sigma} t^*_{\mu\sigma})$ from (3.3.6) without changing the result of the product. To see this, suppose that $\mu\sigma \in \vee E \setminus \vee G$ and $\xi_{\mu\sigma} \neq \mu$. Then Lemma 3.3.9(2) ensures that $\xi_{\mu\sigma} = \mu\alpha$ for some α with $d(\alpha) > 0$. Since $\mu\alpha = \xi_{\mu\sigma} \in \vee G$, it follows that $(t_\mu t^*_\mu - t_{\xi_{\mu\sigma}} t^*_{\xi_{\mu\sigma}})$ is a factor in $Q(t)^{\vee G}_\mu$. Since $t_\mu t^*_\mu - t_{\xi_{\mu\sigma}} t^*_{\xi_{\mu\sigma}} \leq t_\mu t^*_\mu - t_{\mu\sigma} t^*_{\mu\sigma}$, it follows that $t_\mu t^*_\mu - t_{\mu\sigma} t^*_{\mu\sigma} Q(t)^{\vee G}_\mu = Q(t)^{\vee G}_\mu$, and so we can delete such $(t_\mu t^*_\mu - t_{\mu\sigma} t^*_{\mu\sigma})$ from (3.3.6) as claimed. Hence

$$\begin{aligned}
Q(t)^{\vee E}_\mu &= t_\mu t^*_\mu\Big(\prod_{\substack{\mu\nu\in\vee G\\ d(\nu)>0}} (t_\mu t^*_\mu - t_{\mu\nu} t^*_{\mu\nu})\Big)\Big(\prod_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} (t_\mu t^*_\mu - t_{\mu\sigma} t^*_{\mu\sigma})\Big) \\
&= Q(t)^{\vee G}_\mu\Big(\prod_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} (t_\mu t^*_\mu - t_{\mu\sigma} t^*_{\mu\sigma})\Big). \qquad (3.3.7)
\end{aligned}$$

Moreover, Lemma 3.3.9 ensures that if $\mu\sigma \in \vee E \setminus \vee G$, then $\mu\sigma \in \mathrm{MCE}(\xi_{\mu\sigma}, \lambda)$; in particular, when $\xi_{\mu\sigma} = \mu$, we have $d(\mu\sigma) = d(\mu) \vee d(\lambda)$. Hence if $\mu\sigma$ and $\mu\sigma'$

satisfy $\xi_{\mu\sigma} = \mu = \xi_{\mu\sigma'}$, then $d(\mu\sigma) = d(\mu\sigma')$, so $t_{\mu\sigma}t^*_{\mu\sigma}t_{\mu\sigma'}t^*_{\mu\sigma'} = \delta_{\sigma,\sigma'}t_{\mu\sigma}t^*_{\mu\sigma}$ by Lemma 3.1.2. It follows that

$$\prod_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} (t_\mu t^*_\mu - t_{\mu\sigma}t^*_{\mu\sigma}) = t_\mu t^*_\mu - \sum_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} t_{\mu\sigma}t^*_{\mu\sigma},$$

and (3.3.7) shows that for all $\mu \in \vee G$, we have

$$Q(t)^{\vee E}_\mu = Q(t)^{\vee G}_\mu\Big(t_\mu t^*_\mu - \sum_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} t_{\mu\sigma}t^*_{\mu\sigma}\Big). \tag{3.3.8}$$

Substituting (3.3.8) for those terms in $\sum_{\sigma\in\vee E} Q(t)^{\vee E}_\sigma$ for which σ belongs to $\vee G$, we obtain

$$\begin{aligned}\sum_{\sigma\in\vee E} Q(t)^{\vee E}_\lambda &= \Big(\sum_{\mu\in\vee G} Q(t)^{\vee E}_\mu\Big) + \Big(\sum_{\tau\in\vee E\setminus\vee G} Q(t)^{\vee E}_\tau\Big)\\ &= \Big(\sum_{\mu\in\vee G} Q(t)^{\vee G}_\mu\Big(t_\mu t^*_\mu - \sum_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} t_{\mu\sigma}t^*_{\mu\sigma}\Big)\Big) + \Big(\sum_{\tau\in\vee E\setminus\vee G} Q(t)^{\vee E}_\tau\Big). \quad (3.3.9)\end{aligned}$$

Since Lemma 3.3.9 ensures that for $\tau \in \vee E \setminus \vee G$, the path $\xi_\tau \in \vee G$ is uniquely determined by τ, we can rewrite (3.3.9) as

$$\sum_{\sigma\in\vee E} Q(t)^{\vee E}_\lambda = \sum_{\mu\in\vee G}\Big(Q(t)^{\vee G}_\mu\Big(t_\mu t^*_\mu - \sum_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} t_{\mu\sigma}t^*_{\mu\sigma}\Big) + \Big(\sum_{\substack{\tau\in\vee E\setminus\vee G\\ \xi_\tau=\mu}} Q(t)^{\vee E}_\tau\Big)\Big).$$

Lemma 3.3.10 now allows us to replace each $Q(t)^{\vee E}_\tau$ with $Q(t)^{\vee G}_{\xi_\tau}t_\tau t^*_\tau$, yielding

$$\begin{aligned}\sum_{\sigma\in\vee E} Q(t)^{\vee E}_\sigma &= \sum_{\mu\in\vee G}\Big(Q(t)^{\vee G}_\mu\Big(t_\mu t^*_\mu - \sum_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} t_{\mu\sigma}t^*_{\mu\sigma}\Big) + \Big(\sum_{\substack{\tau\in\vee E\setminus\vee G\\ \xi_\tau=\mu}} Q(t)^{\vee G}_{\xi_\tau}t_\tau t^*_\tau\Big)\Big)\\ &= \sum_{\mu\in\vee G}\Big(Q(t)^{\vee G}_\mu\Big(t_\mu t^*_\mu - \sum_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} t_{\mu\sigma}t^*_{\mu\sigma}\Big) + \Big(\sum_{\substack{\tau\in\vee E\setminus\vee G\\ \xi_\tau=\mu}} Q(t)^{\vee G}_\mu t_\tau t^*_\tau\Big)\Big)\\ &= \sum_{\mu\in\vee G} Q(t)^{\vee G}_\mu\Big(\Big(t_\mu t^*_\mu - \sum_{\substack{\mu\sigma\in\vee E\setminus\vee G\\ \xi_{\mu\sigma}=\mu}} t_{\mu\sigma}t^*_{\mu\sigma}\Big) + \Big(\sum_{\substack{\tau\in\vee E\setminus\vee G\\ \xi_\tau=\mu}} t_\tau t^*_\tau\Big)\Big)\\ &= \sum_{\mu\in\vee G} Q(t)^{\vee G}_\mu\end{aligned}$$

and this last is equal to t_v by the inductive hypothesis. $\square$

We can use Proposition 3.3.3 to express the range projections associated to paths in E as sums of the projections $Q(t)^E_\lambda$.

Corollary 3.3.11. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Suppose that $E \subset \Lambda$ is finite and that $\lambda, \mu \in E$ implies* $\mathrm{MCE}(\lambda,\mu) \subset E$. *For each $\mu \in E$, we have*

$$t_\mu t_\mu^* = \sum_{\mu\nu \in E} Q(t)^E_{\mu\nu}.$$

Proof. Let $v := r(\mu)$. By Proposition 3.3.3, we have

$$t_\mu t_\mu^* = t_\mu t_\mu^* \Big(\prod_{\lambda \in vE} (t_v - t_\lambda t_\lambda^*) + \sum_{\lambda \in vE} Q(t)^E_\lambda \Big).$$

Since $\mu\nu \in E$ implies $t_\mu t_\mu^* \geq Q(t)^E_{\mu\nu}$ by definition, we need only show that

(1) $t_\mu t_\mu^* \prod_{\lambda \in vE} (t_v - t_\lambda t_\lambda^*) = 0$; and
(2) $t_\mu t_\mu^* Q(t)^E_\lambda = 0$ for all $\lambda \in E \setminus \mu\Lambda$.

Item (1) above is immediate because $\mu \in vE$ and so

$$t_\mu t_\mu^* \prod_{\lambda \in vE} (t_v - t_\lambda t_\lambda^*) = t_\mu t_\mu^* (t_v - t_\mu t_\mu^*) \prod_{\lambda \in vE \setminus \{\mu\}} (t_v - t_\lambda t_\lambda^*) = 0.$$

For (2), fix $\sigma \in E \setminus \mu\Lambda$. If $\mathrm{MCE}(\mu,\sigma) = \emptyset$, then $t_\mu t_\mu^* Q(t)^E_\sigma \leq t_\mu t_\mu^* t_\sigma t_\sigma^* = 0$ by Lemma 3.1.2(3). On the other hand, if $\mathrm{MCE}(\mu,\sigma) \neq 0$, then $r(\sigma) = r(\mu) = v$, and

$$t_\mu t_\mu^* Q(t)^E_\sigma = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\sigma)} \Big(t_{\sigma\beta} t_{\sigma\beta}^* \prod_{\substack{\sigma\nu \in E \\ d(\nu) > 0}} (t_\sigma t_\sigma^* - t_{\sigma\nu} t_{\sigma\nu}^*) \Big). \tag{3.3.10}$$

Fix $(\alpha,\beta) \in \Lambda^{\min}(\mu,\sigma)$. Since E is closed under minimal common extensions we have $\sigma\beta \in E$. Furthermore, since $\sigma \notin \mu\Lambda$, we have $d(\beta) > 0$. It follows that

$$t_{\sigma\beta} t_{\sigma\beta}^* \prod_{\substack{\sigma\nu \in E \\ d(\nu) > 0}} (t_\sigma t_\sigma^* - t_{\sigma\nu} t_{\sigma\nu}^*) = t_{\sigma\beta} t_{\sigma\beta}^* (t_\sigma t_\sigma^* - t_{\sigma\beta} t_{\sigma\beta}^*) \prod_{\substack{\sigma\nu \in E \setminus \{\sigma\beta\} \\ d(\nu) > 0}} (t_\sigma t_\sigma^* - t_{\sigma\nu} t_{\sigma\nu}^*) = 0,$$

and applying this to each term in (3.3.10) gives (2). □

3.4. Finite dimensional subalgebras of the core

The material in this section is a modification of joint work with Raeburn and Yeend; it represents an expanded version of [**30**, Section 3] up to and including [**30**, Remark 3.4].

For a finitely aligned k-graph (Λ, d) and a Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$, we want to show that $\overline{\mathrm{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda, d(\lambda) = d(\mu)\}$ is an AF algebra by describing it explicitly as the closure of an increasing union of finite-dimensional subalgebras. Furthermore, we want to do this in such a way that the nonzero matrix units in these finite-dimensional subalgebras can be written explicitly in terms of the $t_\lambda t_\mu^*$. In this section, we achieve the first of these goals:

PROPOSITION 3.4.1. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Then $\overline{\operatorname{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\}$ is AF.*

We prove Proposition 3.4.1 at the end of this section. To do so we follow the procedure developed in [**30**, §3]. The idea is to show that given a finite subset E of Λ, the algebra $C^*(\{t_\lambda t_\mu^* : \lambda, \mu \in E, d(\lambda) = d(\mu)\})$ is finite dimensional. The immediate problem is that $\operatorname{span}\{t_\lambda t_\mu^* : \lambda, \mu \in E, d(\lambda) = d(\mu)\}$ is typically not closed under multiplication. Hence the first step is to find a finite set ΠE such that $E \subset \Pi E$, and $\operatorname{span}\{t_\lambda t_\mu^* : \lambda, \mu \in \Pi E, d(\lambda) = d(\mu)\}$ *is* closed under multiplication.

LEMMA 3.4.2. *Let (Λ, d) be a finitely aligned k-graph, and let $E \subset \Lambda$ be finite. Then there exists a set $G \subset \Lambda$ such that $E \subset G$, $\bigvee_{\lambda \in G} d(\lambda) = \bigvee_{\lambda \in E} d(\lambda)$, and*

(3.4.1) $\lambda, \mu, \sigma \in G$ *with* $d(\lambda) = d(\mu)$ *and* $s(\lambda) = s(\mu)$ *implies* $\lambda \operatorname{Ext}(\mu; \{\sigma\}) \subset G$.

PROOF. We prove the lemma by constructing such a set G directly. To do this, we define a map I_Π on subsets of Λ as follows: given a subset E of Λ, let

$$(3.4.2)\quad \begin{aligned} I_\Pi(E) := \big\{\lambda_1(0, d(\lambda_1))\lambda_2(d(\lambda_1), d(\lambda_2)) \dots \lambda_j(d(\lambda_{j-1}), d(\lambda_j)) : j \in \mathbb{N}, \\ \lambda_i \in \vee E, d(\lambda_i) \le d(\lambda_{i+1}), s(\lambda_i) = \lambda_{i+1}(d(\lambda_i)) \text{ for } 1 \le i \le j\big\}. \end{aligned}$$

Write I_Π^n for the n-fold self-composition of I_Π. That is

$$I_\Pi^n = \overbrace{I_\Pi \circ \cdots \circ I_\Pi}^{n \text{ terms}}.$$

We claim that

(a) $E \subset I_\Pi(E)$ for all E;
(b) $\bigvee_{\lambda \in I_\Pi(E)} d(\lambda) = \bigvee_{\lambda \in E} d(\lambda)$ for all finite E;
(c) If E is finite then $I_\Pi(E)$ is finite;
(d) if $\lambda, \mu, \sigma \in E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$, then $\lambda \operatorname{Ext}(\mu; \{\sigma\}) \subset I_\Pi(E)$ for all E; and
(e) for each finite $E \subset \Lambda$, there exists $N \in \mathbb{N}$ such that $I_\Pi^N(E) = I_\Pi^{N+1}(E)$.

For (a), just take $j = 1$ in (3.4.2).

For claim (b) observe that each $\lambda \in I_\Pi(E)$ has degree $d(\lambda_j)$ for some $\lambda_j \in \vee E$, and that each $\lambda_j \in \vee E$ has degree less than or equal to $\bigvee_{\mu \in E} d(\mu)$ by Lemma 3.3.7(4), giving $\bigvee_{\lambda \in I_\Pi(E)} d(\lambda) \le \bigvee_{\lambda \in E} d(\lambda)$. Claim (a) gives the reverse inequality.

To see (c), suppose that E is finite. Then $\vee E$ is finite by statement (2) of Lemma 3.3.7. Fix $N \le \bigvee_{\lambda \in E} d(\lambda)$ and suppose that $0 \le n < N$. For $\lambda \in I_\Pi(E)$ with $d(\lambda) = N$, we have that if $n + e_j \le N$, then $\lambda(n, n + e_j) = \lambda_i(n, n + e_j)$

for some $\lambda_i \in \vee E$ by definition of I_Π. Hence for any path $\lambda \in I_\Pi(E)$ with $d(\lambda) = N$, there are at most $|\vee E|$ possible choices for the segment $\lambda(n, n+e_j)$ whenever $0 \le n \le N - e_j$. Since there are $|N|$ possible choices of n such that $0 \le n \le N - e_j$ for some j, there can be at most $|\vee E|^{|N|}$ distinct paths in $I_\Pi(E) \cap \Lambda^N$. Since $\mu \in I_\Pi(E)$ implies $d(\mu) \le \bigvee_{\lambda \in E} d(\lambda)$ by part (b), it follows that $|I_\Pi(E)| \le \sum_{N \le \bigvee_{\lambda\in E} d(\lambda)} |\vee E|^{|N|} < \infty$.

To check claim (d), let $\lambda, \mu, \sigma \in E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$. Then $\mu\mathrm{Ext}(\mu; \{\sigma\}) = \mathrm{MCE}(\mu, \sigma) \subset \vee E$ by definition of $\vee E$. Hence for each $\alpha \in \mathrm{Ext}(\mu; \{\sigma\})$, taking $j = 2$, $\lambda_1 = \lambda$, and $\lambda_2 = \mu\alpha$ in (3.4.2) shows that $\lambda\alpha \in I_\Pi(E)$ as required.

Finally, we must check (e). For this we first show that if $I^2_\Pi(E) \neq I_\Pi(E)$, then

$$\min\{|d(\lambda)| : \lambda \in I^2_\Pi(E) \setminus I_\Pi(E)\} > \min\{|d(\mu)| : \mu \in I_\Pi(E) \setminus E\}. \tag{3.4.3}$$

Suppose that $I^2_\Pi(E) \neq I_\Pi(E)$. It follows that $I_\Pi(E) \neq E$, so both $\min\{|d(\lambda)| : \lambda \in I^2_\Pi(E) \setminus I_\Pi(E)\}$ and $\min\{|d(\mu)| : \mu \in I_\Pi(E) \setminus E\}$ exist. Let $\lambda \in I^2_\Pi(E) \setminus I_\Pi(E)$. By definition, we have

$$\lambda = \lambda_1(0, d(\lambda_1)) \dots \lambda_j(d(\lambda_{j-1}), d(\lambda_j))$$

where $\lambda_1, \dots, \lambda_j$ all belong to $\vee I_\Pi(E)$. First notice that at least one λ_i must belong to $\vee I_\Pi(E) \setminus I_\Pi(E)$, for if not, then each λ_i can be written as

$$\lambda_i = \lambda_{i,1}(0, d(\lambda_{i,1})) \dots \lambda_{i,h_i}(d(\lambda_{i,h_i - 1}), d(\lambda_{i,h_i})),$$

where each $\lambda_{i,l} \in \vee E$, and then

$$\lambda = \lambda_{1,1}(0, d(\lambda_{1,1})) \dots \lambda_{j,h_j}(d(\lambda_{j,h_j - 1}), d(\lambda_{j,h_j}))$$

belongs to $I_\Pi(E)$, contradicting $\lambda \in I^2_\Pi(E) \setminus I_\Pi(E)$. So fix $i \le j$ such that $\lambda_i \in \vee I_\Pi(E) \setminus I_\Pi(E)$. Then $\lambda_i \in \mathrm{MCE}(F)$ for some $F \subset I_\Pi(E)$, and $d(\lambda) > d(\sigma)$ for all $\sigma \in F$. If $F \subset E$, then $\lambda_i \in \mathrm{MCE}(F) \subset \vee E \subset I_\Pi(E)$, so there must exist $\sigma \in F \setminus E \subset I_\Pi(E) \setminus E$. Since $\sigma \in F$, we have $d(\lambda) > d(\sigma)$, and so $|d(\lambda)| > |d(\sigma)|$, giving

$$|d(\lambda)| > |d(\sigma)| \ge \min_{\mu \in I_\Pi(E)\setminus E} |d(\mu)|.$$

Since $\lambda \in I^2_\Pi(E) \setminus I_\Pi(E)$ was arbitrary, this establishes (3.4.3).

To establish (e), we now notice that (b) ensures that

$$\min\{|d(\lambda)| : \lambda \in I^{n+1}_\Pi(E) \setminus I^n_\Pi(E)\} \le \max\{|d(\lambda)| : \lambda \in I^{n+1}_\Pi(E)\} \le \big|\textstyle\bigvee_{\mu\in E} d(\mu)\big|$$

for all $n > 0$. Hence (3.4.3) ensures that for some $N \le |\bigvee_{\mu \in E} d(\mu)|$, we must have $I^{N+1}_\Pi(E) \setminus I^N_\Pi(E) = \emptyset$. That is, $I^{N+1}_\Pi(E) = I^N_\Pi(E)$ as required. This establishes (e).

Setting $G := I_\Pi^N(E)$, we have G finite by (c), $E \subset G$ by (a),

$$\bigvee_{\lambda \in G} d(\lambda) = \bigvee_{\lambda \in I_\Pi^{N-1}(E)} d(\lambda) = \ldots = \bigvee_{\lambda \in E} d(\lambda)$$

by N applications of (b), and G satisfies (3.4.1) by (d). □

REMARK 3.4.3.

1. If G satisfies (3.4.1), then G is closed under taking minimal common extensions. To see this, just take $\lambda = \mu$ in (3.4.1).
2. Suppose that G satisfies (3.4.1). If λ and μ belong to ΠE with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$, and if $\nu \in s(\lambda)\Lambda$, then $\lambda\nu \in \Pi E$ if and only if $\mu\nu \in \Pi E$. To see this, note that setting $\sigma = \mu\nu$ in (3.4.1) shows that $\mu\nu \in \Pi E$ implies $\lambda\nu \in \Pi E$, and the reverse implication follows by symmetry.
3. Condition (3.4.1) is equivalent to the condition that if $\lambda, \mu, \sigma \in G$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$, and if $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$, then $\lambda\alpha$ belongs to G.

DEFINITION 3.4.4. Let (Λ, d) be a finitely aligned k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family, and suppose that $G \subset \Lambda$ satisfies (3.4.1). Define $M_G^t := \operatorname{span}\{t_\lambda t_\mu^* : \lambda, \mu \in G, d(\lambda) = d(\mu)\}$.

LEMMA 3.4.5. *Let (Λ, d) be a finitely aligned k-graph and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. If $G \subset \Lambda$ satisfies (3.4.1) then M_G^t is closed under multiplication and taking adjoints. In particular, if G is finite, then M_G^t is a finite-dimensional C^*-subalgebra of $C^*(\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\})$.*

PROOF. It suffices to show that M_G^t is closed under taking adjoints and under multiplication. That is, we must show that for λ, μ, σ and τ in G with $d(\lambda) = d(\mu)$, $s(\lambda) = s(\mu)$, $d(\sigma) = d(\tau)$ and $s(\sigma) = s(\tau)$, we have

1. $(t_\lambda t_\mu^*)^* \in M_G^t$; and
2. $t_\lambda t_\mu^* t_\sigma t_\tau^* \in M_G^t$.

It is easy to check (1) because $(t_\lambda t_\mu^*)^* = t_\mu t_\lambda^*$ is another of the generators of M_G^t. To check (2), we use (TCK3) and (TCK2) to calculate

$$t_\lambda t_\mu^* t_\sigma t_\tau^* = t_\lambda \Big(\sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\sigma)} t_\alpha t_\beta^* \Big) t_\tau^* = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\sigma)} t_{\lambda\alpha} t_{\tau\beta}^*.$$

Suppose $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$. We have $d(\lambda\alpha) = d(\mu\alpha) = d(\sigma\beta) = d(\tau\beta)$ because $d(\lambda) = d(\mu)$ and $d(\sigma) = d(\tau)$. Furthermore $\lambda\alpha$ and $\tau\beta$ belong to G by Remark 3.4.3(3). Hence $t_{\lambda\alpha} t_{\tau\beta}^*$ is a generator of M_G^t. Since $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$ was arbitrary, it follows that $t_\lambda t_\mu^* t_\sigma t_\tau^* \in M_G^t$ as required. □

Definition 3.4.6. Let (Λ, d) be a finitely aligned k-graph, and let E be a finite subset of Λ. Define $\Pi E := \bigcap\{G \subset \Lambda : E \subset G \text{ and } G \text{ satisfies (3.4.1)}\}$.

Lemma 3.4.7. *Let (Λ, d) be a finitely aligned k-graph and let E be a finite subset of Λ. Then ΠE is finite, $E \subset \Pi E$, $\bigvee_{\lambda\in\Pi E} d(\lambda) = \bigvee_{\lambda\in E} d(\lambda)$, and ΠE satisfies (3.4.1). In particular, $M^t_{\Pi E}$ is a finite-dimensional C^*-subalgebra of $C^*(\Lambda)$.*

Proof. We have $E \subset \Pi E$ by definition, and it is an immediate corollary of Lemma 3.4.2 that ΠE is finite whenever E is. To see that $\bigvee_{\lambda\in\Pi E} d(\lambda) = \bigvee_{\lambda\in E} d(\lambda)$, notice that we have $\bigvee_{\lambda\in\Pi E} d(\lambda) \geq \bigvee_{\lambda\in E} d(\lambda)$ because $E \subset \Pi E$, and on the other hand, $\bigvee_{\lambda\in\Pi E} d(\lambda) \leq \bigvee_{\lambda\in E} d(\lambda)$ because Lemma 3.4.2 ensures that at least one of the sets G in the intersection in Definition 3.4.6 satisfies $\bigvee_{\lambda\in G} d(\lambda) = \bigvee_{\lambda\in E} d(\lambda)$.

It remains to show that ΠE satisfies (3.4.1). To see this, suppose that λ, μ and σ belong to ΠE and that $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$. Fix G containing E and satisfying (3.4.1). Since $\lambda, \mu, \sigma \in \Pi E$, we know that λ, μ and σ belong to G, and hence $\lambda\mathrm{Ext}(\mu;\sigma) \subset G$ because G satisfies (3.4.1). Since G was chosen arbitrarily from amongst all sets containing E and satisfying (3.4.1), it follows that

$$\lambda\mathrm{Ext}(\mu;\sigma) \subset \bigcap\{G \subset \Lambda : E \subset G \text{ and } G \text{ satisfies (3.4.1)}\}$$

which is equal to ΠE by definition. □

We are now ready to prove the main result of this section.

Proof of Proposition 3.4.1. Let $a_1, \dots a_n \in \overline{\mathrm{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\}$ and let $\varepsilon > 0$. By Lemma 3.1.2(5), there exist finite linear combinations $a_i' = \sum_{\lambda,\mu\in E_i, d(\lambda)=d(\mu)} a^i_{\lambda,\mu} t_\lambda t_\mu^*$ such that $\|a_i - a_i'\| \leq \varepsilon$ for $1 \leq i \leq n$. Let $E := \bigcup_{i=1}^n E_i$. Then $a_1', \dots a_n' \in M^t_{\Pi E}$, which is finite-dimensional. It follows from [**4**, Theorem 2.2] that $\overline{\mathrm{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\}$ is AF. □

3.5. Identifying matrix units

The material in this section is a modification of joint work with Raeburn and Yeend; it represents an expanded and slightly generalised version of [**30**, Section 3] from [**30**, Definition 3.8] to [**30**, Lemma 3.16].

In this section we identify a collection of nonzero matrix units for each $M^t_{\Pi E}$ so that we can decide precisely when two Toeplitz-Cuntz-Krieger families $\{t_\lambda : \lambda \in \Lambda\}$ and $\{t'_\lambda : \lambda \in \Lambda\}$ satisfy

$$\overline{\mathrm{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\} \cong \overline{\mathrm{span}}\{t'_\lambda {t'_\mu}^* : d(\lambda) = d(\mu)\}.$$

DEFINITION 3.5.1. Let (Λ, d) be a finitely aligned k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family, and let $E \subset \Lambda$ be finite. Define partial isometries

$$\Theta(t)^{\Pi E}_{\lambda,\mu} := Q(t)^{\Pi E}_{\lambda} t_\lambda t^*_\mu.$$

for all $\lambda, \mu \in \Pi E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$.

The idea is to show that these $\Theta(t)^{\Pi E}_{\lambda,\mu}$ are matrix units for $M^t_{\Pi E}$, and to provide a criterion for deciding precisely which of them are nonzero.

NOTATION 3.5.2. We write $\Pi E \times_{d,s} \Pi E$ for the collection

$$\{(\lambda, \mu) \in \Pi E \times \Pi E : d(\lambda) = d(\mu), s(\lambda) = s(\mu)\}$$

of pairs of paths in ΠE to which matrix units are associated.

The goal of this section is to prove the following proposition:

PROPOSITION 3.5.3. *Let (Λ, d) be a finitely aligned k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family, and let $E \subset \Lambda$ be finite. Then $\{\Theta(t)^{\Pi E}_{\lambda,\mu} : (\lambda, \mu) \in \Pi E \times_{d,s} \Pi E\}$ is a collection of matrix units for $M^t_{\Pi E}$. Suppose that $t_v \neq 0$ for every $v \in \Lambda^0$, and let $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$. If $\{\nu \in s(\lambda) \setminus \Lambda^0 : \lambda\nu \in \Pi E\}$ is not exhaustive, then $\Theta(t)^{\Pi E}_{\lambda,\mu}$ is nonzero. Otherwise, $\Theta(t)^{\Pi E}_{\lambda,\mu} = 0$ if and only if*

$$\prod_{\lambda\nu \in \Pi E,\, d(\nu) > 0} (t_{s(\lambda)} - t_\nu t^*_\nu) = 0.$$

To prove this result, we need to establish a number of technical results.

LEMMA 3.5.4. *Let (Λ, d) be a finitely aligned k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family, and let $E \subset \Lambda$ be finite. If $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, then*

$$\Theta(t)^{\Pi E}_{\lambda,\mu} = t_\lambda \Big(\prod_{\lambda\nu \in \Pi E,\, d(\nu) > 0} (t_{s(\lambda)} - t_\nu t^*_\nu) \Big) t^*_\mu = t_\lambda t^*_\mu Q(t)^{\Pi E}_{\mu}. \tag{3.5.1}$$

PROOF. Note first that it suffices to establish the first equality in (3.5.1), because the second then follows by taking adjoints in the first and then applying Remark 3.4.3(2). To establish the first equality, fix $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$ and calculate

$$\begin{aligned} \Theta(t)^{\Pi E}_{\lambda,\mu} &= Q(t)^{\Pi E}_{\lambda} t_\lambda t^*_\mu \\ &= t_\lambda t^*_\lambda \Big(\prod_{\lambda\nu \in \Pi E,\, d(\nu) > 0} (t_\lambda t^*_\lambda - t_{\lambda\nu} t^*_{\lambda\nu}) \Big) t_\lambda t^*_\mu \\ &= t_\lambda t^*_\lambda \Big(\prod_{\lambda\nu \in \Pi E,\, d(\nu) > 0} t_\lambda (t_{s(\lambda)} - t_\nu t^*_\nu) t^*_\lambda \Big) t_\lambda t^*_\mu. \end{aligned} \tag{3.5.2}$$

But $t_\lambda^* t_\lambda = t_{s(\lambda)} \geq (t_{s(\lambda)} - t_\nu t_\nu^*)$ for all $\nu \in s(\lambda)\Lambda$, so the $t_\lambda^* t_\lambda$ which occur between factors in (3.5.2) can be deleted, giving

$$\begin{aligned}\Theta(t)^{\Pi E}_{\lambda,\mu} &= t_\lambda t_\lambda^* t_\lambda \Big(\prod_{\lambda\nu \in \Pi E,\, d(\nu)>0} (t_{s(\lambda)} - t_\nu t_\nu^*) \Big) t_\lambda^* t_\lambda t_\mu^* \\ &= t_\lambda \Big(\prod_{\lambda\nu \in \Pi E,\, d(\nu)>0} (t_{s(\lambda)} - t_\nu t_\nu^*) \Big) t_\mu^*. \end{aligned}$$

□

REMARK 3.5.5. Notice that for $\lambda \in \Pi E$,

$$\Theta(t)^{\Pi E}_{\lambda,\lambda} = t_\lambda t_\lambda^* Q(t)^{\Pi E}_\lambda = Q(t)^{\Pi E}_\lambda \tag{3.5.3}$$

since $Q(t)^{\Pi E}_\lambda \leq t_\lambda t_\lambda^*$ by definition. We therefore obtain

$$Q(t)^{\Pi E}_\lambda = t_\lambda \Big(\prod_{\lambda\nu \in \Pi E, d(\nu)>0} (t_{s(\lambda)} - t_\nu t_\nu^*) \Big) t_\lambda^*$$

by combining (3.5.3) with Lemma 3.5.4.

LEMMA 3.5.6. *Let (Λ, d) be a finitely aligned k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family, and let E be a finite subset of Λ. Then*

(1) $\big(\Theta(t)^{\Pi E}_{\lambda,\mu}\big)^* = \Theta(t)^{\Pi E}_{\mu,\lambda}$; *and*
(2) $\Theta(t)^{\Pi E}_{\lambda,\mu} \Theta(t)^{\Pi E}_{\sigma,\tau} = \delta_{\mu\sigma} \Theta(t)^{\Pi E}_{\lambda,\tau}$

for all $(\lambda, \mu), (\sigma, \tau) \in \Pi E \times_{d,s} \Pi E$.

PROOF. To see (1), just note that

$$\big(\Theta(t)^{\Pi E}_{\lambda,\mu}\big)^* = \big(Q(t)^{\Pi E}_\lambda t_\lambda t_\mu^*\big)^* = t_\mu t_\lambda^* \left(Q(t)^{\Pi E}_\lambda\right)^* = t_\mu t_\lambda^* Q(t)^{\Pi E}_\lambda = \Theta(t)^{\Pi E}_{\mu,\lambda}$$

by Lemma 3.5.4.

For (2), note that

$$\Theta(t)^{\Pi E}_{\lambda,\mu} \Theta(t)^{\Pi E}_{\sigma,\tau} = t_\lambda t_\mu^* Q(t)^{\Pi E}_\mu Q(t)^{\Pi E}_\sigma t_\sigma t_\tau^*$$

by Lemma 3.5.4 applied to $\Theta(t)^{\Pi E}_{\lambda,\mu}$, and the definition of $\Theta(t)^{\Pi E}_{\sigma,\tau}$. Proposition 3.3.3 ensures that $Q(t)^{\Pi E}_\mu Q(t)^{\Pi E}_\sigma = \delta_{\mu\sigma} Q(t)^{\Pi E}_\mu$, and hence

$$\begin{aligned}\Theta(t)^{\Pi E}_{\lambda,\mu} \Theta(t)^{\Pi E}_{\sigma,\tau} &= t_\lambda t_\mu^* \delta_{\mu,\sigma} Q(t)^{\Pi E}_\mu t_\sigma t_\tau^* \\ &= \delta_{\mu,\sigma} (t_\lambda t_\mu^* Q(t)^{\Pi E}_\mu) t_\mu t_\tau^* \\ &= \delta_{\mu,\sigma} Q(t)^{\Pi E}_\lambda t_\lambda t_\mu^* t_\mu t_\tau^* \quad \text{by Lemma 3.5.4} \\ &= \delta_{\mu,\sigma} Q(t)^{\Pi E}_\lambda t_\lambda t_\tau^* \quad \text{by Lemma 3.1.2(3)} \\ &= \delta_{\mu,\sigma} \Theta(t)^{\Pi E}_{\lambda,\tau} \quad \text{by definition.} \end{aligned}$$

□

We now show that the partial isometries $\Theta(t)^{\Pi E}_{\lambda,\mu}$ span the finite-dimensional C^*-algebra $M^t_{\Pi E}$.

LEMMA 3.5.7. *Let (Λ, d) be a finitely aligned k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family, and let $E \subset \Lambda$ be finite. Suppose $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$. Then*

$$t_\lambda t_\mu^* = \sum_{\lambda\nu \in \Pi E} \Theta(t)^{\Pi E}_{\lambda\nu,\mu\nu}.$$

PROOF. We have $t_\lambda t_\mu^* = t_\lambda t_\mu^* t_\mu t_\mu^*$, which in turn is equal to $t_\lambda t_\mu^* \sum_{\mu\nu \in \Pi E} Q(t)^{\Pi E}_{\mu\nu}$ by Corollary 3.3.11. Applying Remark 3.5.5 then gives

$$\begin{aligned} t_\lambda t_\mu^* &= \sum_{\mu\nu \in \Pi E} t_\lambda t_\mu^* t_{\mu\nu} \Big(\prod_{\mu\nu\nu' \in \Pi E, d(\nu')>0} (t_{s(\nu)} - t_{\nu'} t_{\nu'}^*) \Big) t_{\mu\nu}^* \\ &= \sum_{\mu\nu \in \Pi E} t_{\lambda\nu} \Big(\prod_{\mu\nu\nu' \in \Pi E, d(\nu')>0} (t_{s(\nu)} - t_{\nu'} t_{\nu'}^*) \Big) t_{\mu\nu}^*, \end{aligned}$$

and Remark 3.4.3(2) and Lemma 3.5.4 combine to show that the last line is equal to $\sum_{\lambda\nu \in \Pi E} \Theta(t)^{\Pi E}_{\lambda\nu,\mu\nu}$. □

PROOF OF PROPOSITION 3.5.3. By Lemmas 3.5.6 and 3.5.7, we know that the $\Theta(t)^{\Pi E}_{\lambda,\mu}$ are matrix units which span $M^t_{\Pi E}$.

Now suppose that $t_v \neq 0$ for all $v \in \Lambda^0$. If $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$ and $T(\lambda) := \{\nu \in s(\lambda)\Lambda \setminus \Lambda^0 : \lambda\nu \in \Pi E\}$ is not exhaustive, then there exists $\xi \in s(\lambda)\Lambda$ such that $\Lambda^{\min}(\xi, \nu) = \emptyset$ for all $\nu \in T(\lambda)$. It follows that for each $\nu \in T(\lambda)$,

$$t_{\lambda\xi} t_{\lambda\xi}^* (t_\lambda t_\lambda^* - t_{\lambda\nu} t_{\lambda\nu}^*) = t_{\lambda\xi} t_{\lambda\xi}^* - \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda\xi, \lambda\nu)} t_{\lambda\xi\alpha} t_{\lambda\xi\alpha}^* = t_{\lambda\xi} t_{\lambda\xi}^*$$

since $\Lambda^{\min}(\lambda\xi, \lambda\nu) = \Lambda^{\min}(\xi, \nu) = \emptyset$ by choice of ξ. Hence

$$t_{\lambda\xi} t_{\lambda\xi}^* \Theta(t)^{\Pi E}_{\lambda,\mu} = t_{\lambda\xi} t_{\lambda\xi}^* \Big(\prod_{\lambda\nu \in \Pi E, d(\nu)>0} (t_\lambda t_\lambda^* - t_{\lambda\nu} t_{\lambda\nu}^*) \Big) t_\lambda t_\mu^* = t_{\lambda\xi} t_{\mu\xi}^*.$$

Since $t_{\lambda\xi} t_{\mu\xi}^* t_{\mu\xi} = t_{\lambda\xi} \neq 0$ by Lemma 3.1.2(6), we have $t_{\lambda\xi} t_{\mu\xi}^* \neq 0$, and it follows that $\Theta(t)^{\Pi E}_{\lambda,\mu} \neq 0$ as required.

If $T(\lambda)$ is exhaustive, such a ξ does not exist. In this case, we use the expression

$$\Theta(t)^{\Pi E}_{\lambda,\mu} = t_\lambda \Big(\prod_{\lambda\nu \in \Pi E, d(\nu)>0} (t_{s(\lambda)} - t_\nu t_\nu^*) \Big) t_\mu^*$$

from Lemma 3.5.4 to decide whether $\Theta(t)^{\Pi E}_{\lambda,\mu}$ is nonzero:

- if $\Theta(t)^{\Pi E}_{\lambda,\mu} = 0$, then multiplying on the left by t_λ^* and on the right by t_μ gives $\prod_{\nu \in T(\lambda)} (t_{s(\lambda)} - t_\nu t_\nu^*) = 0$.
- if $\prod_{\nu \in T(\lambda)} (t_{s(\lambda)} - t_\nu t_\nu^*) = 0$, then multiplying on the left by t_λ and on the right by t_μ^* gives $\Theta(t)^{\Pi E}_{\lambda,\mu} = 0$.

Hence $\Theta(t)^{\Pi E}_{\lambda,\mu} = 0$ if and only if $\prod_{\nu \in T(\lambda)}(t_{s(\lambda)} - t_\nu t_\nu^*) = 0$. □

THEOREM 3.5.8. *Let (Λ, d) be a finitely aligned k-graph and let $\{t_\lambda : \lambda \in \Lambda\}$ and $\{t'_\lambda : \lambda \in \Lambda\}$ be Toeplitz-Cuntz-Krieger Λ-families such that $t_v \neq 0$ and $t'_v \neq 0$ for all $v \in \Lambda^0$. Then there is an isomorphism*

$$\pi_{t,t'} : \overline{\operatorname{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\} \to \overline{\operatorname{span}}\{t'_\lambda {t'_\mu}^* : d(\lambda) = d(\mu)\}$$

such that $\pi_{t,t'}(t_\lambda t_\mu^*) = t'_\lambda {t'_\mu}^*$ for all $\lambda, \mu \in \Lambda$ with $d(\lambda) = d(\mu)$ if and only if for every $E \in v\,\mathrm{FE}(\Lambda)$

$$\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0 \iff \prod_{\lambda \in E}(t'_{r(E)} - t'_\lambda {t'_\lambda}^*) = 0. \tag{3.5.4}$$

PROOF. The necessity of (3.5.4) is obvious, so we need only show that it is sufficient. Suppose, then, that (3.5.4) holds. Then Proposition 3.5.3 shows that for $E \subset \Lambda$ finite and $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, the matrix unit $\Theta(t)^{\Pi E}_{\lambda,\mu}$ is nonzero if and only if the corresponding matrix unit $\Theta(t')^{\Pi E}_{\lambda,\mu}$ is nonzero. It follows that $\Theta(t)^{\Pi E}_{\lambda,\mu} \mapsto \Theta(t')^{\Pi E}_{\lambda,\mu}$ defines a canonical isomorphism of each $M^t_{\Pi E}$ onto $M^{t'}_{\Pi E}$. The result now follows from [**1**, Lemma 1.3]. □

3.6. Matrix algebra inclusions

In Section 3.4, we demonstrated that if (Λ, d) is a finitely aligned k-graph and $\{t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family, then the C^*-algebra $\overline{\operatorname{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\}$ is the closure of an increasing union of finite-dimensional algebras and is therefore AF. In Section 3.5, we identified nonzero matrix units in the finite dimensional subalgebras $M^t_{\Pi E}$ from which $\overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda, d(\lambda) = d(\mu)\}$ is built, giving us an elementary test for deciding whether two different Toeplitz-Cuntz-Krieger Λ-families produce isomorphic AF cores. However, it is not immediately clear how the finite-dimensional subalgebras $M^t_{\Pi E}$ fit together into a Bratteli diagram. In this section, we make the inclusions explicit.

We begin by showing how $M^t_{\Pi E}$ decomposes as a direct sum of copies of $M_n(\mathbb{C})$.

DEFINITION 3.6.1. Let (Λ, d) be a finitely aligned k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family, let $E \subset \Lambda$ be finite, and suppose that $n \in \mathbb{N}^k$ and $v \in \Lambda^0$ are such that $(\Pi E)v \cap \Lambda^n \neq \emptyset$. Define $M^t_{\Pi E}(n, v)$ to be the subalgebra

$$M^t_{\Pi E}(n, v) := \operatorname{span}\{\Theta(t)^{\Pi E}_{\lambda,\mu} : \lambda, \mu \in (\Pi E)v \cap \Lambda^n\}$$

of $M^t_{\Pi E}$. Define $T^{\Pi E}(n, v)$ to be the set

$$T^{\Pi E}(n, v) := \{\nu \in v\Lambda \setminus \Lambda^0 : \lambda\nu \in \Pi E \text{ for } \lambda \in (\Pi E)v \cap \Lambda^n\}$$

of nontrivial tails which extend paths in $(\Pi E)v \cap \Lambda^n$ to larger elements of ΠE.

LEMMA 3.6.2. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Let $E \subset \Lambda$ be finite. Then*

$$M^t_{\Pi E} = \bigoplus_{\substack{v \in s(\Pi E) \\ n \in d((\Pi E)v)}} M^t_{\Pi E}(n, v).$$

Furthermore, for fixed $v \in s(\Pi E)$ and $n \in d((\Pi E)v)$,

$$M^t_{\Pi E}(n, v) \cong \begin{cases} \{0\} & \text{if } \prod_{\nu \in T^{\Pi E}(n,v)} (t_v - t_\nu t^*_\nu) = 0 \\ M_{(\Pi E)v \cap \Lambda^n}(\mathbb{C}) & \text{otherwise.} \end{cases} \tag{3.6.1}$$

PROOF. The first statement of the lemma is obvious because $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$ implies $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$. For the second statement, fix $v \in s(\Pi E)$, and $n \in d((\Pi E)v)$. If $\prod_{\nu \in T^{\Pi E}(n,v)} (t_v - t_\nu t^*_\nu) = 0$, then Proposition 3.5.3 implies that $\Theta(t)^{\Pi E}_{\lambda,\mu} = 0$ for all $\lambda, \mu \in (\Pi E)v \cap \Lambda^n$. Hence if $\prod_{\nu \in T^{\Pi E}(n,v)} (t_v - t_\nu t^*_\nu) = 0$, then $M^t_{\Pi E}(n, v) = \{0\}$. On the other hand, if $\prod_{\nu \in T^{\Pi E}(n,v)} (t_v - t_\nu t^*_\nu) \neq 0$, then Proposition 3.5.3 implies that $\{\Theta(t)^{\Pi E}_{\lambda,\mu} : \lambda, \mu \in (\Pi E)v \cap \Lambda^n\}$ is a family of nonzero matrix units which span $M^t_{\Pi E}(n, v)$ by definition. □

Next we need to show how to express the inclusion map $M^t_{\Pi E} \hookrightarrow M^t_{\Pi G}$ where $E \subset G$ in terms of the matrix units $\Theta(t)^{\Pi E}_{\lambda,\mu}$ and $\Theta(t)^{\Pi G}_{\sigma,\tau}$.

LEMMA 3.6.3. *Let (Λ, d) be a finitely aligned k-graph, and let E, G be finite subsets of Λ with $E \subset G$. Suppose that $\lambda \in \Pi G \setminus \Pi E$ and that there exists $n \leq d(\lambda)$ such that $\lambda(0, n) \in \Pi E$. Then there is a unique path $\iota_\lambda \in \Pi E$ such that*

(1) *$\lambda(0, d(\iota_\lambda)) = \iota_\lambda$; and*
(2) *if $\mu \in \Pi E$ and $\lambda(0, d(\mu)) = \mu$, then $d(\mu) \leq d(\iota_\lambda)$.*

PROOF. Let $N := \bigvee\{n \leq d(\lambda) : \lambda(0, n) \in \Pi E\}$ and let $\iota_\lambda := \lambda(0, N)$. Then $\iota_\lambda \in \Pi E$ because ΠE is closed under taking minimal common extensions by Remark 3.4.3(1), and $\lambda(0, d(\iota_\lambda)) = \iota_\lambda$ by definition. Furthermore, if $\mu \in \Pi E$ and $\lambda(0, d(\mu)) = \mu$, then $d(\iota_\lambda) = N \geq d(\mu)$ by definition. □

DEFINITION 3.6.4. Let (Λ, d) be a finitely aligned k-graph, and let E, G be finite subsets of Λ with $E \subset G$. For those $\lambda \in \Pi G \setminus \Pi E$ such that there exists $n \leq d(\lambda)$ for which $\lambda(0, n) \in \Pi E$, ι_λ is defined by Lemma 3.6.3. For all other $\lambda \in \Pi G$, we define $\iota_\lambda := \lambda$.

Lemma 3.6.5. *Let (Λ, d) be a finitely aligned k-graph, and let E and G be finite subsets of Λ with $E \subset G$. Suppose that $\lambda, \mu \in \Pi E$ with $d(\lambda) = d(\mu)$ and $s(\lambda) = s(\mu)$. Suppose also that $\lambda\nu \in \Pi G$ and that $\iota_{\lambda\nu} = \lambda$. Then $\mu\nu \in \Pi G$ and $\iota_{\mu\nu} = \mu$.*

Proof. We have that $\mu\nu \in \Pi G$ by Remark 3.4.3(2). Suppose for contradiction that $\iota_{\mu\nu} \neq \mu$. Since $(\mu\nu)(0, d(\mu)) = \mu$, and since $\mu \in \Pi E$, Lemma 3.6.3(2) ensures that $\iota_{\mu\nu} = \mu\mu'$ for some $\mu' \in \Lambda \setminus \Lambda^0$. Remark 3.4.3(2) then shows that $\lambda\mu'$ also belongs to ΠE, and since $d(\lambda) = d(\mu)$, we have $d(\lambda\mu') > d(\lambda) = d(\iota_{\lambda\nu})$. Since $\iota_{\mu\nu} = \mu\mu'$, we have $(\mu\nu)(0, d(\mu\mu')) = \mu\mu'$, and the factorisation property then ensures that $\nu = \mu'\nu'$ for some $\nu' \in \Lambda$. It follows from another application of the factorisation property that $(\lambda\nu)(0, d(\lambda\mu')) = \lambda\mu'$. But we now have $\lambda\mu' \in \Pi E$, $d(\lambda\mu') > d(\iota_{\lambda\nu})$, and $(\lambda\nu)(0, d(\lambda\mu')) = \lambda\mu'$, which contradicts Lemma 3.6.3(2). □

Lemma 3.6.6. *Let (Λ, d) be a finitely aligned k-graph, let E, G be finite subsets of Λ with $E \subset G$, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Suppose $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$. Then*

$$\Theta(t)^{\Pi E}_{\lambda,\mu} = \sum_{\lambda\nu \in \Pi G\,,\, \iota_{\lambda\nu} = \lambda} \Theta(t)^{\Pi G}_{\lambda\nu,\mu\nu}.$$

Proof. We begin by showing that

$$\lambda \in \Pi G \text{ and } \mu \in \Pi E \text{ implies } Q(t)^{\Pi E}_\mu Q(t)^{\Pi G}_\lambda = \delta_{\iota_\lambda,\mu} Q(t)^{\Pi G}_\lambda. \tag{3.6.2}$$

First suppose that $\iota_\lambda \neq \mu$. We must show that $Q(t)^{\Pi E}_\mu Q(t)^{\Pi G}_\lambda = 0$. There are two cases:

Case 1: suppose that $\lambda \notin \mu\Lambda$. Then $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ implies $d(\alpha) > 0$. Hence

$$Q(t)^{\Pi E}_\mu Q(t)^{\Pi G}_\lambda \leq \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} t_{\lambda\alpha} t^*_{\lambda\alpha} Q(t)^{\Pi G}_\lambda = 0$$

because for each $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$, $(t_\lambda t^*_\lambda - t_{\lambda\alpha} t^*_{\lambda\alpha})$ is a factor in $Q(t)^{\Pi G}_\lambda$.

Case 2: suppose that $\lambda \in \mu\Lambda$. Since $\iota_\lambda \neq \mu$, we must have $\iota_\lambda = \mu\alpha$ where $d(\alpha) > 0$ by (2) of Lemma 3.6.3. Furthermore, $\iota_\lambda = \mu\alpha \in \Pi E$. So $Q(t)^{\Pi E}_\mu Q(t)^{\Pi G}_\lambda \leq (t_\mu t^*_\mu - t_{\mu\alpha} t^*_{\mu\alpha})(t_\lambda t^*_\lambda) = 0$ since both μ and $\mu\alpha = \iota_\lambda$ are initial segments of λ.

We have now established that $\iota_\lambda \neq \mu$ implies $Q(t)^{\Pi E}_\mu Q(t)^{\Pi G}_\lambda = 0$. Now suppose that $\iota_\lambda = \mu$. Then

$$Q(t)^{\Pi E}_\mu Q(t)^{\Pi G}_\lambda = t_\mu t^*_\mu \prod_{\mu\nu \in \Pi E,\, d(\nu) > 0} (t_\mu t^*_\mu - t_{\mu\nu} t^*_{\mu\nu}) Q(t)^{\Pi G}_\lambda. \tag{3.6.3}$$

Suppose that $\mu\nu \in \Pi E$ with $d(\nu) > 0$. Then since $\iota_\lambda = \mu$, we must have $\lambda \notin \mu\nu\Lambda$, so $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu\nu)$ implies $d(\alpha) > 0$. But then for each $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu\nu)$, we have $(t_\lambda t_\lambda^* - t_{\lambda\alpha} t_{\lambda\alpha}^*)$ a factor in $Q(t)^{\Pi G}_\lambda$, and hence $t_{\lambda\alpha} t_{\lambda\alpha}^* Q(t)^{\Pi G}_\lambda = 0$. It follows that for each $\mu\nu \in \Pi E$ such that $d(\nu) > 0$, we have

$$\begin{aligned}(t_\mu t_\mu^* - t_{\mu\nu} t_{\mu\nu}^*) Q(t)^{\Pi G}_\lambda &= t_\mu t_\mu^* Q(t)^{\Pi G}_\lambda - \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu\nu)} t_{\lambda\alpha} t_{\lambda\alpha}^* Q(t)^{\Pi G}_\lambda \\ &= t_\mu t_\mu^* Q(t)^{\Pi G}_\lambda \\ &= Q(t)^{\Pi G}_\lambda\end{aligned}$$

because $\mu = \iota_\lambda = \lambda(0, d(\iota_\lambda))$, and hence $t_\lambda t_\lambda^* \le t_\mu t_\mu^*$. Applying this reasoning to each factor on the right-hand side of (3.6.3) gives $Q(t)^{\Pi E}_\mu Q(t)^{\Pi G}_\lambda = Q(t)^{\Pi G}_\lambda$ which establishes (3.6.2).

We have that $\Theta(t)^{\Pi E}_{\lambda,\mu} = t_\lambda t_\lambda^* Q(t)^{\Pi E}_\lambda t_\lambda t_\mu^*$ by definition, and an application of Corollary 3.3.11 then gives $\Theta(t)^{\Pi E}_{\lambda,\mu} = \sum_{\lambda\nu\in\Pi G} Q(t)^{\Pi G}_{\lambda\nu} Q(t)^{\Pi E}_\lambda t_\lambda t_\mu^*$. Applying (3.6.2) to each term in this sum, we therefore obtain

$$\Theta(t)^{\Pi E}_{\lambda,\mu} = \sum_{\substack{\lambda\nu\in\Pi G\\ \iota_{\lambda\nu}=\lambda}} Q(t)^{\Pi G}_{\lambda\nu} t_\lambda t_\mu^*. \tag{3.6.4}$$

For $\lambda\nu \in \Pi G$ with $\iota_{\lambda\nu} = \lambda$ we have $Q(t)^{\Pi G}_{\lambda\nu} \le t_{\lambda\nu} t_{\lambda\nu}^*$, and hence $Q(t)^{\Pi G}_{\lambda\nu} t_\lambda t_\mu^* = Q(t)^{\Pi G}_{\lambda\nu} t_{\lambda\nu} t_{\lambda\nu}^* t_\lambda t_\mu^* = Q(t)^{\Pi G}_{\lambda\nu} t_{\lambda\nu} t_{\mu\nu}^*$. It follows from Lemma 3.5.4 that $Q(t)^{\Pi G}_{\lambda\nu} t_\lambda t_\mu^* = \Theta(t)^{\Pi G}_{\lambda\nu,\mu\nu}$ for all $\lambda\nu \in \Pi G$ such that $\iota_{\lambda\nu} = \lambda$. Combining this with (3.6.4) gives

$$\Theta(t)^{\Pi E}_{\lambda,\mu} = \sum_{\substack{\lambda\nu\in\Pi G\\ \iota_{\lambda\nu}=\lambda}} \Theta(t)^{\Pi G}_{\lambda\nu,\mu\nu}$$

as required. □

Definition 3.6.7. Let (Λ, d) be a finitely aligned k-graph. Suppose that E and G are finite subsets of Λ with $E \subset G$. Fix (n, v) such that $M^t_{\Pi E}(n, v)$ is nontrivial. We define $I^G_E(n, v) := \{\nu \in \Lambda : \lambda\nu \in \Pi G \text{ and } \iota_{\lambda\nu} = \lambda \text{ for } \lambda \in (\Pi E)v \cap \Lambda^n\}$.

Lemma 3.6.5 ensures that $I^G_E(n, v)$ is well-defined. For convenience, we will write $I^G_E(\lambda)$ in place of $I^G_E(d(\lambda), s(\lambda))$ for $\lambda \in \Pi E$.

Corollary 3.6.8. *Let (Λ, d) be a finitely aligned k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family, and let E, G be finite subsets of Λ with $E \subset G$. The inclusion map $M^t_{\Pi E} \hookrightarrow M^t_{\Pi G}$ is given by*

$$\Theta(t)^{\Pi E}_{\lambda,\mu} \longrightarrow \bigoplus_{\substack{v\in s(I^G_E(\lambda))\\ n\in d(I^G_E(\lambda)v)}} \sum_{\nu\in I^G_E(\lambda)v\cap\Lambda^n} \Theta(t)^{\Pi G}_{\lambda\nu,\mu\nu}.$$

In particular, for $v \in s(\Pi E)$, $n \in d((\Pi E)v)$, $v' \in s(\Pi G)$ and $n' \in d((\Pi G)v')$ such that $n' \geq n$, the inclusion

$$M^t_{\Pi E}(n, v) \hookrightarrow M^t_{\Pi G}(n', v')$$

has multiplicity $\left|I^G_E(n, v)v' \cap \Lambda^{n'-n}\right|$.

Proof. The result is a direct corollary of Lemmas 3.6.6 and 3.6.2. □

3.7. The path-space representation

In this section we define the path space Λ^* of a k-graph Λ, and define a Toeplitz-Cuntz-Krieger Λ-family $\{T_\lambda : \lambda \in \Lambda\} \subset \mathcal{B}(\ell^2(\Lambda^*))$. We show that the representation $\pi^{\mathcal{T}}_T : \mathcal{T}C^*(\Lambda) \to \mathcal{B}(\ell^2(\Lambda^*))$ determined by $\pi^{\mathcal{T}}_T(s_{\mathcal{T}}(\lambda)) = T_\lambda$ is faithful on $\mathcal{T}C^*(\Lambda)^\gamma$.

Definition 3.7.1. Given k-graphs (Λ_1, d_1) and (Λ_2, d_2), a *graph morphism* $x : (\Lambda_1, d_1) \to (\Lambda_2, d_2)$ is a functor x from the category Λ_1 to the category Λ_2 which respects the degree maps in the sense that $d_2 \circ x = d_1$.

Let (Λ, d) be a finitely aligned k-graph. We define Λ^* to be the collection $\{x : \Omega_{k,m} \to \Lambda : m \in (\mathbb{N} \cup \{\infty\})^k, x \text{ a graph morphism}\}$.

Notice that if $m \in \mathbb{N}^k$ then the graph morphisms $x : \Omega_{k,m} \to \Lambda$ are in bijective correspondence with Λ^m (see [**29**, Examples 2.2(ii)]). When m contains some infinite coordinates, we think of the graph morphisms $x : \Omega_{k,m} \to \Lambda$ as partially-infinite paths in Λ. Extending this analogy, for a graph morphism $x : \Omega_{k,m} \to \Lambda$, we write $r(x)$ for $x(0)$, and we write $d(x)$ for m.

The following simple lemma has been used implicitly to define boundary path space representations in [**18, 29, 30**], but for brevity it was not stated explicitly or proved in any of these papers. The result is not surprising and the proof is straightforward; they are included here only for completeness.

Lemma 3.7.2. *Let (Λ, d) be a finitely aligned k-graph, and let $\{\lambda_i : i \in \mathbb{N}\}$ be a sequence of paths in Λ such that $d(\lambda_{i+1}) \geq d(\lambda_i)$ and $\lambda_{i+1}(0, d(\lambda_i)) = \lambda_i$ for all $i \geq 1$. Define $m \in (\mathbb{N} \cup \{\infty\})^k$ by $m_j := \lim_{i\to\infty} d(\lambda_i)_j \in \mathbb{N} \cup \{\infty\}$ for $1 \leq j \leq k$. Then there is a unique graph morphism $x_\lambda : \Omega_{k,m} \to \Lambda$ such that $x(0, d(\lambda_i)) = \lambda_i$ for all $i \in \mathbb{N}$.*

Proof. To prove existence, we need to define $x_\lambda(n_1, n_2)$ for all $0 \leq n_1 \leq n_2 \leq m$ with $n_i \in \mathbb{N}^k$. For this, fix such a pair n_1, n_2. Since $n_2 \leq m$ there exists $i \in \mathbb{N}$ such that $d(\lambda_i) \geq n_2$. We then define $x_\lambda(n_1, n_2) := \lambda_i(n_1, n_2)$; this is well defined because for $j \geq i$, we have $\lambda_j = \lambda_i\lambda_{i,j}$ for some $\lambda_{i,j} \in \Lambda$, and so $\lambda_j(n_1, n_2) = \big(\lambda_i\lambda_{i,j}\big)(n_1, n_2) = \lambda_i(n_1, n_2)$. It follows by definition that $x_\lambda(0, d(\lambda_i)) = \lambda_i$ for all i. Now we must show that x_λ preserves the degree map and is a functor. To see that x_λ preserves the degree map, just notice that $d(\lambda_i(n_1, n_2)) = n_2 - n_1 = d((n_1, n_2))$ by definition of $\lambda(n_1, n_2)$. To see that x_λ is a functor, let $n_1, n_2, n_3 \in \mathbb{N}^k$ with $n_1 \leq n_2 \leq n_3 \leq m$. Let $i \in \mathbb{N}$ such that $d(\lambda_i) \geq n_3$. By definition, we have $x_\lambda((n_j, n_l)) = \lambda_i(n_j, n_l)$ for $j \leq l$, and so in particular,

$$x_\lambda((n_1, n_2))x_\lambda((n_2, n_3)) = \lambda_i(n_1, n_2)\lambda_i(n_2, n_3) = \lambda_i(n_1, n_3) =: x_\lambda((n_1, n_3)).$$

This establishes existence.

For uniqueness, suppose that $x : \Omega_{k,m} \to \Lambda$ is another such graph morphism, and suppose $n_1, n_2 \in \mathbb{N}^k$ with $0 \leq n_1 \leq n_2 \leq m$. Fix i such that $(\lambda_i) \geq n_2$. Then the morphism $(0, d(\lambda_i)) \in \Omega_{k,m}$ factorises as $(0, d(\lambda_i)) = (0, n_1)(n_1, n_2)(n_2, d(\lambda_i))$. Since x is a functor, it follows that $\lambda_i = x(0, d(\lambda_i)) = x(0, n_1)x(n_1, n_2)x(n_2, d(\lambda_i))$. We have $d(x(0, n_1)) = n_1$, $d(x(n_1, n_2)) = n_2 - n_1$ and $d(x(n_2, d(\lambda_i))) = d(\lambda_i) - n_2$ because x preserves the degree map. But $\lambda_i(0, n_1)$, $\lambda_i(n_1, n_2)$ and $\lambda_i(n_2, d(\lambda_i))$ are the unique morphisms in Λ such that $d(\lambda_i(0, n_1)) = n_1$, $d(\lambda_i(n_1, n_2)) = n_2 - n_1$, $d(\lambda_i(n_2, d(\lambda_i))) = d(\lambda_i) - n_2$, and $\lambda_i = \lambda_i(0, n_1)\lambda_i(n_1, n_2)\lambda_i(n_2, d(\lambda_i))$. It follows that $x(n_1, n_2) = \lambda_i(n_1, n_2) =: x_\lambda(n_1, n_2)$. Since n_1 and n_2 were arbitrary, it follows that $x = x_\lambda$. □

Definition 3.7.3. Let (Λ, d) be a finitely aligned k-graph and let $x \in \Lambda^*$ with $d(x) = m$.

(1) Let $\lambda \in \Lambda$ with $s(\lambda) = r(x)$. For $i \in \mathbb{N}$, define $n_i \in \mathbb{N}^k$ by $(n_i)_j = d(\lambda)_j + \min\{i, m_j\}$ for $1 \leq j \leq k$, $i \in \mathbb{N}$. By Lemma 3.7.2, there is a unique graph morphism $y : \Omega_{k,m+d(\lambda)} \to \Lambda$ such that $y(0, n_i) = \lambda x(0, n_i - d(\lambda))$ for all $i \in \mathbb{N}$. We denote this morphism by λx. We have that $(\lambda x)(d(\lambda), d(\lambda) + n) = x(0, n)$ for all $n \in \mathbb{N}^k$, that $d(\lambda x) = d(\lambda) + d(x)$ and that $r(\lambda x) = r(\lambda)$ by definition.

(2) Let $n \in \mathbb{N}^k$ and let $n' \in (\mathbb{N} \cup \{\infty\})^k$ with $n \leq n' \leq m$. For $i \in \mathbb{N}$, define $n_i \in \mathbb{N}^k$ by $(n_i)_j = \min\{i, (n' - n)_j\}$. By Lemma 3.7.2, there is a unique graph morphism $y : \Omega_{k,n'-n} \to \Lambda$ such that $y(0, n_i) = x(n, n + n_i)$ for all $i \in \mathbb{N}$. In order to distinguish the restricted morphism y from its image in Λ, we denote it by $x|_n^{n'}$. We have that $x|_n^{n'}(0, l) = x(n, n + l)$ for all $l \in \mathbb{N}^k$ with $n + l \leq n'$, that $d(x|_n^{n'}) = n' - n$ and that $r(x|_n^{n'}) = x(n)$.

REMARK 3.7.4. The notation $x|_n^{n'}$ is designedly reminiscent of restriction notation because $x|_n^{n'}$ is precisely the restriction of x to that part of $\Omega_{k,m}$ bounded above by n' and below by n. To ensure that this notation is unambiguous, we take the convention that this restriction applies only to the term immediately preceding it unless parentheses are used to explicitly indicate otherwise. Thus, $\lambda x|_n^{n'}$ is equal to $\lambda\big(x|_n^{n'}\big)$, and we parenthesise when we want to refer instead to $(\lambda x)|_n^{n'}$.

The idea now is to index the basis of a Hilbert space by Λ^*, and use the concatenation and truncation procedures just defined to produce a Toeplitz-Cuntz-Krieger Λ-family on this Hilbert space.

DEFINITION 3.7.5. Let (Λ, d) be a finitely aligned k-graph. For each $\lambda \in \Lambda$, define an operator T_λ on $\ell^2(\Lambda^*)$ by $T_\lambda e_x = \delta_{s(\lambda),r(x)} e_{\lambda x}$.

LEMMA 3.7.6. *Let (Λ, d) be a finitely aligned k-graph. Then the collection of operators $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family in $\mathcal{B}(\ell^2(\Lambda^*))$ called the* path-space representation *of Λ. Furthermore, T_v is nonzero for all $v \in \Lambda^0$, and if $E \in v\,\mathrm{FE}(\Lambda)$, then $\prod_{\lambda \in E}(T_v - T_\lambda T_\lambda^*)$ is also nonzero.*

PROOF. We must first check that the T_λ are partial isometries in $\mathcal{B}(\ell^2(\Lambda^*))$. To see this, notice that $x \mapsto \lambda x$ is an injective map from $\{x \in \Lambda^* : r(x) = s(\lambda)\}$ to $\{y \in \Lambda^* : y(0, d(\lambda)) = \lambda\}$, and hence T_λ is a partial isometry with range projection $P_{\overline{\mathrm{span}}\{e_y : y(0,d(\lambda)) = \lambda\}}$ and source projection $P_{\overline{\mathrm{span}}\{e_x : r(x) = s(\lambda)\}}$. For $v \in \Lambda^0$, the operator T_v is equal to the projection onto $\overline{\mathrm{span}}\{e_x : r(x) = v\}$ by definition, establishing (TCK1). If $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$, and $x \in \Lambda^*$ with $r(x) = s(\mu)$, then an application of Lemma 3.7.2 shows that $(\lambda\mu)x$ and $\lambda(\mu x)$ are equal as graph morphisms from $\Omega_{k,d(\lambda)+d(\mu)+d(x)}$ to Λ. It follows that $T_\lambda T_\mu e_x = \delta_{s(\mu),r(x)} e_{\lambda(\mu x)} = \delta_{s(\mu),r(x)} e_{(\lambda\mu)x} = T_{\lambda\mu} e_x$. If, on the other hand, $s(\lambda) \neq r(\mu)$ then $T_\lambda T_\mu e_x = \delta_{s(\mu),r(x)} T_\lambda e_{\mu x} = \delta_{s(\lambda),r(\mu)} \delta_{s(\mu),r(x)} e_{\lambda\mu x} = 0$, establishing (TCK2).

To check (TCK3), we first need to see how T_λ^* behaves with respect to the standard basis for $\ell^2(\Lambda^*)$. So let $x, y \in \Lambda^*$, and calculate

$$(e_x | T_\lambda^* e_y) = (T_\lambda e_x | e_y) = \delta_{s(\lambda),r(x)} (e_{\lambda x} | e_y) = \delta_{\lambda x, y}.$$

On the other hand, $(e_x|e_y) = \delta_{x,y}$ by definition, so

$$T_\lambda^* y = \begin{cases} e_{y|_{d(\lambda)}^{d(y)}} & \text{if } y(0, d(\lambda)) = \lambda \\ 0 & \text{otherwise.} \end{cases} \tag{3.7.1}$$

Now suppose $\lambda, \mu \in \Lambda$ and $x \in \Lambda^*$. Then

$$T_\lambda^* T_\mu e_x = \delta_{s(\mu),r(x)} T_\lambda^* e_{\mu x} = \begin{cases} e_{(\mu x)|_{d(\lambda)}^{d(\mu x)}} & \text{if } s(\mu) = r(x) \text{ and } (\mu x)(0, d(\lambda)) = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

But $(\mu x)(0, d(\lambda)) = \lambda$ if and only if $(\mu x)(0, d(\lambda) \vee d(\mu)) \in \mathrm{MCE}(\lambda, \mu)$, so we obtain

$$T_\lambda^* T_\mu e_x = \begin{cases} e_{(\mu x)|_{d(\lambda)}^{d(\mu x)}} & \text{if } (\mu x)(0, d(\lambda) \vee d(\mu)) \in \mathrm{MCE}(\lambda, \mu) \\ 0 & \text{otherwise.} \end{cases} \tag{3.7.2}$$

On the other hand,

$$\begin{aligned} \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} T_\alpha T_\beta^* e_x &= \sum_{\substack{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu) \\ x(0,d(\beta))=\beta}} T_\alpha e_{x|_{d(\beta)}^{d(x)}} \\ &= \sum_{\substack{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu) \\ x(0,d(\beta))=\beta}} e_{\alpha x|_{d(\beta)}^{d(x)}}, \end{aligned} \tag{3.7.3}$$

since $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ implies $s(\alpha) = s(\beta)$. If $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$, then $d(\beta) = (d(\lambda) \vee d(\mu)) - d(\mu)$, so there can be at most one term in (3.7.3). Writing m for $(d(\lambda) \vee d(\mu)) - d(\mu)$, this gives

$$\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} T_\alpha T_\beta^* e_x = \begin{cases} e_{\alpha x|_m^{d(x)}} & \text{if there exists } (\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu) \\ & \qquad \text{such that } x(0, m) = \beta \\ 0 & \text{otherwise.} \end{cases}$$

But $x(0, m) = \beta$ for some $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ if and only if $\mu x = \mu\beta x|_m^{d(x)} = \lambda\alpha x|_m^{d(x)}$, which in turn is true if and only if $(\mu x)(0, d(\lambda) \vee d(\mu)) \in \mathrm{MCE}(\lambda, \mu)$. Combining this with (3.7.2) establishes (TCK3).

To establish the last two claims, let $v \in \Lambda^0$, and let $E \subset v\Lambda \setminus \{v\}$ be any finite set of paths. Define $x_v : \Omega_{k,0} \to \Lambda$ by $x_v(0, 0) := v$, so that $x \in \Lambda^*$. Since $v(0, d(\lambda)) = v \neq \lambda$ for all $\lambda \in E$, we have $T_\lambda T_\lambda^* e_v = 0$ for all $\lambda \in v$, and hence

$$T_v e_v = e_v \quad \text{and} \quad \prod_{\lambda\in E} (T_v - T_\lambda T_\lambda^*) e_v = e_v.$$

Since e_v is a basis element in $\ell^2(\Lambda^*)$, and hence has norm 1, it follows that both T_v and $\prod_{\lambda\in E}(T_v - T_\lambda T_\lambda^*)$ are nonzero. □

Corollary 3.7.7. *Let (Λ, d) be a finitely aligned k-graph. Let $\{s_{\mathcal{T}}(\lambda) : \lambda \in \Lambda\} \subset \mathcal{T}C^*(\Lambda)$ be the universal generating Toeplitz-Cuntz-Krieger Λ-family. Let $v \in \Lambda^0$, and let $E \in \mathrm{FE}(\Lambda)$. Then both $s_{\mathcal{T}}(v)$ and $\prod_{\lambda\in E}(s_{\mathcal{T}}(r(E)) - s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\lambda)^*)$ are nonzero.*

PROOF. By Lemma 3.7.6, we have $T_v \neq 0$. Since $T_v = \pi_T^{\mathcal{T}}(s_{\mathcal{T}}(v))$, it follows that $s_{\mathcal{T}}(v)$ is nonzero. Likewise, Lemma 3.7.6 ensures that $\prod_{\lambda \in E}(T_{r(E)} - T_\lambda T_\lambda^*) \neq 0$, and since

$$\prod_{\lambda \in E}(T_v - T_\lambda T_\lambda^*) = \pi_T^{\mathcal{T}}\Big(\prod_{\lambda \in E}(s_{\mathcal{T}}(v) - s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\lambda)^*)\Big),$$

it follows that $\prod_{\lambda \in E}(s_{\mathcal{T}}(r(E)) - s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\lambda)^*)$ is nonzero. □

COROLLARY 3.7.8. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Then the homomorphism $\pi_t^{\mathcal{T}}$ which takes $s_{\mathcal{T}}(\lambda)$ to t_λ is injective on $\mathcal{T}C^*(\Lambda)^\gamma$ if and only if*

(1) *$t_v \neq 0$ for all $v \in \Lambda^0$; and*
(2) *$\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) \neq 0$ for all $E \in \mathrm{FE}(\Lambda)$.*

In particular, the path-space representation $\pi_T^{\mathcal{T}}$ of Lemma 3.7.6 is faithful on $\mathcal{T}C^*(\Lambda)^\gamma$.

PROOF. It follows from Corollary 3.7.7 that (1) and (2) are necessary, and Theorem 3.5.8 shows that they are sufficient. Combining this with Lemma 3.7.6 now proves the last statement of the Corollary. □

COROLLARY 3.7.9. *Let (Λ, d) be a finitely aligned k-graph, and let $E \subset \Lambda$ be finite. Then*

$$M_{\Pi E}^{s_{\mathcal{T}}} \cong \bigoplus_{\substack{v \in s(\Pi E) \\ n \in d((\Pi E)v)}} M_{(\Pi E)v \cap \Lambda^n}(\mathbb{C}).$$

PROOF. It suffices to show that $\prod_{\nu \in T^{\Pi E}(n,v)}(s_{\mathcal{T}}(v) - s_{\mathcal{T}}(\nu)s_{\mathcal{T}}(\nu)^*) \neq 0$ for all $\lambda \in \Pi E$; the result then follows from Lemma 3.6.2. If $T^{\Pi E}(n, v)$ is not exhaustive, then $\prod_{\nu \in T^{\Pi E}(n,v)}(s_{\mathcal{T}}(v) - s_{\mathcal{T}}(\nu)s_{\mathcal{T}}(\nu)^*) \neq 0$ by Proposition 3.5.3 and Corollary 3.7.7. On the other hand, if $T^{\Pi E}(n, v)$ is exhaustive, then $\prod_{\nu \in T^{\Pi E}(n,v)}(s_{\mathcal{T}}(v) - s_{\mathcal{T}}(\nu)s_{\mathcal{T}}(\nu)^*) \neq 0$ by Corollary 3.7.7. □

3.8. Faithful representations of the Toeplitz algebra

In this section we establish that the homomorphism $\pi_t^{\mathcal{T}}$ of $\mathcal{T}C^*(\Lambda)$ determined by a Toeplitz-Cuntz-Krieger Λ family $\{t_\lambda : \lambda \in \Lambda\}$ is injective if and only if it is injective on $\mathcal{T}C^*(\Lambda)^\gamma$; combining this with Corollary 3.7.8 proves a uniqueness theorem for the C^*-algebras generated by Toeplitz-Cuntz-Krieger Λ-families.

PROPOSITION 3.8.1. *Let* (Λ, d) *be a finitely aligned* k*-graph, and let* $\{t_\lambda : \lambda \in \Lambda\}$ *be a Toeplitz-Cuntz-Krieger* Λ*-family such that* $t_v \neq 0$ *for all* $v \in \Lambda^0$ *and* $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) \neq 0$ *for all* $E \in \mathrm{FE}(\Lambda)$*. Then there is a linear map*

$$\Phi_t : C^*(\{t_\lambda : \lambda \in \Lambda\}) \to \overline{\operatorname{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\}$$

such that $\Phi_t \circ \pi_t^{\mathcal{T}} = \pi_t^{\mathcal{T}} \circ \Phi^\gamma$*.*

The following technical result is the guts of Proposition 3.8.1.

LEMMA 3.8.2. *Let* (Λ, d) *be a finitely aligned* k*-graph, and let* $\{t_\lambda : \lambda \in \Lambda\}$ *be a Toeplitz-Cuntz-Krieger* Λ*-family such that* $t_v \neq 0$ *for all* $v \in \Lambda^0$ *and* $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) \neq 0$ *for all* $E \in \mathrm{FE}(\Lambda)$*. Let* $E \subset \Lambda$ *be finite, and let*

$$a = \sum_{\lambda,\mu \in E} a_{\lambda,\mu} t_\lambda t_\mu^* \quad \text{where } a_{\lambda,\mu} \in \mathbb{C} \text{ for all } \lambda, \mu \in E.$$

Then $\big\| \sum_{\lambda,\mu \in E, d(\lambda)=d(\mu)} a_{\lambda,\mu} t_\lambda t_\mu^* \big\| \leq \|a\|$*.*

PROOF. For convenience we denote $\sum_{\lambda,\mu \in E, d(\lambda)=d(\mu)} a_{\lambda,\mu} t_\lambda t_\mu^*$ by $\Phi(a)$ for the duration of this proof. By Proposition 3.5.3, $\Phi(a)$ can be written as $\Phi(a) = \sum_{\lambda,\mu \in \Pi E \times_{d,s} \Pi E} a'_{\lambda,\mu} \Theta(t)^{\Pi E}_{\lambda,\mu}$ for some $\{a'_{\lambda,\mu} : \lambda, \mu \in \Pi E \times_{d,s} \Pi E\} \subset \mathbb{C}$. Since $M^t_{\Pi E}$ decomposes as a direct sum as in Lemma 3.6.2, there exist $v \in s(\Pi E)$ and $n \in d((\Pi E)v)$ such that

$$\|\Phi(a)\| = \Big\| \sum_{\lambda,\mu \in (\Pi E)v \cap \Lambda^n} a'_{\lambda,\mu} \Theta(t)^{\Pi E}_{\lambda,\mu} \Big\|.$$

Define a projection Q by

$$Q := \sum_{\lambda \in (\Pi E)v \cap \Lambda^n} Q(t)^{\Pi E}_\lambda.$$

Since the $Q(t)^{\Pi E}_\lambda$ are mutually orthogonal, Lemma 3.5.4 gives

$$Q\Theta(t)^{\Pi E}_{\lambda,\mu} = \Theta(t)^{\Pi E}_{\lambda,\mu} Q = \begin{cases} \Theta(t)^{\Pi E}_{\lambda,\mu} & \text{if } \lambda, \mu \in (\Pi E)v \cap \Lambda^n \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have

$$\|Q\Phi(a)Q\| = \|\Phi(a)\|. \tag{3.8.1}$$

On the other hand, we have

$$QaQ = \sum_{\lambda,\mu \in E} a_{\lambda,\mu} Q t_\lambda t_\mu^* Q.$$

To investigate this expression further, fix $\lambda \in E$, and calculate

$$\begin{aligned} Qt_\lambda &= Qt_\lambda t_\lambda^* t_\lambda \\ &= Q \sum_{\lambda\nu \in \Pi E} Q(t)^{\Pi E}_{\lambda\nu} t_\lambda \quad \text{by Corollary 3.3.11} \\ &= \sum_{\lambda\nu \in (\Pi E)v \cap \Lambda^n} Q(t)^{\Pi E}_{\lambda\nu} t_\lambda \\ &= \sum_{\lambda\nu \in (\Pi E)v \cap \Lambda^n} t_{\lambda\nu} \Big(\prod_{\lambda\nu\nu' \in \Pi E, d(\nu')>0} (t_{s(\nu)} - t_{\nu'} t_{\nu'}^*) \Big) t_\nu^* \end{aligned}$$

by Remark 3.5.5. Hence (TCK3) gives

$$Qt_\lambda t_\mu^* Q = \sum_{\substack{\lambda\nu, \mu\sigma \in (\Pi E)v \cap \Lambda^n \\ (\alpha,\beta) \in \Lambda^{\min}(\nu,\sigma)}} t_{\lambda\nu} \Big(\prod_{\substack{\lambda\nu\nu' \in \Pi E \\ d(\nu')>0}} (t_v - t_{\nu'} t_{\nu'}^*) \Big) t_\alpha t_\beta^* \Big(\prod_{\substack{\mu\sigma\sigma' \in \Pi E \\ d(\sigma')>0}} (t_v - t_{\sigma'} t_{\sigma'}^*) \Big) t_{\mu\sigma}^*.$$

To eliminate these terms when $d(\lambda) \neq d(\mu)$, we adjust the projection Q as follows: let

$$E' = \bigcup_{\substack{\lambda,\mu \in \Pi E, \\ d(\lambda) \neq d(\mu), d(\lambda), d(\mu) \leq n}} \{\alpha, \beta : \lambda\nu, \mu\sigma \in (\Pi E)v \cap \Lambda^n, (\alpha,\beta) \in \Lambda^{\min}(\nu,\sigma)\},$$

and let

$$Q' := Q \sum_{\lambda \in (\Pi E)v \cap \Lambda^n} \Big(\prod_{\xi \in E'} (s_\lambda s_\lambda^* - s_{\lambda\xi} s_{\lambda\xi}^*) \Big).$$

Since Lemma 3.1.2(3) ensures that the range projections associated to paths λ of degree n are orthogonal, we have that

$$\begin{aligned} Q' &= \sum_{\lambda \in (\Pi E)v \cap \Lambda^n} Q(t)^{\Pi E}_\lambda \prod_{\xi \in E'} (s_\lambda s_\lambda^* - s_{\lambda\xi} s_{\lambda\xi}^*) \\ &= \sum_{\lambda \in (\Pi E)v \cap \Lambda^n} t_\lambda \Big(\prod_{\nu \in T^{\Pi E}(n,v)} (t_v - t_\nu t_\nu^*) \prod_{\xi \in E'} (t_v - t_\xi t_\xi^*) \Big) t_\lambda^*. \end{aligned}$$

Each summand in Q' is a nonzero subprojection of the corresponding summand in Q: on the one hand, if $T^{\Pi E}(n,v) \cup E'$ is exhaustive, we have

$$\prod_{\nu \in T^{\Pi E}(n,v)} (t_v - t_\nu t_\nu^*) \prod_{\xi \in E'} (t_v - t_\xi t_\xi^*) \neq 0$$

by assumption; on the other hand, if $T^{\Pi E}(n,v) \cup E'$ is not exhaustive, there exists $\tau \in v\Lambda$ such that $\Lambda^{\min}(\sigma,\tau) = \emptyset$ for all $\sigma \in T^{\Pi E}(n,v) \cup E'$, and then

$$\prod_{\nu \in T^{\Pi E}(n,v)} (t_v - t_\nu t_\nu^*) \prod_{\xi \in E'} (t_v - t_\xi t_\xi^*) t_\tau t_\tau^* = t_\tau t_\tau^* \neq 0.$$

It follows that $\Theta(t)^{\Pi E}_{\lambda,\mu} \mapsto Q'\Theta(t)^{\Pi E}_{\lambda,\mu}Q'$ determines an isomorphism of $M^t_{\Pi E}(n,v)$. Combining this with (3.8.1) gives

$$\|Q'\Phi(a)Q'\| = \|Q\Phi(a)Q\| = \|\Phi(a)\|. \tag{3.8.2}$$

On the other hand, if $\lambda,\mu \in \Pi E$ and $d(\lambda) \neq d(\mu)$, we have

$$Q't_\lambda t_\mu^* Q' = \sum_{\substack{\lambda\nu,\mu\sigma\in(\Pi E)v\cap\Lambda^n\\ (\alpha,\beta)\in\Lambda^{\min}(\nu,\sigma)}} \Big(t_{\lambda\nu}\Big(\prod_{\lambda\nu\nu'\in\Pi E, d(\nu')>0}(t_v - t_{\nu'}t_{\nu'}^*)\prod_{\xi\in E'}(t_v - t_\xi t_\xi^*)\Big)$$
$$t_\alpha t_\beta^*\Big(\prod_{\zeta\in E'}(t_v - t_\zeta t_\zeta^*)\prod_{\mu\sigma\sigma'\in\Pi E, d(\sigma')>0}(t_v - t_{\sigma'}t_{\sigma'}^*)\Big)t_{\mu\sigma}^*\Big).$$

Since each α occurring in this sum belongs to E' by definition, we have

$$\prod_{\xi\in E'}(t_v - t_\xi t_\xi^*)t_\alpha = \Big(\prod_{\xi\in E'\setminus\{\alpha\}}(t_v - t_\xi t_\xi^*)\Big)(t_v - t_\alpha t_\alpha^*)t_\alpha = 0$$

in every term, and so the sum collapses, giving $Q't_\lambda t_\mu^* Q' = 0$ for all $\lambda,\mu \in \Pi E$ such that $d(\lambda) \neq d(\mu)$. Hence $Q'aQ' = Q'\Phi(a)Q'$, and (3.8.2) gives

$$\|\Phi(a)\| = \|Q'\Phi(a)Q'\| = \|Q'aQ'\| \leq \|Q'\|\,\|a\|\,\|Q'\| = \|a\|. \qquad \square$$

Proof of Proposition 3.8.1. By Lemma 3.8.2 the formula

$$\sum_{\lambda,\mu\in E} a_{\lambda,\mu}t_\lambda t_\mu^* \mapsto \sum_{\lambda,\mu\in E, d(\lambda)=d(\mu)} a_{\lambda,\mu}t_\lambda t_\mu^* \tag{3.8.3}$$

is norm-decreasing on $\operatorname{span}\{t_\lambda t_\mu^* : \lambda,\mu \in \Lambda\}$ which is dense in $C^*(\{t_\lambda : \lambda \in \Lambda\})$. It follows that (3.8.3) determines a well-defined norm-decreasing linear map from $\operatorname{span}\{t_\lambda t_\mu^* : \lambda,\mu \in \Lambda\}$ to $\operatorname{span}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\}$, and hence defines a bounded linear map

$$\Phi^t : \overline{\operatorname{span}}\{t_\lambda t_\mu^* : \lambda,\mu \in \Lambda\} \to \overline{\operatorname{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\}$$

such that $\|\Phi_t\| \leq 1$. Since $\Phi_t \circ \pi_t^{\mathcal{T}}$ agrees with $\pi_t^{\mathcal{T}} \circ \Phi^\gamma$ on the dense subset $\operatorname{span}\{s_{\mathcal{T}}(\lambda)s_{\mathcal{T}}(\mu)^* : \lambda,\mu \in \Lambda\} \subset \mathcal{T}C^*(\Lambda)$, we have that $\Phi_t \circ \pi_t^{\mathcal{T}} = \pi_t^{\mathcal{T}} \circ \Phi^\gamma$ as linear maps from $\mathcal{T}C^*(\Lambda)$ to $\overline{\operatorname{span}}\{t_\lambda t_\mu^* : d(\lambda) = d(\mu)\}$ as required. $\square$

We can now prove the main result of this chapter.

Proof of Theorem 3.1.6. Corollary 3.7.8 establishes necessity, so we need only establish sufficiency. For this, suppose that $\{t_\lambda : \lambda \in \Lambda\}$ satisfies (1) and (2) of Theorem 3.1.6, and suppose that $a \in \mathcal{T}C^*(\Lambda)$ and that $\pi_t^{\mathcal{T}}(a) = 0$. Then $\pi_t^{\mathcal{T}}(a^*a) = 0$ by the C^*-identity, and then $\Phi_t(\pi_t^{\mathcal{T}}(a^*a)) = 0$ since Φ_t is linear. Proposition 3.8.1 ensures that $\Phi_t \circ \pi_t^{\mathcal{T}} = \pi_t^{\mathcal{T}} \circ \Phi^\gamma$, so we deduce that $\pi_t^{\mathcal{T}}(\Phi^\gamma(a^*a)) = 0$. Corollary 3.7.8 shows that $\pi_t^{\mathcal{T}}$ is faithful on $\mathcal{T}C^*(\Lambda)^\gamma$, so we have $\Phi^\gamma(a^*a) = 0$.

But Φ^γ is faithful on positive elements by Proposition 3.2.3, giving $a^*a = 0$. Hence $a = 0$ by the C^*-identity. □

COROLLARY 3.8.3 (The Toeplitz Uniqueness Theorem). *Let (Λ, d) be a finitely aligned k-graph. There exists a Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ such that*

(1) $t_v \neq 0$ *for all* $v \in \Lambda^0$; *and*
(2) $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) \neq 0$ *for all* $E \in \mathrm{FE}(\Lambda)$.

Furthermore, any two Toeplitz-Cuntz-Krieger Λ-families satisfying (1) and (2) generate canonically isomorphic C^*-algebras.

PROOF. The first statement follows from Lemma 3.7.6, and the second then follows from Theorem 3.1.6. □

COROLLARY 3.8.4. *Let (Λ, d) be a finitely aligned k-graph. The representation $\pi_T^{\mathcal{T}} : \mathcal{T}C^*(\Lambda) \to \mathcal{B}(\ell^2(\Lambda^*))$ associated to the path-space representation of Lemma 3.7.6 is faithful.*

PROOF. This is a direct consequence of Lemma 3.7.6 and Theorem 3.1.6. □

CHAPTER 4

Relative Cuntz-Krieger algebras

This chapter introduces the *relative Cuntz-Krieger algebras* associated to a finitely aligned k-graph Λ. Given a subset $\mathcal{E}$ of $\mathrm{FE}(\Lambda)$, the relative Cuntz-Krieger algbra $C^*(\Lambda;\mathcal{E})$ is the universal C^*-algebra generated by a Toeplitz-Cuntz-Krieger family in which $\prod_{\lambda\in E}(t_{r(E)} - t_\lambda t_\lambda^*)$ is equal to zero for $E \in \mathcal{E}$. We call this algebra a *relative Cuntz-Krieger algebra*. When $\mathcal{E} = \emptyset$, the associated relative Cuntz-Krieger algebra $C^*(\Lambda;\emptyset)$ is the Toeplitz algebra of the previous chapter. When $\mathcal{E} = \mathrm{FE}(\Lambda)$ we obtain the Cuntz-Krieger algebra $C^*(\Lambda)$ studied in [**30**]. In general the algebras $C^*(\Lambda;\mathcal{E})$ interpolate between $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$ in the sense that for each $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ there exist ideals $J_{\mathcal{E}} \subset \mathcal{T}C^*(\Lambda)$ (see Definition 4.1.3) and $I_{\mathrm{FE}(\Lambda)\setminus\overline{\mathcal{E}}} \subset C^*(\Lambda;\mathcal{E})$ (see Definition 5.3.1) such that

$$\mathcal{T}C^*(\Lambda)/J_{\mathcal{E}} \underset{\text{can}}{\cong} C^*(\Lambda;\mathcal{E}) \qquad \text{and} \qquad C^*(\Lambda;\mathcal{E})/I_{\mathrm{FE}(\Lambda)\setminus\overline{\mathcal{E}}} \underset{\text{can}}{\cong} C^*(\Lambda).$$

Our primary objective in this chapter is to prove a gauge-invariant uniqueness theorem for $C^*(\Lambda;\mathcal{E})$. We achieve this objective in Theorem 4.3.12.

4.1. Relative Cuntz-Krieger families

In this section we introduce the Cuntz-Krieger relation (CK) and use it to define the relative Cuntz-Krieger algebra $C^*(\Lambda;\mathcal{E})$ associated to a k-graph Λ and a subset $\mathcal{E}$ of $\mathrm{FE}(\Lambda)$. We show that $C^*(\Lambda;\mathcal{E})$ admits a gauge action, and investigate the consequences of this fact.

Definition 4.1.1. Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. A *relative Cuntz-Krieger* $(\Lambda;\mathcal{E})$*-family* is a Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ which satisfies

(CK) $\prod_{\lambda\in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$ for all $E \in \mathcal{E}$.

REMARK 4.1.2. Let (Λ, d) be a finitely aligned k-graph. When $\mathcal{E} = \emptyset$, the new relation (CK) is trivial; hence, the relative Cuntz-Krieger $(\Lambda; \emptyset)$-families are precisely the Toeplitz-Cuntz-Krieger Λ-families of Definition 3.1.1. On the other hand, if $\mathcal{E} = \mathrm{FE}(\Lambda)$, then (CK) applies to every finite exhaustive set; consequently, the relative Cuntz-Krieger $(\Lambda; \mathrm{FE}(\Lambda))$-families are precisely the Cuntz-Krieger Λ-families of [**30**, Definition 2.6].

DEFINITION 4.1.3. Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E}$ be a subset of $FE(\Lambda)$. We define $J_{\mathcal{E}}$ to be the ideal in $\mathcal{T}C^*(\Lambda)$ generated by

$$\big\{ \textstyle\prod_{\lambda \in E} \big(s_{\mathcal{T}}(r(E)) - s_{\mathcal{T}}(\lambda) s_{\mathcal{T}}(\lambda)^*\big) : E \in \mathcal{E} \big\}.$$

We define $C^*(\Lambda; \mathcal{E}) := \mathcal{T}C^*(\Lambda)/J_{\mathcal{E}}$ and $s_{\mathcal{E}}(\lambda) := s_{\mathcal{T}}(\lambda) + J_{\mathcal{E}} \in C^*(\Lambda; \mathcal{E})$ for all $\lambda \in \Lambda$. We call $C^*(\Lambda; \mathcal{E})$ the *relative Cuntz-Krieger algebra* associated to Λ and $\mathcal{E}$.

THEOREM 4.1.4. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *The family $\{s_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family which generates $C^*(\Lambda; \mathcal{E})$. Furthermore, $C^*(\Lambda; \mathcal{E})$ is the universal C^*-algebra generated by a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family in the sense that if $\{t_\lambda : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family in a C^*-algebra B, then there exists a unique homomorphism $\pi_t^{\mathcal{E}} : C^*(\Lambda; \mathcal{E}) \to B$ such that $\pi_t^{\mathcal{E}}(s_{\mathcal{E}}(\lambda)) = t_\lambda$ for all $\lambda \in \Lambda$.*

PROOF. We have that $\{s_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ satisfies (TCK1)–(TCK3) because the quotient map is a homomorphism. Moreover, $\{s_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ satisfies (CK) by definition. It follows that $\{s_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. Since $\{s_{\mathcal{T}}(\lambda) : \lambda \in \Lambda\}$ generates $\mathcal{T}C^*(\Lambda)$, we have that $\{s_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ generates $C^*(\Lambda; \mathcal{E})$.

To see that $C^*(\Lambda; \mathcal{E})$ is universal, let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family in a C^*-algebra B. Since $\{t_\lambda : \lambda \in \Lambda\}$ is in particular a Toeplitz-Cuntz-Krieger Λ-family, Theorem 3.1.5 gives a unique homomorphism $\pi_t^{\mathcal{T}} : \mathcal{T}C^*(\Lambda) \to B$ such that $\pi_t^{\mathcal{T}}(s_{\mathcal{T}}(\lambda)) = t_\lambda$ for all $\lambda \in \Lambda$. Since $\{t_\lambda : \lambda \in \Lambda\}$ satisfies (CK) we have $J_{\mathcal{E}} \subset \ker \pi_t^{\mathcal{T}}$, so π descends to the required homomorphism $\pi_t^{\mathcal{E}}$ of $C^*(\Lambda; \mathcal{E})$. We have that $\pi_t^{\mathcal{E}}$ is the unique homomorphism of $C^*(\Lambda; \mathcal{E})$ satisfying $\pi_t^{\mathcal{E}}(s_{\mathcal{E}}(\lambda)) = t_\lambda$ for all $\lambda \in \Lambda$ because $\{s_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ generates $C^*(\Lambda; \mathcal{E})$. □

REMARK 4.1.5. Muhly and Tomforde study relative graph algebras associated to directed graphs E in [**23**, Section 3]. Given a directed graph E and a subset $V \subset E^0$, the relative graph algebra $C^*(E, V)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ such that $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$ and such that $p_v \geq \sum_{e \in F} s_e s_e^*$ for all $v \in E^0$ and finite $F \subset r^{-1}(v)$, with equality when $F = r^{-1}(v)$ is finite and $v \in V$. Muhly and Tomforde's analysis proceeds by constructing from a graph E and a subset V of E^0 a graph E_V such that $C^*(E_V)$ is canonically isomorphic to the relative graph algebra $C^*(E, V)$. This technique has the great advantage that it allows the extensive theory of graph algebras to be brought to bear on the study of relative graph-algebras. One might therefore hope to obtain a simpler analysis of relative Cuntz-Krieger algebras associated to k-graphs by adapting the techniques of [**23**] to the higher-rank setting as opposed to employing a direct analysis of $C^*(\Lambda; \mathcal{E})$.

We have as yet been unable to find a higher-rank analogue of the construction of E_V from E and V developed in [**23**]. Indeed, since the relative Cuntz-Krieger algebra is determined by a collection $\mathcal{E}$ of finite exhaustive sets rather than a set V of vertices, we would expect that any such construction would be significantly more complicated than the construction in [**23**]. Added to this complication are the combinatorial issues which the factorisation property causes when one tries to modify k-graphs by adding or removing edges. However, when $\Lambda = E^*$ is the 1-graph corresponding to a directed graph E, the relative Cuntz-Krieger algebras associated to Λ by Definition 4.1.3 are precisely the relative graph algebras $C^*(E, V)$ associated to E in [**23**] (see Remark 5.2.8), and the results in this chapter provide an alternative analysis of $C^*(E, V)$ to that given in [**23**].

Proposition 4.1.6. *Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *There exists a strongly continuous action $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda; \mathcal{E}))$, called the* gauge action, *which satisfies $\gamma_z(s_{\mathcal{E}}(\lambda)) = z^{d(\lambda)} s_{\mathcal{E}}(\lambda)$ for all $\lambda \in \Lambda$. Let $C^*(\Lambda; \mathcal{E})^\gamma$ denote the set of fixed points $\{a \in C^*(\Lambda; \mathcal{E}) : \gamma_z(a) = a \text{ for all } z \in \mathbb{T}^k\}$. Then*

$$C^*(\Lambda; \mathcal{E})^\gamma = \overline{\mathrm{span}}\{s_{\mathcal{E}}(\lambda) s_{\mathcal{E}}(\mu)^* : \lambda, \mu \in \Lambda, d(\lambda) = d(\mu)\}.$$

Proof. Since the ideal $J_{\mathcal{E}}$ in $\mathcal{T}C^*(\Lambda)$ is generated by projections which are invariant under the gauge action γ on $\mathcal{T}C^*(\Lambda)$, the gauge action on $\mathcal{T}C^*(\Lambda)$ descends to the required action, also denoted γ, on $C^*(\Lambda; \mathcal{E})$. For the second statement, just notice that $C^*(\Lambda; \mathcal{E})^\gamma$ is precisely $q_{\mathcal{E}}\big(\mathcal{T}C^*(\Lambda)^\gamma\big)$. Since $q_{\mathcal{E}}(s_{\mathcal{T}}(\lambda)) = s_{\mathcal{E}}(\lambda)$ by definition, the result now follows from Proposition 3.2.1. □

PROPOSITION 4.1.7. *Let* (Λ, d) *be a finitely aligned* k*-graph, and let* $\mathcal{E}$ *be a subset of* $\mathrm{FE}(\Lambda)$. *There exists a faithful conditional expectation* $\Phi^\gamma : C^*(\Lambda; \mathcal{E}) \to C^*(\Lambda; \mathcal{E})^\gamma$ *which satisfies*

$$\Phi^\gamma(s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\mu)^*) = \delta_{d(\lambda),d(\mu)}s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\mu)^*.$$

PROOF. The proof is identical to that of Proposition 3.2.3. □

COROLLARY 4.1.8. *Let* (Λ, d) *be a finitely aligned* k*-graph, and let* $\mathcal{E}$ *be a subset of* $\mathrm{FE}(\Lambda)$. *Let* $\{t_\lambda : \lambda \in \Lambda\}$ *be a relative Cuntz-Krieger* $(\Lambda; \mathcal{E})$*-family in a* C^**-algebra* B*, and suppose that there is a strongly continuous action* $\theta : \mathbb{T}^k \to \mathrm{Aut}\, B$ *such that* $\theta \circ \pi_t^{\mathcal{E}} = \pi_t^{\mathcal{E}} \circ \gamma$. *Then* $\pi_t^{\mathcal{E}}$ *is injective if and only if it is injective on* $C^*(\Lambda; \mathcal{E})^\gamma$.

PROOF. The "only if" is trivial, so suppose that $\pi_t^{\mathcal{E}}$ is injective on $C^*(\Lambda; \mathcal{E})^\gamma$, and that $\pi_t^{\mathcal{E}}(a) = 0$. Let $\Phi^\theta : B \to B^\theta$ be the faithful conditional expectation obtained from Proposition 3.2.4. We have that $\Phi^\theta \circ \pi_t^{\mathcal{E}} = \pi_t^{\mathcal{E}} \circ \Phi^\gamma$ because $\pi_t^{\mathcal{E}}$ is equivariant in γ and θ. Since $\pi_t^{\mathcal{E}}$ is a homomorphism, we have $\pi_t^{\mathcal{E}}(a^*a) = 0$, and then $\Phi^\theta(\pi_t^{\mathcal{E}}(a^*a)) = 0$ because Φ^θ is linear. It follows that $\pi_t^{\mathcal{E}}(\Phi^\gamma(a^*a)) = 0$ because $\Phi^\theta \circ \pi_t^{\mathcal{E}} = \pi_t^{\mathcal{E}} \circ \Phi^\gamma$. By Proposition 4.1.7, $\Phi^\gamma(a^*a)$ belongs to $C^*(\Lambda; \mathcal{E})^\gamma$. Since $\pi_t^{\mathcal{E}}$ is injective on $C^*(\Lambda; \mathcal{E})^\gamma$ by assumption, it follows that $\Phi^\gamma(a^*a) = 0$. Since Φ^γ is faithful on positive elements, we then have $a^*a = 0$ and thus $a = 0$ by the C^*-identity. □

4.2. Satiated sets

We want to use Theorem 3.5.8 to decide when relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-families are injective on the core of $C^*(\Lambda; \mathcal{E})$. Corollary 4.1.8 will then give a gauge-invariant uniqueness theorem for $C^*(\Lambda; \mathcal{E})$. To do this, we need to do two things: firstly we need to show that $s_{\mathcal{E}}(v) \neq 0$ for all $v \in \Lambda^0$; and secondly we need to decide precisely which sets $E \in \mathrm{FE}(\Lambda)$ satisfy $\prod_{\lambda \in E} \big(s_{\mathcal{E}}(r(E)) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*\big) \neq 0$.

To achieve these aims, we construct a concrete representation of $C^*(\Lambda; \mathcal{E})$ akin to the path-space representation of $\mathcal{T}C^*(\Lambda)$ (in fact, when $\mathcal{E} = \emptyset$, the two representations are one and the same). The idea is to choose a subset $\partial(\Lambda; \mathcal{E})$ of Λ^* with the property that restricting the boundary-path space to $\ell^2(\partial(\Lambda; \mathcal{E}))$ produces a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. In order to choose such a subset, we will first need to investigate which sets $E \in \mathrm{FE}(\Lambda)$ necessarily satisfy $\prod_{\lambda \in E} \big(s_{\mathcal{E}}(r(E)) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*\big) = 0$. We begin with a technical lemma.

Lemma 4.2.1. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Let $v \in \Lambda^0$, let $\lambda \in v\Lambda$ and suppose that $E \subset s(\lambda)\Lambda$ is finite and satisfies $\prod_{\nu \in E}(t_{s(\lambda)} - t_\nu t_\nu^*) = 0$. Then*

$$t_v - t_\lambda t_\lambda^* = t_v \textstyle\prod_{\nu \in E}(t_v - t_{\lambda\nu} t_{\lambda\nu}^*).$$

Proof. Since $t_{\lambda\mu} t_{\lambda\mu}^* \leq t_\lambda t_\lambda^*$ for all $\mu \in s(\lambda)\Lambda$, we have

$$(t_v - t_\lambda t_\lambda^*)(t_v - t_{\lambda\nu} t_{\lambda\nu}^*) = t_v - t_\lambda t_\lambda^*$$

for all $\nu \in E$. Since range projections in $C^*(\Lambda)$ commute by Lemma 3.1.2(2), and since E is finite, it follows that

$$(t_v - t_\lambda t_\lambda^*) \prod_{\nu \in E}(t_v - t_{\lambda\nu} t_{\lambda\nu}^*) = t_v - t_\lambda t_\lambda^*. \tag{4.2.1}$$

On the other hand,

$$\begin{aligned}
(t_v - t_\lambda t_\lambda^*)\Big(\prod_{\nu \in E}(t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\Big) &= t_v\Big(\prod_{\nu \in E}(t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\Big) - t_\lambda t_\lambda^*\Big(\prod_{\nu \in E}(t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\Big) \\
&= \Big(t_v \prod_{\nu \in E}(t_v - t_{\lambda\nu} t_{\lambda\nu}^*)\Big) - t_\lambda\Big(\prod_{\nu \in E}(t_{s(\lambda)} - t_\nu t_\nu^*)\Big)t_\lambda^* \\
&= t_v \prod_{\nu \in E}(t_v - t_{\lambda\nu} t_{\lambda\nu}^*)
\end{aligned}$$

because $\prod_{\nu \in E}(t_{s(\lambda)} - t_\nu t_\nu^*) = 0$ by hypothesis. □

The following lemma is intended to provide motivation for the definitions and lemmas later in the section, particularly Definition 4.2.3.

Lemma 4.2.2. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E}$ be a subset of $\mathrm{FE}(\Lambda)$. Let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. Let $E \subset \Lambda$ be a finite set constructed in any of the following four ways:*

(i) *$E = G \cup F$ for some $G \in \mathcal{E}$ and finite $F \subset r(G)\Lambda$;*
(ii) *$E = \mathrm{Ext}(\mu; G)$ for some $G \in \mathcal{E}$ and $\mu \in r(G)\Lambda \setminus G\Lambda$;*
(iii) *$E = \{\lambda(0, n_\lambda) : \lambda \in G\}$ where $G \in \mathcal{E}$ and $0 < n_\lambda \leq d(\lambda)$ for all $\lambda \in G$; or*
(iv) *$E = G \setminus G' \cup \big(\bigcup_{\lambda \in G'} \lambda G'_\lambda\big)$ where $G \in \mathcal{E}$, $G' \subset G$ and $G'_\lambda \in s(\lambda)\mathcal{E}$ for all $\lambda \in G'$.*

Then $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$.

Proof. We prove the Lemma by demonstrating that if E is constructed from $G \in \mathcal{E}$ as in any of (i)–(iv), then $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*)$ is dominated by $\prod_{\lambda \in G}(t_{r(G)} - t_\lambda t_\lambda^*)$. Since $\{t_\lambda : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family, we have $\prod_{\lambda \in G}(t_{r(G)} - t_\lambda t_\lambda^*) = 0$ for $G \in \mathcal{E}$, and hence $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$ as required.

Suppose that $E = G \cup F$ for some $G \in \mathcal{E}$ and finite $F \subset r(G)\Lambda$. Then

$$\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = \Big(\prod_{\lambda \in G}(t_{r(G)} - t_\lambda t_\lambda^*)\Big)\Big(\prod_{\lambda \in F \setminus G}(t_{r(E)} - t_\lambda t_\lambda^*)\Big).$$

Suppose that $E = \operatorname{Ext}(\mu; G)$ for some $G \in \mathcal{E}$ and $\mu \in r(G)\Lambda \setminus G\Lambda$. Then

$$\begin{aligned}\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) &= t_\mu^* t_\mu \Big(\prod_{\lambda \in E}(t_{s(\mu)} - t_\lambda t_\lambda^*)\Big) t_\mu^* t_\mu \\ &= t_\mu^* \Big(\prod_{\lambda \in \operatorname{Ext}(\mu;G)}(t_\mu t_\mu^* - t_{\mu\lambda} t_{\mu\lambda}^*\Big) t_\mu \\ &= t_\mu^* \Big(t_\mu t_\mu^* \prod_{\sigma \in G}(t_{r(\mu)} - t_\sigma t_\sigma^*)\Big) t_\mu.\end{aligned}$$

Suppose that $E = \{\lambda(0, n_\lambda) : \lambda \in G\}$ for some $G \in \mathcal{E}$ and $0 < n_\lambda \le d(\lambda)$ for each $\lambda \in G$. Then since $t_{r(E)} - t_{\lambda(0,n_\lambda)} t_{\lambda(0,n_\lambda)}^* \le t_{r(E)} - t_\lambda t_\lambda^*$ for all $\lambda \in E$, we have

$$\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = \prod_{\mu \in G}(t_{r(G)} - t_{\mu(0,n_\mu)} t_{\mu(0,n_\mu)}^*) \le \prod_{\mu \in G}(t_{r(G)} - t_\mu t_\mu^*).$$

Finally, suppose that $E = G \setminus G' \cup \big(\bigcup_{\lambda \in G'} \lambda G'_\lambda\big)$ for some $G \in \mathcal{E}$, $G' \subset G$, and $G'_\lambda \in s(\lambda)\mathcal{E}$ for each $\lambda \in G'$. Then Lemma 4.2.1 implies that for $\lambda \in G'$, we have $t_{r(G)} - t_\lambda t_\lambda^* = \prod_{\mu \in G'_\lambda}(t_{r(G)} - t_{\lambda\mu} t_{\lambda\mu}^*)$. Hence

$$\begin{aligned}\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) &= \Big(\prod_{\lambda \in G \setminus G'}(t_{r(E)} - t_\lambda t_\lambda^*)\Big) \prod_{\lambda \in G'}\Big(\prod_{\mu \in G'_\lambda}(t_{r(E)} - t_{\lambda\mu} t_{\lambda\mu}^*)\Big) \\ &= \Big(\prod_{\lambda \in G \setminus G'}(t_{r(E)} - t_\lambda t_\lambda^*)\Big)\Big(\prod_{\lambda \in G'}(t_{r(E)} - t_\lambda t_\lambda^*)\Big) \\ &= \prod_{\lambda \in G}(t_{r(G)} - t_\lambda t_\lambda^*).\end{aligned}$$

□

Definition 4.2.3. Let (Λ, d) be a finitely aligned k-graph. We say that a subset $\mathcal{E}$ of $\operatorname{FE}(\Lambda)$ is *satiated*† if it satisfies

(S1) If $G \in \mathcal{E}$ and $E \subset r(G)\Lambda \setminus \Lambda^0$ is finite with $G \subset E$ then $E \in \mathcal{E}$;

(S2) If $G \in \mathcal{E}$ with $r(G) = v$ and if $\mu \in v\Lambda \setminus G\Lambda$ then $\operatorname{Ext}(\mu; G) \in \mathcal{E}$;

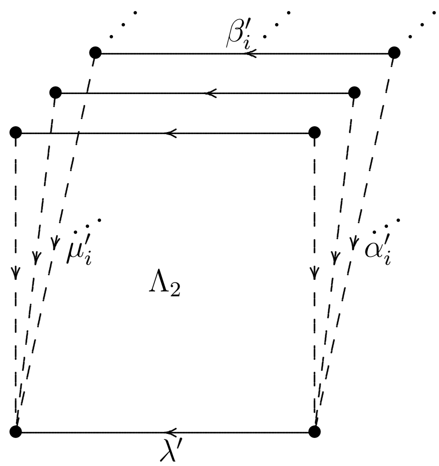
(S3) If $G \in \mathcal{E}$ and $0 < n_\lambda \le d(\lambda)$ for each $\lambda \in G$, then $\{\lambda(0, n_\lambda) : \lambda \in G\} \in \mathcal{E}$; and

(S4) If $G \in \mathcal{E}$, $G' \subset G$ and $G'_\lambda \in s(\lambda)\mathcal{E}$ for each $\lambda \in G'$, then $\big((G \setminus G') \cup \big(\bigcup_{\lambda \in G'} \lambda G'_\lambda\big)\big) \in \mathcal{E}$.

We write $\overline{\mathcal{E}}$ for the smallest collection of subsets of Λ which contains $\mathcal{E}$ and satisfies (S1)–(S4), and refer to $\overline{\mathcal{E}}$ as the *satiation* of E.

†The author considered using the term "normal" rather than "satiated," but is given to understand that Quigg holds the trademark on this terminology.

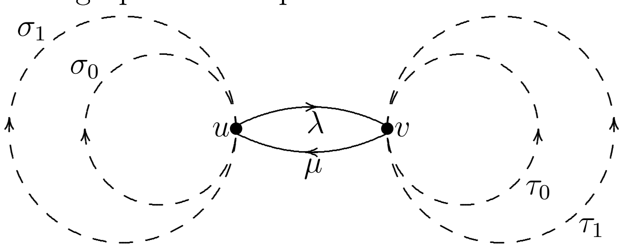
EXAMPLE 4.2.4. We provide here examples of each of the four conditions (S1)–(S4). Consider the 2-graph Λ_2 of Example 2.4.8 with 1-skeleton



Condition (S1) insists that if, for example, the exhaustive set $\{\lambda'\}$ belongs to $\overline{\mathcal{E}}$, then so must all sets of the form $\{\lambda', \mu_i' : i \in I\}$ where I is a finite subset of $\mathbb{N}$. C^*-algebraically, this is because if $s_{\mathcal{E}}(r(\lambda')) - s_{\mathcal{E}}(\lambda')s_{\mathcal{E}}(\lambda')^*$ is equal to zero, then any product in which this is a factor must also be equal to zero.

Condition (S2) says that if $\{\lambda'\} \in \overline{\mathcal{E}}$, then each $\{\beta_i'\} = \operatorname{Ext}(\mu_i, \{\lambda'\})$ must also belong to $\overline{\mathcal{E}}$. C^*-algebraically, this is because if $s_{\mathcal{E}}(r(\lambda')) - s_{\mathcal{E}}(\lambda')s_{\mathcal{E}}(\lambda')^*$ is equal to zero, then for each i, we have $s_{\mathcal{E}}(\mu_i')^*\big(s_{\mathcal{E}}(r(\lambda')) - s_{\mathcal{E}}(\lambda')s_{\mathcal{E}}(\lambda')^*\big)s_{\mathcal{E}}(\mu_i')$ equal to zero, and this last is equal to $s_{\mathcal{E}}(\mu_i')^*s_{\mathcal{E}}(\mu_i') - s_{\mathcal{E}}(\mu_i')^*s_{\mathcal{E}}(\lambda')s_{\mathcal{E}}(\lambda')^*s_{\mathcal{E}}(\mu_i')$ which is equal to $s_{\mathcal{E}}(r(\beta_i')) - s_{\mathcal{E}}(\beta_i')s_{\mathcal{E}}(\beta_i')^*$.

Now consider the 2-graph of Example 2.3.3 with 1-skeleton



and factorisation property determined by bi-coloured squares

$$S = \big\{\{(\lambda, \sigma_i), (\tau_i, \lambda)\}, \{(\mu, \tau_i), (\sigma_i, \mu)\}\big\}.$$

Condition (S3) says that if, for example, the set $\{\mu\tau_0, \mu\tau_1\}$ belongs to $\overline{\mathcal{E}}$, then so does $\{\mu\}$ (as well as $\{\mu\tau_0, \mu\}$ and $\{\mu, \mu\tau_1\}$). C^*-algebraically, this is because the projection $s_{\mathcal{E}}(\mu)s_{\mathcal{E}}(\mu)^*$ dominates $s_{\mathcal{E}}(\mu\tau_i)s_{\mathcal{E}}(\mu\tau_i)^*$ for $i = 0, 1$, so $s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\mu)s_{\mathcal{E}}(\mu)^*$ is dominated by $s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\mu\tau_i)s_{\mathcal{E}}(\mu\tau_i)^*$ for $i = 0, 1$. It follows that $s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\mu)s_{\mathcal{E}}(\mu)^*$ is dominated by the product $\prod_{i=1}^{2}(s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\mu\tau_i)s_{\mathcal{E}}(\mu\tau_i)^*)$. Hence if this product is equal to zero, then so is $s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\mu)s_{\mathcal{E}}(\mu)^*$.

Finally, condition (S4) says that if, for example, the sets $\{\mu\}$ and $\{\sigma_0, \sigma_1\}$ belong to $\overline{\mathcal{E}}$, then so does $\{\sigma_0, \sigma_1\mu\}$ (as well as $\{\sigma_0\mu, \sigma_1\}$ and $\{\sigma_0\mu, \sigma_1\mu\}$). C^*-algebraically, this is because if $s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\mu)s_{\mathcal{E}}(\mu)^*$ is equal to zero, then $s_{\mathcal{E}}(u) = s_{\mathcal{E}}(\mu)s_{\mathcal{E}}(\mu)^*$, and hence each

$$s_{\mathcal{E}}(\sigma_i\mu)s_{\mathcal{E}}(\sigma_i\mu)^* = s_{\mathcal{E}}(\sigma_i)s_{\mathcal{E}}(\mu)s_{\mathcal{E}}(\mu)^*s_{\mathcal{E}}(\sigma_i)^* = s_{\mathcal{E}}(\sigma_i)s_{\mathcal{E}}(\sigma_i)^*.$$

Hence if $\prod_{i=0}^{1}(s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\sigma_i)s_{\mathcal{E}}(\sigma_i)^*)$ is also equal to zero, then $\prod_{i=0}^{1}(s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\sigma_i\mu)s_{\mathcal{E}}(\sigma_i\mu)^*) = 0$.

We will construct $\overline{\mathcal{E}}$ directly using a technique inspired by Szymański's construction of the saturation of a set H of vertices in a 1-graph [**38**]. As a by-product of this process, we show that $\overline{\mathcal{E}} \subset \mathrm{FE}(\Lambda)$.

LEMMA 4.2.5. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Define*

$$\Sigma_{(\mathrm{S1})}(\mathcal{E}) := \{G \cup F : G \in \mathcal{E} \text{ and } F \subset r(G)\Lambda \setminus \Lambda^0 \text{ is finite}\}. \tag{4.2.2}$$

Then $\mathcal{E} \subset \Sigma_{(\mathrm{S1})}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$.

PROOF. To see that $\Sigma_{(\mathrm{S1})}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$, notice that each set $E \in \Sigma_{(\mathrm{S1})}(\mathcal{E})$ is finite and contains an element $G \in \mathcal{E}$ by definition; since G is exhaustive, it follows immediately that E is exhaustive, and $E \cap \Lambda^0 = \emptyset$ by definition. Hence $E \in \mathrm{FE}(\Lambda)$. To see that $\Sigma_{(\mathrm{S1})}(\mathcal{E})$ contains $\mathcal{E}$, just take $F = \emptyset$ in (4.2.2) for each $G \in \mathcal{E}$. □

LEMMA 4.2.6. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Define*

$$\Sigma_{(\mathrm{S2})}(\mathcal{E}) := \{\mathrm{Ext}(\mu; G) : G \in \mathcal{E}, \mu \in r(G)\Lambda \setminus G\Lambda\}. \tag{4.2.3}$$

Then $\mathcal{E} \subset \Sigma_{(\mathrm{S2})}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$.

The following lemma is really part of the proof of Lemma 4.2.6, but it is convenient to state it separately.

LEMMA 4.2.7. *Let (Λ, d) be a finitely aligned k-graph, let $v \in \Lambda^0$, let $E \subset v\Lambda$ be finite and exhaustive, and let $\mu \in v\Lambda$. Then $\mathrm{Ext}(\mu; E)$ is a finite exhaustive subset of $s(\mu)\Lambda$. If $\mu \notin E\Lambda$ then $\mathrm{Ext}(\mu; E) \in \mathrm{FE}(\Lambda)$.*

PROOF. Let $E' := \mathrm{Ext}(\mu; E)$. Since E is finite and Λ is finitely aligned we know that E' is finite, and $E' \subset s(\mu)\Lambda$ by definition. So we need only check that

E' is exhaustive. Let $\sigma \in s(\mu)\Lambda$. Since E is exhaustive, there exists $\lambda \in E$ with $\Lambda^{\min}(\lambda, \mu\sigma) \neq \emptyset$, say $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu\sigma)$. So $\lambda\alpha = \mu\sigma\beta$, and hence

$$\big(\alpha(0, (d(\lambda) \vee d(\mu)) - d(\lambda)), (\sigma\beta)(0, (d(\lambda) \vee d(\mu)) - d(\mu))\big) \in \Lambda^{\min}(\lambda, \mu).$$

It follows that $\tau := (\sigma\beta)(0, (d(\lambda) \vee d(\mu)) - d(\mu))$ belongs to E', and we have

$$\big((\sigma\beta)(d(\sigma), d(\sigma) \vee d(\tau)), (\sigma\beta)(d(\tau), d(\sigma) \vee d(\tau))\big) \in \Lambda^{\min}(\sigma, \tau)$$

by definition. Since $\sigma \in s(\mu)\Lambda$ was arbitrary, it follows that E' is exhaustive. For the final statement, suppose that $\mu \notin E\Lambda$. Then $\alpha \in \mathrm{Ext}(\mu; E)$ implies $d(\alpha) > 0$, giving $\mathrm{Ext}(\mu; E) \cap \Lambda^0 = \emptyset$. Since we have already concluded that $\mathrm{Ext}(\mu; E)$ is finite exhaustive, it follows that $\mathrm{Ext}(\mu; E) \in \mathrm{FE}(\Lambda)$. □

Proof of Lemma 4.2.6. By Lemma 4.2.7, we know that each $\mathrm{Ext}(\mu; G)$ belongs to $\mathrm{FE}(\Lambda)$, giving $\Sigma_{(\mathrm{S2})}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$. To see that $\mathcal{E} \subset \Sigma_{(\mathrm{S2})}(\mathcal{E})$, take $\mu = r(G)$ in (4.2.3) for each $G \in \mathcal{E}$; since $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, we have $G \cap \Lambda^0 = \emptyset$, and hence $r(G) \in r(G)\Lambda \setminus G\Lambda$. □

Lemma 4.2.8. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Define*

$$\Sigma_{(\mathrm{S3})}(\mathcal{E}) := \big\{\{\lambda(0, n_\lambda) : \lambda \in G\} : G \in \mathcal{E}, 0 < n_\lambda \leq d(\lambda) \textit{ for all } \lambda \in G\big\}. \tag{4.2.4}$$

Then $\mathcal{E} \subset \Sigma_{(\mathrm{S3})}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$.

Proof. Let $E \in \Sigma_{(\mathrm{S3})}(\mathcal{E})$; say $E = \{\lambda(0, n_\lambda) : \lambda \in G\}$ where $G \in \mathcal{E}$ and $0 < n_\lambda \leq d(\lambda)$ for all $\lambda \in G$. Then $|E| \leq |G| < \infty$, proving that E is finite. To see that it is exhaustive, suppose that $\mu \in r(E)\Lambda$. Then there exists $\lambda \in G$ with $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$ because $G \in \mathcal{E} \subset \mathrm{FE}(\Lambda)$; say $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$. But then $\lambda(0, n_\lambda)(\lambda(n_\lambda, d(\lambda))\alpha = \mu\beta$ is a common extension of $\lambda(0, n_\lambda)$ and μ, giving $\Lambda^{\min}(\lambda(0, n_\lambda), \mu) \neq \emptyset$. Since $\mu \in r(E)\Lambda$ was arbitrary, it follows that E is exhaustive. Let $\lambda \in G$. Since $n_\lambda \leq d(\lambda)$ we have $d(\lambda(0, n_\lambda)) = n_\lambda$. Since $0 < n_\lambda$, it follows that $\lambda(0, n_\lambda) \notin \Lambda^0$. Since $\lambda \in G$ was arbitrary, we therefore have $E \cap \Lambda^0 = \emptyset$, so $E \in \mathrm{FE}(\Lambda)$. We have therefore established that $\Sigma_{(\mathrm{S3})}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$.

Next we must show that $\mathcal{E} \subset \Sigma_{(\mathrm{S3})}(\mathcal{E})$. For this, let $G \in \mathcal{E}$ and let $n_\lambda := d(\lambda)$ for every $\lambda \in G$. Since $G \in \mathcal{E} \subset \mathrm{FE}(\Lambda)$, we have $G \cap \Lambda^0 = \emptyset$, which ensures that $0 < n_\lambda \leq d(\lambda)$ for all $\lambda \in G$. Then $G = \{\lambda(0, n_\lambda) : \lambda \in G\} \in \Sigma_{(\mathrm{S3})}(\mathcal{E})$. □

Lemma 4.2.9. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Define*

$$\begin{aligned}\Sigma_{(\mathrm{S4})}(\mathcal{E}) := \big\{(G \setminus G') \cup \big(\textstyle\bigcup_{\lambda \in G'} \lambda G'_\lambda\big) : G \in \mathcal{E}, G' \subset G,\\ \textit{and } G'_\lambda \in s(\lambda)\mathcal{E} \textit{ for all } \lambda \in G'\big\}.\end{aligned} \tag{4.2.5}$$

Then $\mathcal{E} \subset \Sigma_{(S4)}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$.

PROOF. Suppose that $E \in \Sigma_{(S4)}(\mathcal{E})$; say $E = G \setminus G' \cup \big(\bigcup_{\lambda \in G'} \lambda G'_\lambda\big)$ where $G \in \mathcal{E}$. Then $|E| \leq |G| \cdot \max\{|G'_\lambda| : \lambda \in G'\} < \infty$, so E is finite. To see that E is exhaustive, let $\mu \in r(E)$. Since $G \in \mathcal{E} \subset \mathrm{FE}(\Lambda)$, there exists $\lambda \in G$ with $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. If $\lambda \notin G'$, then $\lambda \in E$, and we are done. On the other hand, if $\lambda \in G'$, let $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$. By definition of G'_λ, there exists $\sigma \in G'_\lambda$ such that $\Lambda^{\min}(\alpha, \sigma) \neq \emptyset$, say $(\eta, \zeta) \in \Lambda^{\min}(\alpha, \sigma)$. Then $\lambda\sigma\zeta = \lambda\alpha\eta = \mu\beta\eta$ is a common extension of $\lambda\sigma$ and μ, so $\Lambda^{\min}(\lambda\sigma, \mu) \neq \emptyset$. But $\lambda\sigma \in \lambda G'_\lambda \subset E$. Since $\mu \in r(E)\Lambda$ was arbitrary, it follows that E is exhaustive. Since $G \in \mathcal{E} \subset \mathrm{FE}(\Lambda)$, we have that $G \cap \Lambda^0 = \emptyset$. Hence $\mu \in G \setminus G'$ implies $d(\mu) > 0$, and $\mu \in \lambda G'_\lambda$ for some $\lambda \in G'$ implies $d(\mu) > d(\lambda) > 0$, giving $E \cap \Lambda^0 = \emptyset$. It follows that $\Sigma_{(S4)}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$. To see that $\Sigma_{(S4)}(\mathcal{E})$ contains $\mathcal{E}$, just take $G' = \emptyset$ in (4.2.5). □

NOTATION 4.2.10. We write $(\Sigma_{(S4)}\Sigma_{(S3)}\Sigma_{(S2)}\Sigma_{(S1)})^n$ for the n-fold iterated application of the composition $\Sigma_{(S4)} \circ \Sigma_{(S3)} \circ \Sigma_{(S2)} \circ \Sigma_{(S1)}$. That is, for a subset $\mathcal{E}$ of $\mathrm{FE}(\Lambda)$, the set $(\Sigma_{(S4)}\Sigma_{(S3)}\Sigma_{(S2)}\Sigma_{(S1)})^n(\mathcal{E})$ is equal to

$$\overbrace{\big(\Sigma_{(S4)} \circ \Sigma_{(S3)} \circ \Sigma_{(S2)} \circ \Sigma_{(S1)}\big) \circ \cdots \circ \big(\Sigma_{(S4)} \circ \Sigma_{(S3)} \circ \Sigma_{(S2)} \circ \Sigma_{(S1)}\big)}^{n \text{ terms}}(\mathcal{E})$$

for all $n \in \mathbb{N}$. Since Lemmas 4.2.5–4.2.9 show that $\mathcal{E} \subset \Sigma_{(S4)} \circ \Sigma_{(S3)} \circ \Sigma_{(S2)} \circ \Sigma_{(S1)}(\mathcal{E})$ for all $\mathcal{E}$, we write $(\Sigma_{(S4)}\Sigma_{(S3)}\Sigma_{(S2)}\Sigma_{(S1)})^\infty(\mathcal{E})$ for $\bigcup_{n=1}^\infty (\Sigma_{(S4)}\Sigma_{(S3)}\Sigma_{(S2)}\Sigma_{(S1)})^n(\mathcal{E})$.

COROLLARY 4.2.11. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Then $\mathcal{E} \subset (\Sigma_{(S4)}\Sigma_{(S3)}\Sigma_{(S2)}\Sigma_{(S1)})^\infty(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$ and $(\Sigma_{(S4)}\Sigma_{(S3)}\Sigma_{(S2)}\Sigma_{(S1)})^\infty(\mathcal{E})$ satisfies* (S1)–(S4).

PROOF. For convenience, we will write $\Sigma_{(S1\text{–}4)}$ in place of $\Sigma_{(S4)}\Sigma_{(S3)}\Sigma_{(S2)}\Sigma_{(S1)}$ for the duration of this proof. Lemmas 4.2.5–4.2.9 show immediately that $\mathcal{E} \subset \Sigma^\infty_{(S1\text{–}4)}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$. For the second statement, notice first that Lemmas 4.2.5–4.2.9 also establish that for $1 \leq i \leq 4$, and $n \in \mathbb{N}$, we have

$$\Sigma_{(Si)}\big(\Sigma^n_{(S1\text{–}4)}(\mathcal{E})\big) \subset \Sigma^{n+1}_{(S1\text{–}4)}(\mathcal{E}). \tag{4.2.6}$$

To see that $\Sigma^\infty_{(S1\text{–}4)}(\mathcal{E})$ satisfies (S1), let $G \in \Sigma^\infty_{(S1\text{–}4)}(\mathcal{E})$ and suppose that F is a finite subset of $r(F)\Lambda \setminus \Lambda^0$. Then $G \in \Sigma^n_{(S1\text{–}4)}(\mathcal{E})$ for some $n \in \mathbb{N}$, and then the definition of $\Sigma_{(S1)}$ combined with (4.2.6) shows that $G \cup F \in \Sigma^{n+1}_{(S1\text{–}4)}(\mathcal{E}) \subset \Sigma^\infty_{(S1\text{–}4)}(\mathcal{E})$.

To see that $\Sigma^\infty_{(S1\text{–}4)}(\mathcal{E})$ satisfies (S2), let $G \in \Sigma^\infty_{(S1\text{–}4)}(\mathcal{E})$ and suppose that $\lambda \in r(G)\Lambda \setminus G\Lambda$. Then $G \in \Sigma^n_{(S1\text{–}4)}(\mathcal{E})$ for some $n \in \mathbb{N}$, and then the definition of $\Sigma_{(S2)}$ combined with (4.2.6) shows that $\mathrm{Ext}(\lambda; G) \in \Sigma^{n+1}_{(S1\text{–}4)}(\mathcal{E}) \subset \Sigma^\infty_{(S1\text{–}4)}(\mathcal{E})$.

To see that $\Sigma^\infty_{\text{(S1–4)}}(\mathcal{E})$ satisfies (S3), let $G \in \Sigma^\infty_{\text{(S1–4)}}(\mathcal{E})$ and suppose that $0 < n_\lambda \le d(\lambda)$ for all $\lambda \in G$. Then $G \in \Sigma^n_{\text{(S1–4)}}(\mathcal{E})$ for some $n \in \mathbb{N}$, and then the definition of $\Sigma_{\text{(S3)}}$ combined with (4.2.6) shows that $\{\lambda(0, n_\lambda) : \lambda \in G\} \in \Sigma^{n+1}_{\text{(S1–4)}}(\mathcal{E}) \subset \Sigma^\infty_{\text{(S1–4)}}(\mathcal{E})$.

Finally, to see that $\Sigma^\infty_{\text{(S1–4)}}(\mathcal{E})$ satisfies (S4), let $G \in \Sigma^\infty_{\text{(S1–4)}}(\mathcal{E})$, suppose that $G' \subset G$ and that $G'_\lambda \in s(\lambda)\Sigma^\infty_{\text{(S1–4)}}(\mathcal{E})$ for each $\lambda \in G'$. Then $G \in \Sigma^n_{\text{(S1–4)}}(\mathcal{E})$ for some $n \in \mathbb{N}$, and for each $\lambda \in G'$, we have $G'_\lambda \in \Sigma^{n_\lambda}_{\text{(S1–4)}}(\mathcal{E})$ for some $n_\lambda \in \mathbb{N}$. Let $m := \max\{n, n_\lambda : \lambda \in G'\}$. Lemmas 4.2.5–4.2.9 show that G and each G'_λ belong to $\Sigma^m_{\text{(S1–4)}}(\mathcal{E})$ and then the definition of $\Sigma_{\text{(S4)}}$ combined with (4.2.6) shows that $(G \setminus G') \cup \big(\bigcup_{\lambda \in G'} \lambda G'_\lambda\big) \in \Sigma^{m+1}_{\text{(S1–4)}}(\mathcal{E}) \subset \Sigma^\infty_{\text{(S1–4)}}(\mathcal{E})$. □

Proposition 4.2.12. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Then $\overline{\mathcal{E}} = (\Sigma_{\text{(S4)}}\Sigma_{\text{(S3)}}\Sigma_{\text{(S2)}}\Sigma_{\text{(S1)}})^\infty(\mathcal{E})$; in particular, $\overline{\mathcal{E}} \subset \mathrm{FE}(\Lambda)$.*

Proof. The previous six results and the definition of $\overline{\mathcal{E}}$ combine to show that

$$\overline{\mathcal{E}} \subset (\Sigma_{\text{(S4)}}\Sigma_{\text{(S3)}}\Sigma_{\text{(S2)}}\Sigma_{\text{(S1)}})^\infty(\mathcal{E}).$$

But the construction of the maps $\Sigma_{\text{(S1)}}$, $\Sigma_{\text{(S2)}}$, $\Sigma_{\text{(S3)}}$ and $\Sigma_{\text{(S4)}}$ ensures that if $\mathcal{F} \subset \mathrm{FE}(\Lambda)$, then $\Sigma_{\text{(S1)}}(\mathcal{F}) \subset \overline{\mathcal{F}}$, $\Sigma_{\text{(S2)}}(\mathcal{F}) \subset \overline{\mathcal{F}}$, $\Sigma_{\text{(S3)}}(\mathcal{F}) \subset \overline{\mathcal{F}}$, and $\Sigma_{\text{(S4)}}(\mathcal{F}) \subset \overline{\mathcal{F}}$. Hence each $\big(\Sigma_{\text{(S4)}}\Sigma_{\text{(S3)}}\Sigma_{\text{(S2)}}\Sigma_{\text{(S1)}}\big)^n(\mathcal{E}) \subset \overline{\mathcal{E}}$, so $(\Sigma_{\text{(S4)}}\Sigma_{\text{(S3)}}\Sigma_{\text{(S2)}}\Sigma_{\text{(S1)}})^\infty(\mathcal{E}) \subset \overline{\mathcal{E}}$. □

Lemma 4.2.13. *Let (Λ, d) be a finitely aligned k-graph, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Toeplitz-Cuntz-Krieger Λ-family. Let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, and suppose that for all $E \in \mathcal{E}$, $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t^*_\lambda) = 0$. Then for every $G \in \overline{\mathcal{E}}$, we have $\prod_{\lambda \in G}(t_{r(G)} - t_\lambda t^*_\lambda) = 0$.*

Proof. Proposition 4.2.12 shows that it suffices to check that if G is a set produced by $\Sigma_{\text{(S1)}}$, $\Sigma_{\text{(S2)}}$, $\Sigma_{\text{(S3)}}$, or $\Sigma_{\text{(S4)}}$ from elements of $\mathcal{E}$, then $\prod_{\lambda \in G}(t_{r(G)} - t_\lambda t^*_\lambda) = 0$; but this is precisely the statement of Lemma 4.2.2. □

Corollary 4.2.14. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E}$ be a subset of $\mathrm{FE}(\Lambda)$. Then $C^*(\Lambda; \mathcal{E}) = C^*(\Lambda; \overline{\mathcal{E}})$.*

Proof. Lemma 4.2.13 ensures that $J_{\mathcal{E}} = J_{\overline{\mathcal{E}}}$. □

Ultimately our aim is to show that the converse of Lemma 4.2.13 holds in the universal algebra $C^*(\Lambda; \mathcal{E})$. That is, we want to show that if $E \in \mathrm{FE}(\Lambda)$ satisfies $\prod_{\lambda \in E}\big(s_{\mathcal{E}}(r(E)) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*\big) = 0$, then $E \in \overline{\mathcal{E}}$. The point is that then we can use Theorem 3.5.8 to decide which relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-families determine homomorphisms of $C^*(\Lambda; \mathcal{E})$ which are injective on the core. We achieve this aim in Theorem 4.3.12, but the proof requires the following technical lemma.

LEMMA 4.2.15. *Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Suppose that $E \in \overline{\mathcal{E}}$ and that $F \subset r(E)\Lambda \setminus \Lambda^0$ is finite and has the property that for all $\mu \in E$, either $\mu \in F\Lambda$ or $\mathrm{Ext}(\mu; F) \in \overline{\mathcal{E}}$. Then $F \in \overline{\mathcal{E}}$.*

PROOF. Define $G := E \setminus F\Lambda$, and for each $\mu \in G$, let $G_\mu := \mathrm{Ext}(\mu; F)$. Then each $G_\mu \in \overline{\mathcal{E}}$ by hypothesis, and hence (S4) gives

$$E' := \big((E \setminus G) \cup \big(\textstyle\bigcup_{\mu \in G} \mu G_\mu\big)\big) \in \overline{\mathcal{E}}.$$

For $\lambda \in E \setminus G$ we have $\lambda(0, n) \in F$ for some n; in this case, let $n_\lambda := n$. For $\mu \in G$ and $\lambda \in \mu G_\mu$, we have $\lambda = \mu\beta$ for some $\beta \in \mathrm{Ext}(\mu; F)$, so there exists $\sigma \in F$ and $\alpha \in \Lambda$ such that $(\alpha, \beta) \in \Lambda^{\min}(\sigma, \mu)$. Hence

$$\lambda(0, d(\sigma)) = (\mu\beta)(0, d(\sigma)) = (\sigma\alpha)(0, \sigma) = \sigma \in F;$$

in this case, set $n_\lambda := d(\sigma)$. Define $E'' := \{\lambda(0, n_\lambda) : \lambda \in E'\} \subset F$. We have $E'' \subset F$ by the previous paragraph. But $E' \in \overline{\mathcal{E}}$, and since $F \cap \Lambda^0 = \emptyset$, each n_λ is strictly greater than zero. Hence (S3) ensures that $E'' \in \overline{\mathcal{E}}$. Since $E'' \subset F \subset \Lambda \setminus \Lambda^0$ and since F is finite, it now follows from (S1) that $F \in \overline{\mathcal{E}}$. □

4.3. The relative boundary-path representation

In this section we identify a subset $\partial(\Lambda; \mathcal{E})$ of Λ^* with the property that the restriction of the partial isometries T_λ of the path space representation to $\ell^2(\partial(\Lambda; \mathcal{E}))$ yields a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family $\{S_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$. We show that for $E \in \mathrm{FE}(\Lambda)$, the projection $\prod_{\lambda \in E}(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^*)$ is equal to zero if and only if E belongs to the satiation $\overline{\mathcal{E}}$ of $\mathcal{E}$. We then produce a version of the gauge-invariant uniqueness theorem for $C^*(\Lambda; \mathcal{E})$.

DEFINITION 4.3.1. Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E}$ be a subset of $\mathrm{FE}(\Lambda)$. We say that $x \in \Lambda^*$ is an *$\mathcal{E}$-relative boundary path* of Λ if for every $n \in \mathbb{N}^k$ such that $n \leq d(x)$, and every E in the satiation $\overline{\mathcal{E}}$ of $\mathcal{E}$ such that $r(E) = x(n)$, there exists $\lambda \in E$ such that $x(n, n + d(\lambda)) = \lambda$. We denote the collection of all $\mathcal{E}$-relative boundary paths of Λ by $\partial(\Lambda; \mathcal{E})$.

EXAMPLE 4.3.2. Let (Λ, d) be the 3-graph of Example 4.4.9 and let $\mathcal{E}$ be as in the same example. So $\overline{\mathcal{E}}$ is the set described in the final paragraph of Example 4.4.9. Consider the $\mathcal{E}$-relative boundary paths with range $v_{(0,0,0)}$. By definition, each must have either $\lambda_{(1,0,0)}$ or else $(\lambda\mu\sigma)_{(-1,1,1)}$ as an initial segment. Moreover a boundary path which passes through the vertex $v_{(1,0,1)}$ must have either $\mu_{(1,1,1)}$ or $\mu_{(1,-1,1)}$ as its final segment. In this example these two conditions identify all the $\mathcal{E}$-relative boundary paths with range $v_{(0,0,0)}$. That is,

$$\big\{(\lambda\mu\sigma)_{(-1,1,1)}, \lambda_{(1,0,0)}, (\lambda\mu)_{(1,1,0)}, (\lambda\sigma)_{(1,0,-1)}, (\lambda\mu\sigma)_{(1,1,-1)}, (\lambda\mu\sigma)_{(1,1,1)}, (\lambda\mu\sigma)_{(1,-1,1)}\big\}$$

is a listing of all the elements of $v_{(0,0,0)}\partial(\Lambda;\mathcal{E})$.

In order to build a representation from the $\mathcal{E}$-relative boundary paths, we need to know that the maps $x \mapsto \lambda x$ and $x \mapsto x|_n^{d(x)}$ defined on Λ^* in Definition 3.7.3 restrict to maps from $\partial(\Lambda;\mathcal{E})$ to $\partial(\Lambda;\mathcal{E})$.

LEMMA 4.3.3. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *Suppose that $x \in \partial(\Lambda;\mathcal{E})$. If $n \in \mathbb{N}^k$ with $n \le d(x)$, then $x|_n^{d(x)} \in \partial(\Lambda;\mathcal{E})$. If $\lambda \in \Lambda$ with $s(\lambda) = r(x)$, then $\lambda x \in \partial(\Lambda;\mathcal{E})$.*

PROOF. For the first statement, let $m \le d(x|_n^{d(x)})$, and suppose $E \in \overline{\mathcal{E}}$ with $r(E) = (x|_n^{d(x)})(m) = x(n+m)$. Since x is an $\mathcal{E}$-relative boundary path, we know that there exists $\lambda \in E$ such that $x((n+m),(n+m)+d(\lambda)) = \lambda$. But $x((n+m),(n+m)+d(\lambda)) = \big(x|_n^{d(x)}\big)(m, m+d(\lambda))$, and since $m \le d(x|_n^{d(x)})$ was arbitrary, it follows that $x|_n^{d(x)} \in \partial(\Lambda;\mathcal{E})$.

Now suppose that $n \in \mathbb{N}^k$ is such that $n \le d(\lambda x)$, and suppose $E \in \overline{\mathcal{E}}$ with $r(E) = (\lambda x)(n)$. Let $\lambda' = (\lambda x)(n, n \vee d(\lambda))$, and let $x' = x|_{(n\vee d(\lambda))-d(\lambda)}^{d(x)}$, so that $(\lambda x)|_n^{d(\lambda x)} = \lambda' x'$, and $x' \in \partial(\Lambda;\mathcal{E})$ by the previous paragraph. We must show that there exists $\mu \in E$ such that $(\lambda' x')(0, d(\mu)) = \mu$. If there exists $\mu \in E$ with $d(\mu) \le d(\lambda')$ and $\lambda'(0, d(\mu)) = \mu$, we are done, so we may assume that $\lambda' \notin E\Lambda$. By (S2), we have $\mathrm{Ext}(\lambda', E) \in \overline{\mathcal{E}}$, and $r(\mathrm{Ext}(\lambda', E)) = s(\lambda') = r(x')$ by definition. Since $x' \in \partial(\Lambda;\mathcal{E})$, it follows that there exists $\alpha \in \mathrm{Ext}(\lambda'; E)$ such that $x'(0, d(\alpha)) = \alpha$; equivalently, there exists $\mu \in E$ and $(\alpha, \beta) \in \Lambda^{\min}(\lambda', \mu)$ such that $\alpha = x'(0, d(\alpha))$. But now $\lambda'\alpha = \mu\beta$, and in particular,

$$(\lambda' x')(0, d(\mu)) = \big(\lambda' x'(0, d(\alpha))\big)(0, d(\mu)) = (\mu\beta)(0, d(\mu)) = \mu. \qquad \square$$

REMARK 4.3.4. Condition (S2) of Definition 4.2.3 is indispensable in the proof of the second statement of Lemma 4.3.3. This is why we defined the $\mathcal{E}$-relative boundary path space in terms of $\overline{\mathcal{E}}$ rather than $\mathcal{E}$. This in turn provides some justification of all the hard work that went into defining $\overline{\mathcal{E}}$ in the previous section. In fact, Lemma 4.3.3 is the first step towards the ultimate justification of conditions (S1)–(S4) in Lemma 4.3.9.

Definition 4.3.5. Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Define operators $\{S_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\} \subset \mathcal{B}(\ell^2(\partial(\Lambda; \mathcal{E})))$ by

$$S_{\mathcal{E}}(\lambda)e_x := \begin{cases} e_{\lambda x} & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise} \end{cases}$$

for all $\lambda \in \Lambda$.

Remark 4.3.6. Lemma 4.3.3 shows that

$$S_{\mathcal{E}}(\lambda) = T_\lambda|_{\ell^2(\partial(\Lambda;\mathcal{E}))} \qquad \text{and} \qquad S^*_{\mathcal{E}}(\lambda) = T^*_\lambda|_{\ell^2(\partial(\Lambda;\mathcal{E}))}$$

for all $\lambda \in \Lambda$. It follows that for all $\lambda \in \Lambda$, we have $S_{\mathcal{E}}(\lambda)^* e_x = \delta_{x(0,d(\lambda)),\lambda} e_{x|^{d(x)}_{d(\lambda)}}$

We now need to show that the partial isometries of Definition 4.3.5 form a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family.

Lemma 4.3.7. *Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. The family of isometries $\{S_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\} \subset \mathcal{B}(\ell^2(\partial(\Lambda; \mathcal{E})))$ of Definition 4.3.5 is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family which we call the* $\mathcal{E}$-relative boundary path representation.

Proof. Since $\{T_\lambda : \lambda \in \Lambda\}$ satisfies (TCK1)–(TCK3) by Lemma 3.7.6, Remark 4.3.6 shows that $\{S_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ also satisfies (TCK1)–(TCK3). To check that $\{S_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ satisfies (CK), fix $E \in \mathcal{E}$ and $x \in \partial(\Lambda; \mathcal{E})$. If $r(x) \neq r(E)$, then $S_{\mathcal{E}}(r(E)) \prod_{\lambda \in E}(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^*)e_x$ is trivially equal to zero. On the other hand, if $r(x) = r(E)$, then there exists $\lambda \in E$ such that $x(0, d(\lambda)) = \lambda$ because $E \in \mathcal{E} \subset \overline{\mathcal{E}}$, and because $x \in \partial(\Lambda; \mathcal{E})$. But then

$$\begin{aligned}\textstyle\prod_{\mu \in E}&(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\mu)S_{\mathcal{E}}(\mu)^*)e_x \\ &= \Big(\textstyle\prod_{\mu \in E \setminus \{\lambda\}}(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\mu)S_{\mathcal{E}}(\mu)^*)\Big)\big(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^*\big)e_x,\end{aligned}$$

which is equal to zero since $S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^* e_x = e_{\lambda x|^{d(x)}_{d(\lambda)}} = e_x$. Since $E \in \mathcal{E}$ and $x \in \partial(\Lambda; \mathcal{E})$ were arbitrary, this establishes (CK). □

We now prove a technical lemma which will allow us to show that the $\mathcal{E}$-relative boundary path representation is always faithful on $C^*(\Lambda)^\gamma$. To prove this result, we need to use a lemma due to Farthing, Muhly, and Yeend. The author is grateful to Trent Yeend for providing him with a draft copy of [**10**] where this lemma appears. We give a proof of this lemma (basically that found in the current version of [**10**], modified slightly for notation and efficiency) in Appendix A.

Lemma 4.3.8 ([**10**, Lemma 1.5]). *Let (Λ, d) be a k-graph. If $v \in \Lambda^0$, $E \subset v\Lambda$, $\lambda_1 \in v\Lambda$ and $\lambda_2 \in s(\lambda_1)\Lambda$, then* $\mathrm{Ext}(\lambda_2; \mathrm{Ext}(\lambda_1; E)) = \mathrm{Ext}(\lambda_1\lambda_2; E)$.

LEMMA 4.3.9. *Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E}$ be a subset of $\mathrm{FE}(\Lambda)$. Then we have*

(1) *$v\partial(\Lambda;\mathcal{E})$ is nonempty for each $v \in \Lambda^0$.*
(2) *$v\partial(\Lambda;\mathcal{E}) \setminus F\partial(\Lambda;\mathcal{E})$ is nonempty whenever $v \in \Lambda^0$ and $F \in v\,\mathrm{FE}(\Lambda) \setminus v\overline{\mathcal{E}}$.*

The proofs of both statements of Lemma 4.3.9 proceed by constructing an $\mathcal{E}$-relative boundary path with the desired properties. The two constructions have a great deal in common, but the construction for statement (2) is somewhat more complicated. To avoid duplication, the author has chosen to present the full text of the proof of statement (2) below, but has typeset those parts of the proof which are germane only to statement (2) in slanted text, and enclosed them in square brackets. The idea is that by reading the proof below, but ignoring text [*like this*], one obtains a proof of statement (1), whilst by reading the whole proof including the slanted and bracketed text one obtains a proof of statement (2).

PROOF OF LEMMA 4.3.9. Define a function $P : (\mathbb{N} \setminus \{0\})^2 \to (\mathbb{N} \setminus \{0\})$ by

$$P(m,n) := \frac{(m+n-1)(m+n-2)}{2} + m.$$

Then P is the position function corresponding to the diagonal listing

$$\begin{array}{lllll} (1,1), & & & & \\ (1,2), & (2,1), & & & \\ (1,3), & (2,2), & (3,1), & & \\ (1,4), & (2,3), & (3,2), & (4,1), & \\ \vdots & & & & \ddots \end{array}$$

of $(\mathbb{N}\setminus\{0\})^2$. That is, if (m,n) is the l^{th} term in the above sequence, then $P(m,n) = l$. For all $l \in \mathbb{N} \setminus \{0\}$, define (i_l, j_l) to be the unique element of $(\mathbb{N} \setminus \{0\})^2$ such that $P(i_l, j_l) = l$.

Fix $v \in \Lambda^0$ $\big[$*and fix $F \in v\,\mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$*$\big]$.

Claim 1: There exist a sequence $\{\lambda_l : l \geq 1\} \subset v\Lambda$ such that $\lambda_l(0, d(\lambda_{l-1})) = \lambda_{l-1}$ for all $l \geq 2$, and listings $\{E_{l,j} : j \geq 1\}$ of $s(\lambda_l)\overline{\mathcal{E}}$ for all $l \geq 1$ satisfying

(i) $\lambda_{l+1}\big(d(\lambda_{i_l}), d(\lambda_{l+1})\big)$ belongs to $E_{i_l,j_l}\Lambda$ for all $l \geq 1$.
$\big[$*(ii) $\mathrm{Ext}(\lambda_{l+1}; F)$ belongs to $\mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$ for all $l \geq 0$.*$\big]$

Proof of Claim 1. We proceed by induction on l. For a base case, we set $\lambda_1 := v$. For each $w \in \Lambda^0$, the collection of finite subsets of $w\Lambda$ is countable because Λ is countable. In particular, $w\overline{\mathcal{E}}$ is countable. Let $\{E_{1,j} : j \in \mathbb{N} \setminus \{0\}\}$ be any listing of $v\overline{\mathcal{E}}$. Note that (i) is trivial in this case because $l = 0$ $\big[$*and (ii) is trivial because* $\mathrm{Ext}(v; F) = F\big]$.

Now suppose as an inductive hypothesis that $l \geq 1$, and that λ_n and $\{E_{n,j} : j \geq 1\}$ exist and satisfy (i) $\big[$*and (ii)*$\big]$ for $1 \leq n \leq l$. We now construct λ_{l+1} satisfying (i) $\big[$*and (ii)*$\big]$ (noticing that $i_l < l$, so that the set E_{i_l,j_l} appearing in (i) has already been defined by the inductive hypothesis), and choose $\{E_{i_{l+1},j} : j \geq 1\}$ to be any listing of $s(\lambda_{l+1})\overline{\mathcal{E}}$. The details are as follows.

Let $\lambda^l_{i_l} := \lambda_l\big(d(\lambda_{i_l}), d(\lambda_l)\big)$ be the final segment of λ_l whose range $s(\lambda_{i_l})$ is equal to the range of E_{i_l,j_l}. Suppose first that $\lambda^l_{i_l}$ belongs to $E_{i_l,j_l}\Lambda$. Then we don't need to do anything at this step, so we define $\lambda_{l+1} := \lambda_l$, and $E_{i_{l+1},j} := E_{i_l,j}$ for all $j \geq 1$. We have that λ_{l+1} satisfies (i) because we supposed $\lambda^l_{i_l}$ to belong to $E_{i_l,j_l}\Lambda$. $\big[$*We have that λ_{l+1} satisfies (ii) because λ_l satisfies (ii) by the inductive hypothesis.*$\big]$

Now suppose that $\lambda^l_{i_l}$ does not belong to $E_{i_l,j_l}\Lambda$. Then we must extend λ_l to $\lambda_{l+1} := \lambda_l\nu_{l+1}$ so that $\lambda^l_{i_l}\nu_{l+1}$ *does* have an initial segment from E_{i_l,j_l} $\big[$*and we must do so in such a way as to ensure that* $\operatorname{Ext}(\lambda_{l+1}; F)$ *belongs to* $\operatorname{FE}(\Lambda)$ *but not to* $\overline{\mathcal{E}}\,\big]$. For this, let $E := \operatorname{Ext}(\lambda^l_{i_l}; E_{i_l,j_l})$. For any element $\alpha \in E$ we have that $\lambda^l_{i_l}\alpha = \mu\beta$ for some $\mu \in E$ and $(\alpha, \beta) \in \Lambda^{\min}(\lambda^l_{i_l}, \mu)$. It follows that any choice of ν_{l+1} from E will ensure that $\lambda_{l+1} := \lambda_l\nu_{l+1}$ satisfies (i).

$\big[$*To complete the construction of λ_{l+1}, we need only show that there exists a choice of $\nu_{l+1} \in E$ such that $\lambda_{l+1} := \lambda_l\nu_{l+1}$ also satisfies (ii). Since λ_l satisfies (ii), we have that* $F_l := \operatorname{Ext}(\lambda_l; F)$ *belongs to* $\operatorname{FE}(\Lambda) \setminus \overline{\mathcal{E}}$. *By the contrapositive statement of Lemma 4.2.15, it follows that there exists $\alpha \in E \setminus F_l\Lambda$ such that* $\operatorname{Ext}(\alpha; F_l)$ *does not belong to* $\overline{\mathcal{E}}$. *On the other hand, Lemma 4.2.7 ensures that* $\operatorname{Ext}(\alpha; F_l) \in \operatorname{FE}(\Lambda)$. *That is,* $\operatorname{Ext}(\alpha; F_l) \in \operatorname{FE}(\Lambda) \setminus \overline{\mathcal{E}}$. *Let $\nu_{l+1} := \alpha$, and define $\lambda_{l+1} := \lambda_l\nu_{l+1}$. Then*

$$\operatorname{Ext}(\lambda_{l+1}; F) = \operatorname{Ext}(\lambda_l\nu_{l+1}; F) = \operatorname{Ext}(\nu_{l+1}; \operatorname{Ext}(\lambda_l; F)) \tag{4.3.1}$$

by Lemma 4.3.8. But $\operatorname{Ext}(\lambda_l; F) = F_l$ *by definition, so* (4.3.1) *gives* $\operatorname{Ext}(\lambda_{l+1}; F) = \operatorname{Ext}(\nu_{l+1}; F_l)$ *which belongs to* $\operatorname{FE}(\Lambda) \setminus \overline{\mathcal{E}}$ *by choice of ν_{l+1}. Hence λ_{l+1} satisfies (ii) as required.*$\big]$

This choice of λ_{l+1} satisfies $\lambda_{l+1}(0, d(\lambda_l)) = \lambda_l$ by definition. $\square$ Claim 1

Let $m := \lim_{l\to\infty} d(\lambda_l) \in (\mathbb{N} \cup \{\infty\})^k$. By Lemma 3.7.2, there is a unique graph morphism $x : \Omega_{k,m} \to \Lambda$ such that $x(0, d(\lambda_l)) = \lambda_l$ for all $l \in \mathbb{N} \setminus \{0\}$. To complete the proof, we show that x belongs to $v\partial(\Lambda; \mathcal{E})$ $\big[$*and that x does not belong to $F\partial(\Lambda; \mathcal{E})$*$\big]$.

We have that $r(x) = v$ by definition, so to see that $x \in v\partial(\Lambda; \mathcal{E})$, suppose that $M \in \mathbb{N}^k$ with $M \leq m$. Let $E \in x(M)\overline{\mathcal{E}}$. We must show that there exists $N \geq M$ such that $x(M, N) \in E$. By definition of x there exists $l \geq 1$ such that

$M \leq d(\lambda_l)$. If $\lambda_l(M, d(\lambda_l))$ belongs to $E\Lambda$, then we are done, so suppose that $\lambda_l(M, d(\lambda_l)) \notin E\Lambda$. By (S3), it follows that $G := \operatorname{Ext}(\lambda_l(M, d(\lambda_l)); E) \in s(\lambda_l)\overline{\mathcal{E}}$, and hence that $G = E_{i_l,j}$ for some $j \geq 1$. But then property (i) ensures that $\lambda_{P(i_l,j)+1}(M, N) \in E$ for some N, and it follows that $x(M, N) \in E$ as required.

$\big[$*Finally we must show that* $x \notin F\Lambda$. *Suppose for contradiction that* $x \in F\Lambda$. *Then* $x(0, N) \in F$ *for some* N, *and it follows from the definition of* x *that there exists* $l \geq 1$ *such that* $\lambda_l(0, N) = x(0, N) \in F$. *Hence* $s(\lambda_l)$ *belongs to* $\operatorname{Ext}(\lambda_l; F)$. *But* $G \in \operatorname{FE}(\Lambda)$ *implies that* $G \cap \Lambda^0 = \emptyset$ *by Definition 2.4.3, so* $s(\lambda_l) \in \operatorname{Ext}(\lambda_l; F)$ *contradicts (ii). Hence* $x \notin F\Lambda$.$\big]$ □

Corollary 4.3.10. *Let* (Λ, d) *be a finitely aligned* k*-graph, and let* $\mathcal{E} \subset \operatorname{FE}(\Lambda)$. *The* $\mathcal{E}$*-relative boundary-path representation satisfies* $S_{\mathcal{E}}(v) \neq 0$ *for all* $v \in \Lambda^0$, *and* $\prod_{\lambda \in E}\big(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^*\big) \neq 0$ *for all* $E \in \operatorname{FE}(\Lambda) \setminus \overline{\mathcal{E}}$.

Proof. Let $v \in \Lambda^0$. By Lemma 4.3.9(1), there exists $x \in \partial(\Lambda; \mathcal{E})$ such that $r(x) = v$. But then $S_{\mathcal{E}}(v)e_x = e_x \neq 0$, and hence $S_{\mathcal{E}}(v) \neq 0$. Now let $E \in \operatorname{FE}(\Lambda) \setminus \overline{\mathcal{E}}$. By Lemma 4.3.9(2), there exists $x \in r(E)\partial(\Lambda; \mathcal{E}) \setminus E\partial(\Lambda; \mathcal{E})$. But then $S_{\mathcal{E}}(r(E))e_x = e_x$, while $S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^* e_x = 0$ for all $\lambda \in E$. Hence $\prod_{\lambda \in E}\big(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^*\big)e_x = e_x \neq 0$. □

Corollary 4.3.11. *Let* (Λ, d) *be a finitely aligned* k*-graph and let* $\mathcal{E} \subset \operatorname{FE}(\Lambda)$. *Let* $\{s_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ *be the universal generating relative Cuntz-Kreiger* $(\Lambda; \mathcal{E})$*-family in* $C^*(\Lambda; \mathcal{E})$. *Then*

(1) $s_{\mathcal{E}}(v) \neq 0$ *for all* $v \in \Lambda^0$; *and*
(2) *if* $E \in \operatorname{FE}(\Lambda)$ *then* $\prod_{\lambda \in E}\big(s_{\mathcal{E}}(r(E)) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*\big) = 0$ *if and only if* E *belongs to* $\overline{\mathcal{E}}$.

Proof. Let $\pi^{\mathcal{E}}_{S_{\mathcal{E}}} : C^*(\Lambda; \mathcal{E}) \to \mathcal{B}\big(\ell^2(\partial(\Lambda; \mathcal{E}))\big)$ be the unique representation of $C^*(\Lambda; \mathcal{E})$ such that $\pi^{\mathcal{E}}_{S_{\mathcal{E}}}(s_{\mathcal{E}}(\lambda)) = S_{\mathcal{E}}(\lambda)$ for all $\lambda \in \Lambda$. Corollary 4.3.10 shows that for all $v \in \Lambda^0$ and for all $E \in \operatorname{FE}(\Lambda) \setminus \overline{\mathcal{E}}$, the projections $S_{\mathcal{E}}(v) = \pi^{\mathcal{E}}_{S_{\mathcal{E}}}(s_{\mathcal{E}}(v))$ and $\prod_{\lambda \in E}\big(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^*\big) = \pi^{\mathcal{E}}_{S_{\mathcal{E}}}\big(\prod_{\lambda \in E}\big(s_{\mathcal{E}}(r(E)) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*\big)\big)$ are both nonzero, establishing condition (1) and the "only if" direction of condition (2). But Lemma 4.2.13 shows that every relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family satisfies the "if" direction of condition (2), and the result follows. □

Theorem 4.3.12 (The gauge-invariant uniqueness theorem). *Let* (Λ, d) *be a finitely aligned* k*-graph and let* $\mathcal{E} \subset \operatorname{FE}(\Lambda)$. *Suppose that* $\{t_\lambda : \lambda \in \Lambda\}$ *is a relative Cuntz-Krieger* $(\Lambda; \mathcal{E})$*-family. Then the homomorphism* $\pi^{\mathcal{E}}_t : C^*(\Lambda; \mathcal{E}) \to C^*(\{t_\lambda :$

$\lambda \in \Lambda\})$ *such that* $\pi_t^{\mathcal{E}}(s_{\mathcal{E}}(\lambda)) = t_\lambda$ *for all* $\lambda \in \Lambda$ *is injective on* $C^*(\Lambda; \mathcal{E})^\gamma$ *if and only if*

(1) $t_v \neq 0$ *for all* $v \in \Lambda^0$*; and*
(2) $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda) \neq 0$ *for all* $E \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$.

Moreover, $\pi_t^{\mathcal{E}}$ *is injective on all of* $C^*(\Lambda; \mathcal{E})$ *if and only if* (1) *and* (2) *hold and additionally,*

(3) *there exists a strongly continuous action* $\theta : \mathbb{T}^k \to \mathrm{Aut}(C^*(\{t_\lambda : \lambda \in \Lambda\})$ *satisfying* $\theta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ *for all* $\lambda \in \Lambda$.

PROOF. If $\pi_t^{\mathcal{E}}$ is faithful on $C^*(\Lambda; \mathcal{E})^\gamma$, then (1) and (2) follow immediately from Corollary 4.3.11. Now suppose that (1) and (2) hold. Lemma 4.2.13 shows that $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$ for all $E \in \overline{\mathcal{E}}$. Combining this with (2), we therefore have that for $E \in \mathrm{FE}(\Lambda)$, the projection $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*)$ is equal to zero if and only if $E \in \overline{\mathcal{E}}$. By Corollary 4.3.11, we therefore have that for $E \in \mathrm{FE}(\Lambda)$,

$$\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0 \quad \text{if and only if} \quad \prod_{\lambda \in E}\big(s_{\mathcal{E}}(r(E)) - s_{\mathcal{E}}(\lambda) s_{\mathcal{E}}(\lambda)^*\big) = 0.$$

Since assumption (1) and statement (1) of Corollary 4.3.11 show that both t_v and $s_{\mathcal{E}}(v)$ are nonzero for all $v \in \Lambda^0$, all the hypotheses of Theorem 3.5.8 are now satisfied, and we can therefore use Theorem 3.5.8 to deduce that $\pi_t^{\mathcal{E}}$ is injective on $C^*(\Lambda; \mathcal{E})^\gamma$.

Now suppose that $\pi_t^{\mathcal{E}}$ is faithful on all of $C^*(\Lambda; \mathcal{E})$. Then setting $\theta := \pi_t^{\mathcal{E}} \circ \gamma$ establishes (3), and since $\pi_t^{\mathcal{E}}$ is, in particular, injective on $C^*(\Lambda; \mathcal{E})^\gamma$, (1) and (2) follow from the previous paragraph. Now suppose that (1), (2) and (3) all hold. Then the previous paragraph shows that $\pi_t^{\mathcal{E}}$ is faithful on $C^*(\Lambda; \mathcal{E})^\gamma$, and then it follows from Corollary 4.1.8 and (3) that $\pi_t^{\mathcal{E}}$ is injective on all of $C^*(\Lambda; \mathcal{E})$. □

COROLLARY 4.3.13. *Let* (Λ, d) *be a finitely aligned* k*-graph, and let* $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. *The* $\mathcal{E}$*-relative boundary-path representation of* $C^*(\Lambda; \mathcal{E})$ *is faithful on the core.*

PROOF. The result follows from Corollary 4.3.10 and Corollary 4.3.11. □

NOTATION 4.3.14. Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Let $E \subset \Lambda$ be finite. We define $(d \times s)_{\mathcal{E}}^{+}(\Pi E)$ to be the set of pairs $(n, v) \in \mathbb{N}^k \times \Lambda^0$ such that $(\Pi E)v \cap \Lambda^n$ is nonempty and such that $T^{\Pi E}(n, v) \notin \overline{\mathcal{E}}$, where $T^{\Pi E}(n, v) = \{\nu \in v\Lambda \setminus \Lambda^0 : \lambda\nu \in \Pi E \text{ for } \lambda \in (\Pi E)v \cap \Lambda^n\}$ as in Definition 3.6.1.

COROLLARY 4.3.15. *Let (Λ, d) be a finitely aligned k-graph, let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, and let $E \subset \Lambda$ be finite. Then*

$$M^{s_{\mathcal{E}}}_{\Pi E} \cong \bigoplus_{(n,v)\in(d\times s)^+_{\mathcal{E}}(\Pi E)} M_{(\Pi E)v\cap\Lambda^n}(\mathbb{C}).$$

PROOF. By Corollary 4.3.11(2), we know that when $T^{\Pi E}(n, v) \in \mathrm{FE}(\Lambda)$, we have $\prod_{\lambda\in T^{\Pi E}(n,v)} \big(s_{\mathcal{E}}(v) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*\big) = 0$ if and only if $T^{\Pi E}(n, v) \in \overline{\mathcal{E}}$. Proposition 3.5.3 shows that for $\lambda, \mu \in (\Pi E)v \cap \Lambda^n$, the matrix unit $\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu}$ is nonzero if $T^{\Pi E}(n, v)$ is not in $\mathrm{FE}(\Lambda)$, and that if $T^{\Pi E}(n, v) \in \mathrm{FE}(\Lambda)$, then $\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu} = 0$ if and only if $\prod_{\lambda\in T^{\Pi E}(n,v)} \big(s_{\mathcal{E}}(v) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*\big) = 0$. Hence combining these two results shows that for $\lambda, \mu \in (\Pi E)v \cap \Lambda^n$, we have $\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu} = 0$ if and only if $T^{\Pi E}(n, v) \in \overline{\mathcal{E}}$; that is, if and only if $(n, v) \notin (d \times s)^+_{\mathcal{E}}(\Pi E)$. Hence the nontrivial direct summands in the decomposition of $M^t_{\Pi E}$ provided by Lemma 3.6.2 are precisely those for which $(n, v) \in (d \times s)^+_{\mathcal{E}}(\Pi E)$, proving the result. □

4.4. Satiations: a more efficient construction

In this section we develop a somewhat more efficient construction of $\overline{\mathcal{E}}$ from $\mathcal{E}$ than that provided by Proposition 4.2.12. By more efficient, we mean that to apply the construction in this section to a given example requires less steps. The tradeoff is that we must work much harder to prove that the construction given actually produces $\overline{\mathcal{E}}$. The additional work we need to do to is quite technical, so readers who are not interested in applying the satiation construction to specific examples of k-graphs may wish to skip this section. We finish the section with an example in which we apply our new construction.

LEMMA 4.4.1. *Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *Then $\Sigma_{(\mathrm{S1})}(\mathcal{E})$ satisfies* (S1).

PROOF. Suppose that $E \in \Sigma_{(\mathrm{S1})}(\mathcal{E})$, and that $E \subset G$ where G is a finite subset of $r(E)\Lambda \setminus \{r(E)\}$. Since $E \in \Sigma_{(\mathrm{S1})}$, there exists $F \subset E$ with $F \in \mathcal{E}$. But now we have $F \subset E \subset G$ and hence $G \in \Sigma_{(\mathrm{S1})}$ by definition. □

LEMMA 4.4.2. *Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *Then $\Sigma_{(\mathrm{S2})}(\mathcal{E})$ satisfies* (S2). *If $\mathcal{E}$ satisfies* (S1) *then so does $\Sigma_{(\mathrm{S2})}(\mathcal{E})$.*

PROOF. To see that $\Sigma_{(\mathrm{S2})}(\mathcal{E})$ satisfies (S2), let $E \in \Sigma_{(\mathrm{S2})}(\mathcal{E})$, and suppose that $\mu \in r(E)\Lambda \setminus E\Lambda$. If $E \in \mathcal{E}$, then $\mathrm{Ext}(\mu; E) \in \Sigma_{(\mathrm{S2})}(\mathcal{E})$ by definition, so we may suppose that $E = \mathrm{Ext}(\lambda; G)$ for some $G \in \mathcal{E}$ and $\lambda \in r(G)\Lambda\setminus G\Lambda$. By Lemma 4.3.8, we have $\operatorname{Ext}(\mu; E) = \operatorname{Ext}(\lambda\mu; G)$. Since $\mu \notin E\Lambda$, we have $s(\mu) \notin \operatorname{Ext}(\mu; E)$; that is $s(\lambda\mu) \notin \operatorname{Ext}(\lambda\mu; G)$, so $\lambda\mu \in r(G)\Lambda \setminus G\Lambda$. It follows that $\operatorname{Ext}(\mu; E) = \operatorname{Ext}(\lambda\mu; G)$ belongs to $\Sigma_{(\mathrm{S2})}(\mathcal{E})$, so $\Sigma_{(\mathrm{S2})}(\mathcal{E})$ satisfies (S2) as required.

Suppose now that $\mathcal{E}$ satisfies (S1), and suppose that $G \in \Sigma_{(\mathrm{S2})}(\mathcal{E})$, and that $E = G \cup F$ for some finite $F \subset r(G)\Lambda \setminus \Lambda^0$. If $G \in \mathcal{E}$, then $E \in \mathcal{E}$ because $\mathcal{E}$ satisfies (S1), so suppose that $G = \operatorname{Ext}(\mu; G')$ for some $G' \in \mathcal{E}$ and $\mu \in r(G')\Lambda \setminus G'\Lambda$. Then $G' \cup \{\mu(E \setminus G)\} \in \mathcal{E}$ because $\mathcal{E}$ satisfies (S1), and hence $E = \operatorname{Ext}(\mu; G' \cup \{\mu(E \setminus G)\}) \in \Sigma_{(\mathrm{S2})}(\mathcal{E})$. □

LEMMA 4.4.3. *Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *Then $\Sigma_{(\mathrm{S3})}(\mathcal{E})$ satisfies* (S3). *If $\mathcal{E}$ satisfies* (S1) *then so does $\Sigma_{(\mathrm{S3})}(\mathcal{E})$. If $\mathcal{E}$ satisfies* (S1) *and* (S2) *then so does $\Sigma_{(\mathrm{S3})}(\mathcal{E})$.*

Before proving Lemma 4.4.3, we need the following technical lemma:

LEMMA 4.4.4. *Let (Λ, d) be a finitely aligned k-graph, and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *Then $\Sigma_{(\mathrm{S3})}(\mathcal{E})$ is equal to the collection*

$$\begin{aligned}\big\{\{\lambda(0, n_\lambda) : \lambda \in G\} : G \in \mathcal{E}, 0 < n_\lambda \le d(\lambda) \textit{ for all } \lambda \in G,\\ \lambda(0, n_\lambda) \in G \textit{ implies } n_\lambda = d(\lambda)\big\}.\end{aligned} \tag{4.4.1}$$

PROOF. Let $E \in \Sigma_{(\mathrm{S3})}(\mathcal{E})$, say $E = \{\lambda(0, n_\lambda) : \lambda \in E'\}$ where $E' \in \mathcal{E}$ and $0 < n_\lambda \le d(\lambda)$ for all $\lambda \in E'$. We must show that E belongs to the set (4.4.1). We proceed by induction on $|\{\lambda \in E' \cap E : n_\lambda \ne d(\lambda)\}|$. If $|\{\lambda \in E' \cap E : n_\lambda \ne d(\lambda)\}| = 0$ then E belongs to the set (4.4.1) by definition. Now suppose that E belongs to (4.4.1) whenever $|\{\lambda \in E' \cap E : n_\lambda \ne d(\lambda)\}| \le N$ where $N \ge 0$, and fix $E = \{\lambda(0, n_\lambda) : \lambda \in E'\}$ where $E' \in \mathcal{E}$, $0 < n_\lambda \le d(\lambda)$ for all $\lambda \in E$ such that $|\{\lambda \in E' \cap E : n_\lambda \ne d(\lambda)\}| = N + 1$. We show that there exists a subset $\{n'_\lambda : \lambda \in E'\}$ of $\mathbb{N}^k$ such that

(1) $0 < n'_\lambda \le d(\lambda)$ for all $\lambda \in E'$;
(2) $\{\lambda(0, n'_\lambda) : \lambda \in E'\} = E$; and
(3) $|\{\lambda \in E' \cap E : n'_\lambda \ne d(\lambda)\}| = N$.

This will show that E belongs to (4.4.1) by the inductive hypothesis.

Since $N + 1 > 0$, we may fix $\lambda \in E' \cap E$ with $n_\lambda \ne d(\lambda)$. We have $\lambda = \mu(0, n_\mu)$ for some $\mu \in E'$, and we must have $d(\mu) > d(\lambda)$ for if not then $n_\mu = n_\lambda \ne d(\lambda)$ contradicting $\mu(0, n_\mu) = \lambda$. Furthermore, we have $0 < n_\lambda < d(\lambda)$ because $n < n_\lambda \le d(\lambda)$ by definition, and $n_\lambda \ne d(\lambda)$ by assumption. Define $n'_\lambda = d(\lambda)n_\mu$, and define $n'_\mu := n_\lambda$. For $\sigma \in E' \setminus \{\lambda, \mu\}$, let $n'_\sigma = n_\sigma$.

To check (1), note that for $\sigma \in E' \setminus \{\lambda, \mu\}$ we have $0 < n'_\sigma \leq d(\sigma)$ because $n'_\sigma = n_\sigma$. We have $0 < n'_\lambda \leq d(\lambda)$ because $\lambda \in E'$ so $d(\lambda) > 0$. We have $0 < n'_\mu \leq d(\mu)$ because $n'_\mu = n_\lambda$ and $0 < n_\lambda \leq d(\lambda) < d(\mu)$. This establishes (1).

For (2), notice that $\{\sigma(0, n'_\sigma) : \sigma \in E'\} = \{\sigma(0, n_\sigma) : \sigma \in E' \setminus \{\lambda, \mu\}\} \cup \{\lambda(0, n'_\lambda), \mu(0, n'_\mu)\}$. But $\lambda(0, n'_\lambda) = \lambda(0, d(\lambda)) = \lambda$ and $\mu(0, n'_\mu) = \mu(0, n_\lambda) = \lambda(0, n_\lambda)$ because $\mu(0, n_\mu) = \lambda$.

Finally, for (3), notice that $\{\sigma \in (E' \setminus \{\lambda, \mu\}) \cap E : n'_\sigma \neq \sigma\} = \{\sigma \in (E' \setminus \{\lambda, \mu\}) \cap E : n_\sigma \neq \sigma\}$ because $\sigma \in E' \setminus \{\lambda, \mu\}$ implies $n_\sigma = n'_\sigma$. Since $n'_\mu < n_\mu < d(\mu)$ we have $\mu \in \{\sigma \in E' \cap E : n'_\sigma \neq \sigma\}$ if and only if $\mu \in \{\sigma \in E' \cap E : n_\sigma \neq \sigma\}$. Finally since $\lambda \in E' \cap E$ by choice, we have $\lambda \in \{\sigma \in E' \cap E : n_\sigma \neq d(\sigma)\} \setminus \{\sigma \in E' \cap E : n'_\sigma \neq d(\sigma)\}$. This establishes (3). □

Proof of Lemma 4.4.3. To see that $\Sigma_{(S3)}(\mathcal{E})$ satisfies (S3), suppose $E \in \Sigma_{(S3)}(\mathcal{E})$ and that $0 < n_\lambda \leq d(\lambda)$ for every $\lambda \in E$. Since $E \in \Sigma_{(S3)}(\mathcal{E})$, there exists $G \in \mathcal{E}$ and a collection $\{m_\mu : \mu \in G\}$ such that $0 < m_\mu \leq d(\mu)$ for all $\mu \in G$ for which $E = \{\mu(0, m_\mu) : \mu \in G\}$. But now $0 < n_{\mu(0,m_\mu)} \leq m_\mu \leq d(\mu)$ for all $\mu \in G$, and $\{\lambda(0, n_\lambda) : \lambda \in E\} = \{\mu(0, n_{\mu(0,m_\mu)}) : \mu \in G\} \in \Sigma_{(S3)}(\mathcal{E})$ since $G \in \mathcal{E}$.

We now need to show that $\Sigma_{(S3)}$ preserves (S1) and (S2). Suppose that $\mathcal{E}$ satisfies (S1), let $E \in \Sigma_{(S3)}(\mathcal{E})$. By Lemma 4.4.4, we have that $E = \{\lambda(0, n_\lambda) : \lambda \in E'\}$ for some $E' \in \mathcal{E}$ where $0 < n_\lambda \leq d(\lambda)$ for all $\lambda \in E'$ and such that $\lambda \in E' \cap E$ implies $n_\lambda = d(\lambda)$. Suppose that $F = E \cup G$ for some finite $G \subset r(E)\Lambda \setminus \Lambda^0$, and suppose without loss of generality that $G \cap E = \emptyset$.

We must show that $F \in \Sigma_{(S3)}(\mathcal{E})$. Let

$$H := \big((E' \setminus G) \cup \{\lambda(0, n_\lambda) : \lambda \in E' \cap G\}\big) \sqcup G;$$

the second union is disjoint because $E \cap G = \emptyset$ by assumption. For $\lambda \in E' \setminus G$, define $m_\lambda := n_\lambda$ and for $\mu \in E' \cap G$ and $\lambda = \mu(0, n_\mu)$, define $m_\lambda := n_\mu$. To see that this is well defined, suppose that λ belongs to both $E' \setminus G$ and $\{\sigma(0, n_\sigma) : \sigma \in E' \cap G\}$; say $\lambda = \mu(0, n_\mu)$ where $\mu \in E' \cap G$. Then we have $\lambda \in E$ because $\mu \in E'$ so $\mu(0, d(\mu)) \in E$, and we have $\lambda \in E' \setminus G \subset E'$ by choice of λ. It follows from our use of Lemma 4.4.4 in choosing E' and $\{n_\sigma : \sigma \in E'\}$ that $n_\lambda = d(\lambda) = d(\mu(0, n_\mu)) = n_\mu$, so the two expressions given for m_λ agree. For $\lambda \in G$, define $m_\lambda = d(\lambda)$

Now $E' \subset H$ by definition of H, and since $H \subset E' \cup E \cup G$, we have $H \cap \Lambda^0 = \emptyset$ definition of E', E and G. Since $\mathcal{E}$ satisfies (S1) and $E' \in \mathcal{E}$, it follows that $H \in \mathcal{E}$. But then $F = \{\lambda(0, m_\lambda) : \lambda \in H\}$ belongs to $\Sigma_{(S3)}(\mathcal{E})$ by definition.

Now suppose that $\mathcal{E}$ satisfies (S2). Suppose that $G \in \mathcal{E}$ and that $\{n_\lambda : \lambda \in G\} \subset \mathbb{N}^k$ satisfies $0 < n_\lambda \le d(\lambda)$ for all $\lambda \in G$, so that $E := \{\lambda(0, n_\lambda) : \lambda \in G\}$ is a typical element of $\Sigma_{(\mathrm{S3})}(\mathcal{E})$. Suppose that $\mu \in r(E)\Lambda \setminus E\Lambda$. We must show that $\operatorname{Ext}(\mu; E) \in \Sigma_{(\mathrm{S3})}(\mathcal{E})$.

First notice that we have $\mu(0, n) \notin G$ for all $n \le d(\mu)$; for if $\mu(0, n) \in G$, then $\mu(0, n_{\mu(0,n)}) \in E$ by definition, contradicting our choice of μ. Hence $\operatorname{Ext}(\mu; G) \in \mathcal{E}$ because $\mathcal{E}$ satisfies (S2). By definition of $\operatorname{Ext}(\mu; G)$, for each $\beta \in \operatorname{Ext}(\mu; G)$, there exist $\lambda_\beta \in G$ and $\alpha_\beta \in s(\lambda_\beta)\Lambda$ such that $(\alpha_\beta, \beta) \in \Lambda^{\min}(\lambda_\beta, \mu)$. For $\beta \in \operatorname{Ext}(\mu; G)$, set

$$n_\beta := (d(\mu) \vee n_{\lambda_\beta}) - d(\mu).$$

Since $\mu \notin E\Lambda$, we have $d(\mu) \vee n_{\lambda_\beta} > d(\mu)$, and so $n_\beta > 0$ for all β. Moreover,

$$n_\beta = (d(\mu) \vee n_{\lambda_\beta}) - d(\mu) \le (d(\mu) \vee d(\lambda_\beta)) - d(\mu) = d(\beta).$$

It follows that $0 < n_\beta \le d(\beta)$ for all $\beta \in \operatorname{Ext}(\mu; G)$, giving

$$\{\beta(0, n_\beta) : \beta \in \operatorname{Ext}(\mu; G)\} \in \Sigma_{(\mathrm{S3})}(\mathcal{E}) \tag{4.4.2}$$

by definition.

Claim 1: $\{\beta(0, n_\beta) : \beta \in \operatorname{Ext}(\mu; G)\} \subset \operatorname{Ext}(\mu; E)$.

Proof of Claim 1. Suppose $\beta \in \operatorname{Ext}(\mu; G)$. Then $(\alpha_\beta, \beta) \in \Lambda^{\min}(\lambda_\beta, \mu)$ for some $\lambda_\beta \in G$ and $\alpha_\beta \in s(\lambda_\beta)\Lambda$. Hence $\lambda_\beta(0, n_{\lambda_\beta}) \in E$ by definition of E, and $\lambda_\beta(0, n_{\lambda_\beta})\lambda_\beta(n_{\lambda_\beta}, d(\lambda_\beta)) = \mu\beta$. Thus $(\mu\beta)(0, n_{\lambda_\beta} \vee d(\mu)) \in \operatorname{MCE}(\lambda_\beta(0, n_{\lambda_\beta}), \mu)$, giving $\beta\big(0, (n_{\lambda_\beta} \vee d(\mu)) - d(\mu)\big) \in \operatorname{Ext}(\mu; E)$. Since $n_\beta = n_{\lambda_\beta} - d(\mu)$ by definition, it follows that $\beta(0, n_\beta) \in \operatorname{Ext}(\mu; E)$ for all $\beta \in \operatorname{Ext}(\mu; G)$. □ Claim 1

We have already established that if $\mathcal{E}$ satisfies (S1), then so does $\Sigma_{(\mathrm{S3})}(\mathcal{E})$. Combining this with (4.4.2) and with Claim 1 shows that $\operatorname{Ext}(\mu; E) \in \mathcal{E}$ as required. □

Lemma 4.4.5. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \operatorname{FE}(\Lambda)$. Define*

$$\begin{aligned}\Sigma'_{(\mathrm{S4})}(\mathcal{E}) := \Big\{(G \setminus G') \cup \Big(\textstyle\bigcup_{\lambda \in G'} \lambda G'_\lambda\Big) : G \in \mathcal{E}, G' \subset G,\\ \text{and } G'_\lambda \in s(\lambda)\mathcal{E} \text{ or } s(\lambda) \in G'_\lambda \text{ for all } \lambda \in G'\Big\}.\end{aligned} \tag{4.4.3}$$

Then $\mathcal{E} \subset \Sigma'_{(\mathrm{S4})}(\mathcal{E}) \subset \operatorname{FE}(\Lambda)$. If $\mathcal{E}$ satisfies (S1) *then so does $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$. If $\mathcal{E}$ satisfies* (S1) *and* (S2) *then so does $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$.*

Proof. To see that $\mathcal{E} \subset \Sigma'_{(\mathrm{S4})}(\mathcal{E})$, take $G' = \emptyset$ in (4.4.3). To see that $\Sigma'_{(\mathrm{S4})}(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$, let $E := (G \setminus G') \cup \big(\bigcup_{\lambda \in G'} \lambda G'_\lambda\big)$ where $G \in \mathcal{E}$, G' is a subset of G, and for each $\lambda \in G'$, the set G'_λ belongs to $s(\lambda)\mathcal{E}$ or contains $s(\lambda)$. Let $G'' := \{\lambda \in G' : s(\lambda) \notin G'_\lambda\}$. Then Lemma 4.2.9 shows that $E' := (G \setminus G'') \cup \big(\bigcup_{\lambda \in G''} \lambda G'_\lambda\big)$ belongs to $\mathrm{FE}(\Lambda)$, and since $E' \subset E$, lemma 4.2.5 then shows that $E \in \mathrm{FE}(\Lambda)$.

Next suppose that $\mathcal{E}$ satisfies (S1), suppose that $E \in \Sigma'_{(\mathrm{S4})}(\mathcal{E})$, say $E = (E' \setminus F) \cup \big(\bigcup_{\lambda \in F} \lambda F_\lambda\big)$, and suppose that $G \subset \Lambda$ is finite with $E \subset G$. Define $G' = (G \setminus E) \cup E'$. Then $E' \subset G'$, and so $G' \in \mathcal{E}$ because $\mathcal{E}$ satisfies (S1). For $\lambda \in F \cap G$, let $F'_\lambda := F_\lambda \cup \{s(\lambda)\}$ and for $\lambda \in F \setminus G$, let $F'_\lambda := F_\lambda$. Then $G = (G' \setminus F) \cup \big(\bigcup_{\lambda \in F} \lambda F'_\lambda\big)$ and so belongs to $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$ by definition. Hence $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$ satisfies (S1).

Now suppose that $\mathcal{E}$ satisfies (S1) and (S2). We have that $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$ satisfies (S1) by the previous paragraph, so it suffices to show that $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$ satisfies (S2). Suppose that $E \in \mathcal{E}$, that $F \subset E$, and that for each $\lambda \in F$, either F_λ belongs to $s(\lambda)\mathcal{E}$ or $s(\lambda) \in F_\lambda$, so

$$E' := (E \setminus F) \cup \big(\textstyle\bigcup_{\lambda \in F} \lambda F_\lambda\big)$$

is a typical element of $\Sigma'_{(S4)}(\mathcal{E})$. Fix $\mu \in r(E')\Lambda \setminus E'\Lambda$. We must show that $\mathrm{Ext}(\mu; E') \in \Sigma'_{(\mathrm{S4})}\mathcal{E}$. We consider two cases.

Case 1: Suppose that $\mu \in E\Lambda$. Then there exists $N \leq d(\mu)$ such that $\mu(0, N) \in E$. Write $\xi := \mu(0, N)$ and $\nu := \mu(N, d(\mu))$. Since $\mu \notin E'\Lambda$, we have $\xi \in F$ and $s(\xi) \notin F_\xi$. By definition, we then have $F_\xi \in r(\nu)\mathcal{E}$. We have $\nu(0, n) \notin F_\xi$ for all $n \leq d(\nu)$ for otherwise we would have $\mu(0, N + n) = \xi\nu(0, n) \in \xi F_\xi \in E'$. Hence $\mathrm{Ext}(\nu; F_\xi) \in \mathcal{E}$ because $\mathcal{E}$ satisfies (S2). Since $\mu = \xi\nu$, we have $\mathrm{Ext}(\mu; \xi F_\xi) = \mathrm{Ext}(\nu; F_\xi) \in \mathcal{E}$, and since $\mathcal{E}$ satisfies (S1) we have $\mathrm{Ext}(\mu; E') \in \mathcal{E}$ because $\mathrm{Ext}(\mu; E') \in \mathrm{FE}(\Lambda)$ with $\mathrm{Ext}(\mu; \xi F_\xi) \subset \mathrm{Ext}(\mu; E')$. But we have already established that $\mathcal{E} \subset \Sigma'_{(\mathrm{S4})}(\mathcal{E})$, so $\mathrm{Ext}(\mu; E') \in \Sigma'_{(\mathrm{S4})}(\mathcal{E})$ as required. □ Case 1

Case 2: Suppose that $\mu \notin E\Lambda$. Since $\mathcal{E}$ satisfies (S2), we have $\mathrm{Ext}(\mu; E) \in \mathcal{E}$. By definition of E', we have

$$\mathrm{Ext}(\mu; E') = \mathrm{Ext}(\mu; (E \setminus F)) \cup \big(\textstyle\bigcup_{\lambda \in F} \mathrm{Ext}(\mu; \lambda F_\lambda)\big).$$

We aim to show that for each $\lambda \in F$ there exist sets $\{G^\lambda_\beta \in s(\beta)\Lambda : \beta \in \mathrm{Ext}(\mu; \{\lambda\})\}$ such that for all $\beta \in \mathrm{Ext}(\mu; \{\lambda\})$, either $G^\lambda_\beta \in \mathcal{E}$ or $s(\beta) \in G^\lambda_\beta$, and such that

$$\mathrm{Ext}(\mu; \lambda F_\lambda) = \textstyle\bigcup_{\beta \in \mathrm{Ext}(\mu; \{\lambda\})} \beta G^\lambda_\beta. \tag{4.4.4}$$

Suppose that such sets G^λ_β exist. For each $\beta \in \operatorname{Ext}(\mu; F)$, fix $\lambda_\beta \in F$ such that $\beta \in \operatorname{Ext}(\mu; \{\lambda_\beta\})$, and let $G_\beta := G^{\lambda_\beta}_\beta$. Then

$$\begin{aligned}
\operatorname{Ext}(\mu; E') &= \operatorname{Ext}(\mu; (E \setminus F)) \cup \big(\textstyle\bigcup_{\lambda\in F} \operatorname{Ext}(\mu; \lambda F_\lambda)\big) \\
&\supset \big(\operatorname{Ext}(\mu; E) \setminus \operatorname{Ext}(\mu; F)\big) \cup \big(\textstyle\bigcup_{\lambda\in F} \bigcup_{\beta\in\operatorname{Ext}(\mu;\{\lambda\})} \beta G^\lambda_\beta\big) \\
&\supset \big(\operatorname{Ext}(\mu; E) \setminus \operatorname{Ext}(\mu; F)\big) \cup \big(\textstyle\bigcup_{\beta\in\operatorname{Ext}(\mu;F)} \beta G^{\lambda_\beta}_\beta\big). \qquad (4.4.5)
\end{aligned}$$

Since $\mathcal{E}$ satisfies (S2), we have $\operatorname{Ext}(\mu; E) \in \mathcal{E}$, and we selected each G_β so that either $G_\beta \in \mathcal{E}$, or $s(\beta) \in G_\beta$. It follows by definition that (4.4.5) belongs to $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$. Since $\mathcal{E}$ satisfies (S1), we have already established that $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$ satisfies (S1). It follows that $\operatorname{Ext}(\mu; E') \in \Sigma'_{(\mathrm{S4})}(\mathcal{E})$, so $\Sigma'_{(\mathrm{S4})}(\mathcal{E})$ satisfies (S2) as required.

So it suffices to produce sets $\{G^\lambda_\beta : \lambda \in F, \beta \in \operatorname{Ext}(\mu; \{\lambda\})\}$ satisfying (4.4.4). For this, fix $\lambda \in F$. We have $F_\lambda \in \mathcal{E}$ or else $s(\lambda) \in F_\lambda$. For each $\beta \in \operatorname{Ext}(\mu; \{\lambda\})$, define $G^\lambda_\beta := \operatorname{Ext}(\mu\beta; \lambda F_\lambda)$.

We establish first that for all $\lambda \in F$ and all $\beta \in \operatorname{Ext}(\mu; \{\lambda\})$, we have either $G^\lambda_\beta \in \mathcal{E}$ or $s(\beta) \in G^\lambda_\beta$. We consider three possibilities. First suppose that $s(\lambda) \in F$. Since $\beta \in \operatorname{Ext}(\mu; \{\lambda\})$, we have $\mu\beta \in \lambda\Lambda \subset F\Lambda$, and hence we have $s(\beta) \in \operatorname{Ext}(\mu\beta; \lambda F_\lambda) = G^\lambda_\beta$. Next suppose that $s(\lambda) \notin F_\lambda$ and that $\alpha := (\mu\beta)(d(\lambda), d(\lambda) \vee d(\mu)) \in F_\lambda\Lambda$. Since $\mu\beta = \lambda\alpha$, we have $\operatorname{Ext}(\mu\beta; \lambda F_\lambda) = \operatorname{Ext}(\alpha; F_\lambda)$ which contains $s(\alpha)$ by our assumption that $\alpha \in F_\lambda\Lambda$. Since $s(\alpha) = s(\beta)$, it follows that $s(\beta) \in G^\lambda_\beta$. Finally, suppose that $s(\lambda) \notin F_\lambda$ and that $\alpha := (\mu\beta)(d(\lambda), d(\lambda)\vee d(\mu)) \notin F_\lambda\Lambda$. Since $s(\lambda) \notin F_\lambda$, we have $F_\lambda \in \mathcal{E}$. It follows that since $\alpha \notin F_\lambda\Lambda$ and $\mathcal{E}$ satisfies (S2), we have $\operatorname{Ext}(\alpha; F_\lambda) \in \mathcal{E}$ as well. But since $\lambda\alpha = \mu\beta$, we have $\operatorname{Ext}(\alpha; F_\lambda) = \operatorname{Ext}(\mu\beta; \lambda F_\lambda)$, and it follows that G^λ_β belongs to $\mathcal{E}$.

It remains to establish (4.4.4). Fix $\lambda \in F$ and $\beta \in \operatorname{Ext}(\mu; \{\lambda\})$. Then

$$\begin{aligned}
\sigma \in G^\lambda_\beta &\iff (\rho, \sigma) \in \Lambda^{\min}(\lambda\tau, \mu\beta) \text{ for some } \tau \in F_\lambda \text{ and } \rho \in s(\tau)\Lambda \\
&\iff \lambda\tau\rho = \mu\beta\sigma \text{ for some } \tau \in F_\lambda, \text{ and } d(\mu\beta\sigma) = d(\lambda\tau) \vee d(\mu\beta). \qquad (4.4.6)
\end{aligned}$$

We claim that $d(\lambda\tau)\vee d(\mu\beta) = d(\lambda\tau)\vee d(\mu)$. To see this, notice that $\beta \in \operatorname{Ext}(\mu; \lambda)$, so $d(\mu\beta) = d(\lambda) \vee d(\mu)$, and hence $d(\mu\beta)_i = \max\{d(\lambda)_i, d(\mu)_i\}$. But now we have

$$\begin{aligned}
(d(\lambda\tau) \vee d(\mu\beta))_i &= \max\{d(\lambda\tau)_i, d(\mu\beta)_i\} \\
&= \max\big\{d(\lambda\tau)_i, \max\{d(\lambda)_i, d(\mu)_i\}\big\} \\
&= \max\{d(\lambda\tau)_i, d(\lambda)_i, d(\mu)_i\} \\
&= \max\{d(\lambda\tau)_i, d(\mu)_i\} \quad \text{since } d(\lambda\tau) \geq d(\lambda) \\
&= (d(\lambda\tau) \vee d(\mu))_i,
\end{aligned}$$

establishing the claim. But now (4.4.6) shows that $\sigma \in G^\lambda_\beta$ if and only if $(\lambda\tau)\rho = \mu(\beta\sigma)$ and $d(\mu(\beta\sigma)) = d(\lambda\tau) \vee d(\mu)$ for some $\tau \in F_\lambda$. That is, $\sigma \in G^\lambda_\beta$ if and only if $\beta\sigma \in \operatorname{Ext}(\mu; \{\lambda\tau\})$ for some $\tau \in F_\lambda$. So we have

$$\sigma \in G^\lambda_\beta \iff \text{there exists } \tau \in F_\lambda \text{ such that } \beta\sigma \in \operatorname{Ext}(\mu; \{\lambda\tau\}). \tag{4.4.7}$$

This gives $\bigcup_{\beta \in \operatorname{Ext}(\mu;\{\lambda\})} \beta G^\lambda_\beta \subset \operatorname{Ext}(\mu; \lambda F_\lambda)$ immediately. Furthermore, if $\sigma \in \operatorname{Ext}(\mu; \lambda F_\lambda)$, then there exists $\tau \in F_\lambda$ such that $\mu\sigma = \lambda\tau\rho$ where $d(\mu\sigma) = d(\mu) \vee d(\lambda\tau)$. Taking $\beta := \sigma\big(0, (d(\lambda) \vee d(\mu)) - d(\mu)\big)$, and $\sigma' = \sigma(d(\beta), d(\sigma))$, (4.4.7) gives $\sigma' \in G^\lambda_\beta$, so $\operatorname{Ext}(\mu; \lambda F_\lambda) \subset \bigcup_{\beta \in \operatorname{Ext}(\mu;\{\lambda\})} \beta G^\lambda_\beta$, establishing (4.4.4). □ Case 2

It follows that $\Sigma'_{(S4)}(\mathcal{E})$ satisfies (S2) as required, completing the proof of Lemma 4.4.5. □

Notation 4.4.6. We write $(\Sigma'_{(S4)}\Sigma_{(S3)})^n$ for the n-fold iterated application of the composition $\Sigma'_{(S4)} \circ \Sigma_{(S3)}$. That is,

$$(\Sigma'_{(S4)}\Sigma_{(S3)})^n(\mathcal{E}) = \overbrace{(\Sigma'_{(S4)} \circ \Sigma_{(S3)}) \circ \cdots \circ (\Sigma'_{(S4)} \circ \Sigma_{(S3)})}^{n \text{ terms}}(\mathcal{E})$$

for all $n \in \mathbb{N}$. Since Lemmas 4.2.8 and 4.4.5 show that $\mathcal{E} \subset (\Sigma'_{(S4)} \circ \Sigma_{(S3)})(\mathcal{E})$ for all $\mathcal{E}$, we write $(\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$ for $\bigcup_{n=1}^\infty (\Sigma'_{(S4)}\Sigma_{(S3)})^n(\mathcal{E})$.

Corollary 4.4.7. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \operatorname{FE}(\Lambda)$. Then $\mathcal{E} \subset (\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E}) \subset \operatorname{FE}(\Lambda)$ and $(\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$ satisfies* (S3) *and* (S4)*. If $\mathcal{E}$ satisfies* (S1) (*respectively* (S1) *and* (S2)) *then so does $(\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$.*

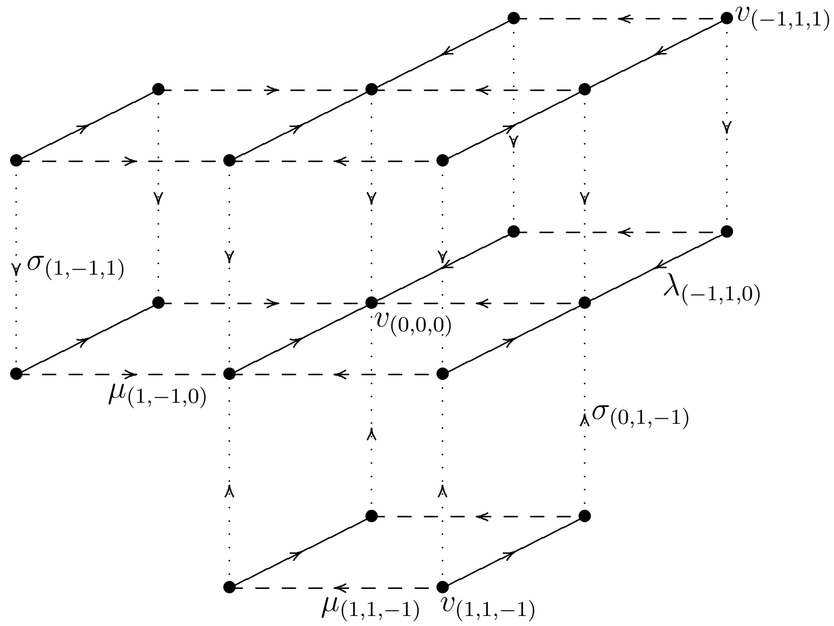
Proof. Lemmas 4.4.5 and 4.4.3 establish everything but the assertion that $(\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$ satisfies (S3) and (S4). To see that it satisfies (S3), suppose that $E \in (\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$ and that $0 < n_\lambda \le d(\lambda)$ for all $\lambda \in E$. By definition, we have $E \in (\Sigma'_{(S4)}\Sigma_{(S3)})^n(\mathcal{E})$ for some n, and then $\{\lambda(0, n_\lambda) : \lambda \in E\} \in \Sigma_{(S3)}((\Sigma'_{(S4)}\Sigma_{(S3)})^n(\mathcal{E}))$ by definition. But Lemma 4.4.5 shows that $\mathcal{E} \subset \Sigma'_{(S4)}(\mathcal{E})$ for all $\mathcal{E}$, and it follows that $\Sigma_{(S3)}((\Sigma'_{(S4)}\Sigma_{(S3)})^n(\mathcal{E})) \subset (\Sigma'_{(S4)}\Sigma_{(S3)})^{n+1}(\mathcal{E})$, giving $\{\lambda(0, n_\lambda) : \lambda \in E\} \in (\Sigma'_{(S4)}\Sigma_{(S3)})^{n+1}(\mathcal{E}) \subset (\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$ as required.

Now we must show that $(\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$ satisfies (S4). So suppose that $E \in (\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$, that $F \subset E$, and that $F_\lambda \in s(\lambda)(\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$ for all $\lambda \in F$. Then we have $E \in (\Sigma'_{(S4)}\Sigma_{(S3)})^N(\mathcal{E})$ and $F_\lambda \in (\Sigma'_{(S4)}\Sigma_{(S3)})^{N_\lambda}(\mathcal{E})$ for some $N, N_\lambda \in \mathbb{N}$. With $M := \max\{N, N_\lambda : \lambda \in F\}$, Lemmas 4.2.8 and 4.4.5 show that $E, F_\lambda \in (\Sigma'_{(S4)}\Sigma_{(S3)})^M(\mathcal{E})$. By another application of Lemma 4.2.8, it follows that $E, F_\lambda \in \Sigma_{(S3)}((\Sigma'_{(S4)}\Sigma_{(S3)})^M(\mathcal{E}))$, and it follows that $E \setminus F \cup \big(\bigcup_{\lambda \in F} \lambda F_\lambda\big) \in (\Sigma'_{(S4)}\Sigma_{(S3)})^{M+1}(\mathcal{E}) \subset (\Sigma'_{(S4)}\Sigma_{(S3)})^\infty(\mathcal{E})$ by definition of the map $\Sigma'_{(S4)}$. □

PROPOSITION 4.4.8. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Then $\overline{\mathcal{E}} = (\Sigma'_{(\mathrm{S4})}\Sigma_{(\mathrm{S3})})^\infty \circ \Sigma_{(\mathrm{S2})} \circ \Sigma_{(\mathrm{S1})}(\mathcal{E})$.*

PROOF. The results in this section combined with those in Section 4.2 show that $\overline{\mathcal{E}} \subset (\Sigma'_{(\mathrm{S4})}\Sigma_{(\mathrm{S3})})^\infty \circ \Sigma_{(\mathrm{S2})} \circ \Sigma_{(\mathrm{S1})}(\mathcal{E})$. On the other hand the construction of the maps $\Sigma_{(\mathrm{S1})}$, $\Sigma_{(\mathrm{S2})}$, $\Sigma_{(\mathrm{S3})}$ ensures that if $\mathcal{F} \subset \mathrm{FE}(\Lambda)$, then $\Sigma_{(\mathrm{S1})}(\mathcal{F}) \subset \overline{\mathcal{F}}$, $\Sigma_{(\mathrm{S2})}(\mathcal{F}) \subset \overline{\mathcal{F}}$, and $\Sigma_{(\mathrm{S3})}(\mathcal{F}) \subset \overline{\mathcal{F}}$. An argument identical to that used in the first paragraph of the proof of Lemma 4.4.5 shows that $\Sigma'_{(\mathrm{S4})}(\mathcal{F}) \subset \overline{\mathcal{F}}$. Hence each $\big(\Sigma_{(\mathrm{S4})}\Sigma_{(\mathrm{S3})}\big)^n \circ \Sigma_{(\mathrm{S2})} \circ \Sigma_{(\mathrm{S1})}(\mathcal{E}) \subset \overline{\mathcal{E}}$, giving $\big(\Sigma_{(\mathrm{S4})}\Sigma_{(\mathrm{S3})}\big)^\infty \circ \Sigma_{(\mathrm{S2})} \circ \Sigma_{(\mathrm{S1})}(\mathcal{E})\big) \subset \overline{\mathcal{E}}$. □

EXAMPLE 4.4.9. We now present an example which illustrates the construction of $\overline{\mathcal{E}}$ from $\mathcal{E}$. The example is quite long, and the calculation of $\overline{\mathcal{E}}$ is tedious, but it does demonstrate the effect of each step in the construction, and shows how to apply the procedure developed in this section to a concrete example. Let (Λ, d) be the unique 3-graph with 1-skeleton:



To label the edges in this 1-skeleton, and hence the paths in Λ, we denote solid edges (that is, those of degree $(1,0,0)$) by λ_n, dashed edges (that is, those of degree $(0,1,0)$) by μ_n, and dotted edges (those of degree $(0,0,1)$) by σ_n. The subscript n gives the cartesian coordinates of the source of the edge; the origin is located at the central vertex $v_{(0,0,0)}$ common to all four tri-coloured cubes in the 3-graph. A few of the edges and vertices in the 1-skeleton have been labelled in this fashion to illustrate the labelling method. To label paths in the k-graph, we use a word in λ, μ, σ to indicate the degree of the path, and a subscript to indicate its source. For example, the path with source $v_{(1,-1,1)}$ and degree $(1,0,1)$ is denoted $(\lambda\sigma)_{(1,-1,1)}$, and has range $v_{(0,-1,0)}$.

Let $E := \{\lambda_{(1,0,0)}, (\lambda\mu\sigma)_{(-1,1,1)}\}$, let $E' := \{\mu_{(1,-1,1)}, \mu_{(1,1,1)}\}$, and let $\mathcal{E} := \{E, E'\}$. We will apply Proposition 4.4.8 to determine $\overline{\mathcal{E}}$. We begin by applying $\Sigma_{(\mathrm{S1})}$. That is, we replace $\mathcal{E}$ with $\Sigma_{(\mathrm{S1})}(\mathcal{E})$ which consists of E' along with all finite subsets of $v_{(0,0,0)}\Lambda \setminus \Lambda^0$ which contain E; an application of $\Sigma_{(\mathrm{S1})}$ does not change E' because $E' = v_{(1,0,1)}\Lambda \setminus \Lambda^0$. So

$$\Sigma_{(\mathrm{S1})}(\mathcal{E}) = \{E \cup F : F \subset v_{(0,0,0)}\Lambda \setminus \{v_{(0,0,0)}\}\} \cup \{E'\}.$$

Next we apply $\Sigma_{(\mathrm{S2})}$. Let $\mathcal{E}_2 = \Sigma_{(\mathrm{S2})}(\{E\})$. Since $\mathrm{Ext}(\lambda; F \cup G) = \mathrm{Ext}(\lambda; F) \cup \mathrm{Ext}(\lambda; G)$ for all λ, F and G, we have $\Sigma_{(\mathrm{S2})}(\Sigma_{(\mathrm{S1})}(\mathcal{E})) = \{F \cup G : F \in \mathcal{E}_2, G \in r(E)\Lambda \setminus \Lambda^0\} \cup \{E'\}$; an application of $\Sigma_{(\mathrm{S2})}$ does not have any effect on E' because $r(E')\Lambda \setminus E'\Lambda$ is empty. To calculate $\mathcal{E}_2$ explicitly, we list the elements of $v_{(0,0,0)}\Lambda \setminus E\Lambda$, and the corresponding elements of $\mathcal{E}_2$:

$$\begin{array}{lcl}
v_{(0,0,0)} & \longrightarrow & E \\
\lambda_{(-1,0,0)} & \longrightarrow & \{(\mu\sigma)_{(-1,1,1)}\} \\
(\lambda\mu)_{(-1,1,0)} & \longrightarrow & \{\sigma_{(-1,1,1)}\} \\
(\lambda\sigma)_{(-1,0,1)} & \longrightarrow & \{\mu_{(-1,1,1)}\} \\
\mu_{(0,1,0)} & \longrightarrow & \{\lambda_{(1,1,0)}, (\lambda\sigma)_{(-1,1,1)}\} \\
(\mu\sigma)_{(0,1,1)} & \longrightarrow & \{\lambda_{(1,1,1)}, \lambda_{(-1,1,1)}\} \\
(\mu\sigma)_{(0,1,-1)} & \longrightarrow & \{\lambda_{(1,1,-1)}\} \\
\mu_{(0,-1,0)} & \longrightarrow & \{\lambda_{(1,-1,0)}\} \\
(\mu\sigma)_{(0,-1,1)} & \longrightarrow & \{\lambda_{(1,-1,1)}\} \\
\sigma_{(0,0,1)} & \longrightarrow & \{\lambda_{(1,0,1)}, (\lambda\mu)_{(-1,1,1)}\} \\
\sigma_{(0,0,-1)} & \longrightarrow & \{\lambda_{(1,0,-1)}\}
\end{array}$$

Next we apply the iterates of $(\Sigma'_{(\mathrm{S4})}\Sigma_{(\mathrm{S3})})$. We do this one step at a time: first we apply $\Sigma_{(\mathrm{S3})}$. We have that $\Sigma_{(\mathrm{S3})}$ has no effect on E' because E' contains only elements of degree e_2. Hence, setting $\mathcal{E}_3 := \Sigma_{(\mathrm{S3})}(\mathcal{E}_2)$, we have that $\Sigma_{(\mathrm{S3})} \circ \Sigma_{(\mathrm{S2})} \circ \Sigma_{(\mathrm{S1})}(\mathcal{E}) = \{E \cup F : E \in \mathcal{E}_3, F \subset r(E)\Lambda \setminus \{r(E)\}\} \cup \{E'\}$. Again, to make this explicit, we list the elements of $\mathcal{E}_3$ in the same way as we listed the elements of $\mathcal{E}_2$ above:

$$\begin{array}{lcl}
E & \longrightarrow & \{\lambda_{(1,0,0)}, \lambda_{(-1,0,0)}\}, \{\lambda_{(1,0,0)}, \mu_{(0,1,0)}\}, \\
 & & \{\lambda_{(1,0,0)}, \sigma_{(0,0,1)}\}, \{\lambda_{(1,0,0)}, (\lambda\mu)_{(-1,1,0)}\}, \\
 & & \{\lambda_{(1,0,0)}, (\lambda\sigma)_{(-1,0,1)}\}, \{\lambda_{(1,0,0)}, (\mu\sigma)_{(0,1,1)}\}, E \\
\{(\mu\sigma)_{(-1,1,1)}\} & \longrightarrow & \{\mu_{(-1,1,0)}\}, \{\sigma_{(-1,0,1)}\}, \{(\mu\sigma)_{(-1,1,1)}\} \\
\{\sigma_{(-1,1,1)}\} & \longrightarrow & \{\sigma_{(-1,1,1)}\} \\
\{\mu_{(-1,1,1)}\} & \longrightarrow & \{\mu_{(-1,1,1)}\} \\
\{\lambda_{(1,1,0)}, (\lambda\sigma)_{(-1,1,1)}\} & \longrightarrow & \{\lambda_{(1,1,0)}, \lambda_{(-1,1,0)}\}, \{\lambda_{(1,1,0)}, \sigma_{(0,1,1)}\}, \\
 & & \{\lambda_{(1,1,0)}, (\lambda\sigma)_{(-1,1,1)}\} \\
\{\lambda_{(1,1,1)}, \lambda_{(-1,1,1)}\} & \longrightarrow & \{\lambda_{(1,1,1)}, \lambda_{(-1,1,1)}\} \\
\{\lambda_{(1,1,-1)}\} & \longrightarrow & \{\lambda_{(1,1,-1)}\}
\end{array}$$

$$\begin{array}{lll}
\{\lambda_{(1,-1,0)}\} & \longrightarrow & \{\lambda_{(1,-1,0)}\} \\
\{\lambda_{(1,-1,1)}\} & \longrightarrow & \{\lambda_{(1,-1,1)}\} \\
\{\lambda_{(1,0,1)}, (\lambda\mu)_{(-1,1,1)}\} & \longrightarrow & \{\lambda_{(1,0,1)}, \lambda_{(-1,0,1)}\},\ \{\lambda_{(1,0,1)}, \mu_{(0,1,1)}\}, \\
 & & \{\lambda_{(1,0,1)}, (\lambda\mu)_{(-1,1,1)}\} \\
\{\lambda_{(1,0,-1)}\} & \longrightarrow & \{\lambda_{(1,0,-1)}\}
\end{array}$$

Next we have to apply $\Sigma'_{(\mathrm{S4})}$. It is once again clear that we need only calculate the elements of $\Sigma'_{(\mathrm{S4})}(\mathcal{E}_3 \cup E')$, for then $\Sigma`_{(\mathrm{S4})}(\mathcal{E}) = \{E \cup F : E \in \Sigma`_{(\mathrm{S4})}(\mathcal{E}_3 \cup \{E'\}), F \subset r(E)\Lambda\}$. A quick inspection shows that applying $\Sigma'_{(\mathrm{S4})}$ to sets taken exclusively from $\mathcal{E}_3$ only produces supersets of elements of $\mathcal{E}_3$, so the only additional sets obtained from applying $\Sigma'_{(\mathrm{S4})}$ are those which arise when E' is appended to a path in an element of $\mathcal{E}_3$, or supersets of such sets. The only elements E of $\mathcal{E}_3$ for which we can append E' to a path in E are the three which contain $\lambda_{(1,0,0)}$. We list these additional elements as above:

$$\begin{array}{lll}
\{\lambda_{(1,0,1)}, \lambda_{(-1,0,1)}\} & \longrightarrow & \{(\lambda\mu)_{(1,1,1)}, (\lambda\mu)_{(1,-1,1)}, \lambda_{(-1,0,1)}\} \\
\{\lambda_{(1,0,1)}, \mu_{(0,1,1)}\} & \longrightarrow & \{(\lambda\mu)_{(1,1,1)}, (\lambda\mu)_{(1,-1,1)}, \mu_{(0,1,1)}\} \\
\{\lambda_{(1,0,1)}, (\lambda\mu)_{(-1,1,1)}\} & \longrightarrow & \{(\lambda\mu)_{(1,1,1)}, (\lambda\mu)_{(1,-1,1)}, (\lambda\mu)_{(-1,1,1)}\}
\end{array}$$

Thus, if we write $\mathcal{E}_4$ for the collection of all sets appearing in the right-hand columns of any of the above three tables, together with the set E', we have $(\Sigma'_{(\mathrm{S4})}\Sigma_{(\mathrm{S3})}) \circ \Sigma_{(\mathrm{S2})} \circ \Sigma_{(\mathrm{S1})}(\mathcal{E}) = \{F \cup G : F \in \mathcal{E}_4, G \subset r(F)\Lambda\}$.

Since $\mathcal{E}_3$ satisfied (S1)–(S3), another application of $\Sigma_{(\mathrm{S3})}$ adds only those sets obtained from operating on the three sets added in the last table. That is, we obtain the additional sets

$$\begin{array}{lll}
\{(\lambda\mu)_{(1,1,1)}, (\lambda\mu)_{(1,-1,1)}, \lambda_{(-1,0,1)}\} & \longrightarrow & \{(\lambda\mu)_{(1,1,1)}, \mu_{(0,-1,1)}, \lambda_{(-1,0,1)}\}, \\
 & & \{\mu_{(0,1,1)}, (\lambda\mu)_{(1,-1,1)}, \lambda_{(-1,0,1)}\}, \\
 & & \{\mu_{(0,1,1)}, \mu_{(0,-1,1)}, \lambda_{(-1,0,1)}\} \\
\{(\lambda\mu)_{(1,1,1)}, (\lambda\mu)_{(1,-1,1)}, \mu_{(0,1,1)}\} & \longrightarrow & \{(\lambda\mu)_{(1,1,1)}, \mu_{(0,-1,1)}, \mu_{(0,1,1)}\}, \\
 & & \{\mu_{(0,1,1)}, (\lambda\mu)_{(1,-1,1)}, \mu_{(0,1,1)}\}, \\
 & & \{\mu_{(0,1,1)}, \mu_{(0,-1,1)}, \mu_{(0,1,1)}\} \\
\{(\lambda\mu)_{(1,1,1)}, (\lambda\mu)_{(1,-1,1)}, (\lambda\mu)_{(-1,1,1)}\} & \longrightarrow & \{(\lambda\mu)_{(1,1,1)}, \mu_{(0,-1,1)}, (\lambda\mu)_{(-1,1,1)}\}, \\
 & & \{\mu_{(0,1,1)}, (\lambda\mu)_{(1,-1,1)}, (\lambda\mu)_{(-1,1,1)}\}, \\
 & & \{\mu_{(0,1,1)}, \mu_{(0,-1,1)}, (\lambda\mu)_{(-1,1,1)}\}
\end{array}$$

These are not all of the sets obtained when $\Sigma_{(S3)}$ is applied to $\mathcal{E}_4 \setminus \mathcal{E}_3$; for example, $\{(\lambda\mu)_{(1,1,1)}, (\lambda\mu)_{(1,-1,1)}, \lambda_{(-1,0,1)}\}$ also yields $\{\lambda_{(1,0,1)}, \mu_{(0,-1,1)}, \lambda_{(-1,0,1)}\}$. However all elements of $\Sigma_{(\mathrm{S3})}(\mathcal{E}_4 \setminus \mathcal{E}_3)$ which are not listed above contain a subset belonging to $\mathcal{E}_3$ (in the case of $\{\lambda_{(1,0,1)}, \mu_{(0,-1,1)}, \lambda_{(-1,0,1)}\}$ for example, the subset $\{\lambda_{(1,0,1)}, \lambda_{(-1,0,1)}\}$ belongs to $\mathcal{E}_3$). Hence the additional sets need not be listed separately.

Write $\mathcal{E}_5$ for the set obtained by adding the entries in the right-hand column of the last table to $\mathcal{E}_4$. Since all of the new sets have range $v_{(0,0,1)}$, and since no elements of $\mathcal{E}_5$ contain any paths with source $v_{(0,0,1)}$, another application of $\Sigma'_{(S4)}$

has no further effect, and since Lemma 4.4.3 ensures that $\Sigma_{(\mathrm{S3})}(\mathcal{F})$ satisfies (S3) for all $\mathcal{F} \subset \mathrm{FE}(\Lambda)$, further applications of $\Sigma_{(\mathrm{S3})}$ will also have no effect. It follows that $\overline{\mathcal{E}} = \{E \cup G : E \in \mathcal{E}_5, G \subset r(E)\Lambda\}$; that is, $\overline{\mathcal{E}}$ consists of all sets containing either the set E' or one of the entries in the right-hand column of any one of the tables in this example.

4.5. A generalised Cuntz-Krieger Uniqueness theorem

In this section we develop a Cuntz-Krieger uniqueness theorem for the relative Cuntz-Krieger algebras $C^*(\Lambda; \mathcal{E})$. We show that this theorem contains [**30**, Theorem 4.5] as a special case. To state our Cuntz-Krieger uniqueness theorem, we need a definition.

Definition 4.5.1. Let (Λ, d) be a k-graph, and let $x, y \in \Lambda^*$. We say that $z \in \Lambda^*$ is a *minimal common extension* of x and y if it satisfies

(1) $d(z)_j = \max\{d(x)_j, d(y)_j\}$ for $1 \leq j \leq k$; and
(2) $z|_0^{d(x)} = x$ and $z|_0^{d(y)} = y$.

We write $\mathrm{MCE}(x, y)$ for the collection of all minimal common extensions of x and y in Λ^*.

To obtain a Cuntz-Krieger uniqueness theorem for relative Cuntz-Krieger algebras, the appropriate analogue of an aperiodic path is a path $x \in \Lambda^*$ such that

$$\text{for distinct } \lambda, \mu \in \Lambda r(x), \text{ we have } \mathrm{MCE}(\lambda x, \mu x) = \emptyset. \tag{4.5.1}$$

Theorem 4.5.2 (The Cuntz-Krieger uniqueness theorem). *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Suppose that the pair $(\Lambda, \mathcal{E})$ satisfies*

(C) *For all $v \in \Lambda^0$ there exists $x \in v\partial(\Lambda; \mathcal{E})$ satisfying (4.5.1); and for all $F \in v\,\mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$ there exists $x \in v\partial(\Lambda; \mathcal{E}) \setminus F\partial(\Lambda; \mathcal{E})$ satisfying (4.5.1).*

Let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family such that $t_v \neq 0$ for all $v \in \Lambda^0$, and $\prod_{\lambda \in F}(t_{r(F)} - t_\lambda t_\lambda^*) \neq 0$ for all $F \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$. Then the representation $\pi_t^{\mathcal{E}}$ of $C^*(\Lambda; \mathcal{E})$ such that $\pi_t^{\mathcal{E}}(s_{\mathcal{E}}(\lambda)) = t_\lambda$ for all $\lambda \in \Lambda$ is faithful.

The remainder of the section is devoted to proving Theorem 4.5.2. We first need some technical lemmas.

Lemma 4.5.3. *Let (Λ, d) be a finitely aligned k-graph, and suppose that $x \in \Lambda^*$ satisfies (4.5.1). Suppose that λ and μ are distinct elements of $\Lambda r(x)$. Then there exists $n_{\lambda,\mu}^x \in \mathbb{N}^k$ such that $n_{\lambda,\mu}^x \leq d(x)$ and $\Lambda^{\min}(\lambda x(0, n_{\lambda,\mu}^x), \mu x(0, n_{\lambda,\mu}^x)) = \emptyset$.*

Proof. If $d(x) \in \mathbb{N}^k$ then taking $n^x_{\lambda,\mu} := d(x)$ suffices, so suppose that $d(x)_j = \infty$ for at least one j, and suppose for contradiction that for all $n \in \mathbb{N}^k$ with $n \le d(x)$, we have $\Lambda^{\min}(\lambda x(0,n), \mu x(0,n)) \neq \emptyset$. For each $i \in \mathbb{N}$, define $n(i) \in \mathbb{N}^k$ by $n(i)_j := \min\{d(x)_j, i\}$. Since $d(x)_j = \infty$ for some j, we have $n(i) > n(i-1)$ for all $i \ge 1$.

Claim 1: There exists a strictly increasing sequence $\{i_l : l \in \mathbb{N}\} \subset \mathbb{N}$, and a sequence of paths $\{\sigma_l \in \operatorname{MCE}\big(\lambda x(0, n(i_l), \mu x(0, n(i_l))\big) : l \in \mathbb{N}\}$ such that $\sigma_{l+1}(0, d(\sigma_l)) = \sigma_l$ for all $l \in \mathbb{N}$.

Proof of Claim 1. By assumption, for each $i \in \mathbb{N}$ there is a pair (α_i, β_i) belonging to $\Lambda^{\min}(\lambda x(0, n(i)), \mu x(0, n(i)))$. Let $i_0 = 0$. Since $\Lambda^{\min}(\lambda, \mu)$ is finite, there must exist a pair (η_0, ζ_0) belonging to $\Lambda^{\min}(\lambda, \mu)$ and an infinite subset $I_0 \subset \mathbb{N} \setminus \{i_0\} = \mathbb{N} \setminus \{0\}$ such that for all $i \in I_0$,

$$(\lambda x(0, n(i))\alpha_i)(0, d(\lambda) \vee d(\mu)) = \lambda\eta_0 = \mu\zeta_0 = (\mu x(0, n(i))\beta_i)(0, d(\lambda) \vee d(\mu)).$$

Let $\sigma_0 := \lambda\eta_0$.

Now suppose that $N \in \mathbb{N}$ with $N \ge 0$ and that we have chosen paths $\sigma_0, \dots, \sigma_N$ such that

(1) $\sigma_l \in \operatorname{MCE}\big(\lambda x(0, n(i_l)), \mu x(0, n(i_l))\big)$ for $0 \le l \le N$ where $0 = i_0 < i_1 < \dots < i_N$ belong to $\mathbb{N}$;
(2) $\sigma_{l+1}(0, d(\sigma_l)) = \sigma_l$ for $0 \le l < N$; and
(3) there exists an infinite subset $I_N \subset \{i \in \mathbb{N} : i > i_N\}$ with the property that $\big(\lambda x(0, n(j))\alpha_i\big)\big(0, d(\lambda) \vee d(\mu) + n(i_N)\big) = \sigma_N$ for all $j \in I_N$.

Note that we have already established (1), (2) and (3) for $N = 0$. We will show that there exist i_{N+1}, σ_{N+1}, and I_{N+1} with the same properties.

Let $i_{N+1} := \min I_N$, and let $J := I_N \setminus \{i_{N+1}\}$. For $j \in J$, we have

$$\big(\lambda x(0, n(j))\alpha_j\big)\big(0, d(\lambda) + n(i_{N+1})\big) = \lambda x(0, n(i_{N+1})) \quad \text{and}$$
$$\big(\lambda x(0, n(j))\alpha_j\big)\big(0, d(\mu) + n(i_{N+1})\big) = \mu x(0, n(i_{N+1}))$$

by (1) and (3) because $j \in J \subset I_N$. Equivalently, $\big(\lambda x(0, n(j))\alpha_j\big)\big(0, d(\lambda) \vee d(\mu) + n(i_{N+1})\big) \in \operatorname{MCE}\big(\lambda x(0, n(i_{N+1})), \mu x(0, n(i_{N+1}))\big)$ for all $j \in J$. But Λ is finitely aligned, so $\operatorname{MCE}\big(\lambda x(0, n(i_{N+1})), \mu x(0, n(i_{N+1}))\big)$ is finite. Since J is infinite, it follows that there exists $(\eta_{N+1}, \zeta_{N+1}) \in \Lambda^{\min}\big(\lambda x(0, n(i_{N+1})), \mu x(0, n(i_{N+1}))\big)$ and an infinite subset I_{N+1} of J such that for all $i \in I_{N+1}$,

$$\big(\lambda x(0, n(j))\alpha_j\big)\big(0, d(\lambda) \vee d(\mu) + n(i_{N+1})\big) = \lambda x(0, n(i_{N+1}))\eta_{N+1}.$$

Let $\sigma_{N+1} := \lambda x(0, n(i_{N+1}))\eta_{N+1}$.

Then σ_{N+1} satisfies (1) and I_{N+1} satisfies (3) by definition, so we only need to show that σ_{N+1} satisfies (2). For this, recall that $I_{N+1} \subset J \subset I_N$, so taking j to be any element of I_{N+1}, we can calculate

$$\begin{aligned}
&\sigma_{N+1}(0, d(\sigma_N)) \\
&\qquad = \sigma_{N+1}(0, d(\lambda) \vee d(\mu) + n(i_N)) \\
&\qquad = \big(\lambda x(0, n(i_{N+1})\eta_{N+1})\big)(0, d(\lambda) \vee d(\mu) + n(i_N)) \\
&\qquad = \big(\lambda x(0, n(j))\alpha_j\big)(0, d(\lambda) \vee d(\mu) + n(i_{N+1})\big)(0, d(\lambda) \vee d(\mu) + n(i_N)). \qquad (4.5.2)
\end{aligned}$$

Since $i_{N+1} > i_N$, we have $n(i_{N+1}) > n(i_N)$ and hence (4.5.2) gives

$$\sigma_{N+1}(0, d(\sigma_N)) = \big(\lambda x(0, n(j))\alpha_j\big)\big)(0, d(\lambda) \vee d(\mu) + n(i_N))$$

for any $j \in I_{N+1}$. But if $j \in I_{N+1}$, then $\big(\lambda x(0, n(j))\alpha_j)\big)(0, d(\lambda) \vee d(\mu) + n(i_N))$ is equal to σ_N by (3). Hence σ_{N+1} satisfies (2). $\square$ Claim 1

By Lemma 3.7.2 and Claim 1, there is a unique graph morphism $y \in \Lambda^*$ such that

$$d(y) = \lim_{l \to \infty} (d(\lambda) \vee d(\mu)) + n(i_l) = (d(\lambda) \vee d(\mu)) + d(x) = d(\lambda x) \vee d(\mu x),$$

and such that $y(0, d(\sigma_l)) = \sigma_l$ for all l. We then have

$$\begin{aligned}
y(0, d(\lambda) + n(i_l)) &= \sigma_l(0, d(\lambda) + n(i_l)) \\
&= \big(\lambda x(0, n(i_l))\eta_{l+1}\big)(0, d(\lambda) + n(i_l)) = \lambda x(0, n(i_l)).
\end{aligned}$$

Since $\{n(i_l) : l \in \mathbb{N}\}$ is an increasing sequence with limit $d(x)$, it follows from the uniqueness claim in Lemma 3.7.2 that $y|_0^{d(\lambda)+d(x)} = \lambda x$. Similarly, we have $y|_0^{d(\mu)+d(x)} = \mu x$. It follows that $y \in \mathrm{MCE}(\lambda x, \mu x)$, contradicting (4.5.1). $\square$

LEMMA 4.5.4. *Let (Λ, d) be a finitely aligned k-graph and suppose that $F \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$. Let $x \in r(F)\partial(\Lambda; \mathcal{E}) \setminus F\partial(\Lambda; \mathcal{E})$, and let $n \in \mathbb{N}^k$ with $n \leq d(x)$. Then $\mathrm{Ext}(x(0, n); F) \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$.*

PROOF. Since $x \notin F\partial(\Lambda; \mathcal{E})$, it follows from Lemma 4.2.7 that $\mathrm{Ext}(x(0, n); F)$ belongs to $\mathrm{FE}(\Lambda)$. Suppose for contradiction that $\mathrm{Ext}(x(0, n); F) \in \overline{\mathcal{E}}$. Since $x \in \partial(\Lambda; \mathcal{E})$, there exists $m > n$ such that $m \leq d(x)$ and $x(n, m) \in \mathrm{Ext}(x(0, n); F)$. So there exists $\lambda \in F$ and $\alpha \in s(\lambda)\Lambda$ such that $(\alpha, x(n, m)) \in \Lambda^{\min}(\lambda, x(0, n))$. But then $x(0, m) = x(0, n)x(n, m) = \lambda\alpha$, contradicting $x \notin F\partial(\Lambda; \mathcal{E})$. $\square$

COROLLARY 4.5.5. *Let (Λ, d) be a finitely aligned k-graph and suppose that $F \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$. Let $x \in r(F)\partial(\Lambda; \mathcal{E}) \setminus F\partial(\Lambda; \mathcal{E})$, and let $n \in \mathbb{N}^k$ with $n \leq d(x)$.*

Let $\{t_\lambda : \lambda \in \Lambda\}$ *be a relative Cuntz-Krieger* $(\Lambda;\mathcal{E})$*-family such that* $t_v \neq 0$ *for all* $v \in \Lambda^0$*, and* $\prod_{\lambda\in F}(t_{r(F)} - t_\lambda t_\lambda^*) \neq 0$ *for all* $F \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$*. Then*

$$\prod_{\lambda\in F}(t_{r(F)} - t_\lambda t_\lambda^*)t_{x(0,n)}t_{x(0,n)}^* = t_{x(0,n)}\Big(\prod_{\beta\in\mathrm{Ext}(x(0,n);F)}(t_{x(n)} - t_\beta t_\beta^*)\Big)t_{x(0,n)}^* \neq 0.$$

PROOF. We first calculate:

$$\begin{aligned}\prod_{\lambda\in F}(t_{r(F)} - t_\lambda t_\lambda^*)t_{x(0,n)}t_{x(0,n)}^* &= \prod_{\lambda\in F}\Big(t_{x(0,n)}t_{x(0,n)}^* - \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,x(0,n))} t_{x(0,n)\beta}t_{x(0,n)\beta}^*\Big)\\ &= t_{x(0,n)}\Big(\prod_{\substack{\lambda\in F\\(\alpha,\beta)\in\Lambda^{\min}(\lambda,x(0,n))}}(t_{s(x(0,n))} - t_\beta t_\beta^*)\Big)t_{x(0,n)}^*\\ &= t_{x(0,n)}\Big(\prod_{\beta\in\mathrm{Ext}(x(0,n);F)}(t_{x(n)} - t_\beta t_\beta^*)\Big)t_{x(0,n)}^*.\end{aligned}$$

Lemma 4.5.4 ensures that $\mathrm{Ext}(x(0,n);F) \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$, and it then follows by hypothesis that $\prod_{\beta\in\mathrm{Ext}(x(0,n);F)}(t_{x(n)} - t_\beta t_\beta^*) \neq 0$ completing the proof. □

LEMMA 4.5.6. *Let* (Λ, d) *be a finitely aligned* k*-graph, let* $\mathcal{E} \subset \mathrm{FE}(\Lambda)$*, and suppose that* $(\Lambda,\mathcal{E})$ *satisfies Condition* (C)*. Let* $\{t_\lambda : \lambda \in \Lambda\}$ *be a relative Cuntz-Krieger* $(\Lambda;\mathcal{E})$*-family such that* $t_v \neq 0$ *for all* $v \in \Lambda^0$*, and* $\prod_{\lambda\in F}(t_{r(F)} - t_\lambda t_\lambda^*) \neq 0$ *for all* $F \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$*. Let* $\pi_t^{\mathcal{E}}$ *be the representation of* $C^*(\Lambda;\mathcal{E})$ *such that* $\pi_t^{\mathcal{E}}(s_{\mathcal{E}}(\lambda)) = t_\lambda$ *for all* $\lambda \in \Lambda$*. Let* $a \in \mathrm{span}\{s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\mu)^* : \lambda,\mu \in \Lambda\} \subset C^*(\Lambda;\mathcal{E})$*. Then* $\|\pi_t^{\mathcal{E}}(\Phi^\gamma(a))\| \leq \|\pi_t^{\mathcal{E}}(a)\|$*.*

PROOF. Express $a = \sum_{\lambda,\mu\in\Pi E} a_{\lambda,\mu}s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\mu)^*$ for some finite $E \subset \Lambda$, and express $\Phi^\gamma(a) = \sum_{(\lambda,\mu)\in\Pi E\times_{d,s}\Pi E} b_{\lambda,\mu}\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu}$. So we have

$$\pi_t^{\mathcal{E}}(a) = \sum_{\lambda,\mu\in\Pi E} a_{\lambda,\mu}t_\lambda t_\mu^* \quad\text{and}\quad \pi_t^{\mathcal{E}}(\Phi^\gamma(a)) = \sum_{(\lambda,\mu)\in\Pi E\times_{d,s}\Pi E} b_{\lambda,\mu}\Theta(t)^{\Pi E}_{\lambda,\mu}.$$

By Corollary 4.6.4, there exists $(n,v) \in (d\times s)^+_{\mathcal{E}}(\Pi E)$ such that

$$\|\pi_t^{\mathcal{E}}(\Phi^\gamma(a))\| = \Big\|\sum_{\lambda,\mu\in(\Pi E)v\cap\Lambda^n} b_{\lambda,\mu}\Theta(t)^{\Pi E}_{\lambda,\mu}\Big\|.$$

By definition of $(d\times s)^+_{\mathcal{E}}(\Pi E)$, we have that $T^{\Pi E}(n,v)$ does not belong to $\overline{\mathcal{E}}$. We claim that there exists $x \in v\partial(\Lambda;\mathcal{E}) \setminus T^{\Pi E}(n,v)\partial(\Lambda;\mathcal{E})$ satisfying (4.5.1). To see this, note that if $T^{\Pi E}(n,v) \in \mathrm{FE}(\Lambda)$, then such an x exists because $(\Lambda,\mathcal{E})$ satisfies Condition (C). On the other hand, if $T^{\Pi E}(n,v) \notin \mathrm{FE}(\Lambda)$, then there exists $\sigma \in v\Lambda$ with $\mathrm{Ext}(\sigma;T^{\Pi E}(n,v)) = \emptyset$. Condition (C) implies that there exists $x' \in s(\sigma)\Lambda$ satisfying (4.5.1). It is then easy to check that $x := \sigma x'$ also satisfies (4.5.1) and does not have an initial segment in $T^{\Pi E}(n,v)$. This establishes the claim.

For all $\lambda \in \Pi E$ with $d(\lambda) \le n$, $\mu \in (\Pi E)s(\lambda)$ with $d(\mu) \ne d(\lambda)$, and $\nu \in s(\lambda)\Lambda$ such that $\lambda\nu \in (\Pi E)v \cap \Lambda^n$, the factorisation property ensures that $\lambda\nu \ne \mu\nu$. Hence Lemma 4.5.3 shows that there exists $n^x_{\lambda\nu,\mu\nu} \in \mathbb{N}^k$ with $n^x_{\lambda\nu,\mu\nu} \le d(x)$ such that $\Lambda^{\min}\big(\lambda\nu x(0, n^x_{\lambda\nu,\mu\nu}), \mu\nu x(n^x_{\lambda\nu,\mu\nu})\big) = \emptyset$. Define

$$N := \bigvee \{n^x_{\lambda\nu,\mu\nu} : \lambda, \mu \in \Pi E, d(\lambda) \ne d(\mu), s(\lambda) = s(\mu), \lambda\nu \in (\Pi E)v \cap \Lambda^n\}.$$

Since each $n^x_{\lambda\nu,\mu\nu} \le d(x)$, we have $N \le d(x)$, and for each λ, μ, ν as above, we have that $\Lambda^{\min}(\lambda\nu x(0, N), \mu\nu x(0, N)) = \emptyset$.

Define a projection $P_1 \in M^t_{\Pi E}(n, v)$ by

$$P_1 := \textstyle\sum_{\lambda\in(\Pi E)v\cap\Lambda^n} \Theta(t)^{\Pi E}_{\lambda,\lambda},$$

and define a projection $P_2 \in C^*(\{t_\sigma t^*_\tau : d(\sigma) = d(\tau) = n + N\})$ by

$$P_2 := \textstyle\sum_{\lambda\in(\Pi E)v\cap\Lambda^n} t_{\lambda x(0,N)} t^*_{\lambda x(0,N)}.$$

Since the $\Theta(t)^{\Pi E}_{\lambda,\mu}$ are matrix units by Lemma 3.5.6(2), we have that $P_1\pi^{\mathcal{E}}_t(\Phi^\gamma(a)) = \sum_{\lambda,\mu\in(\Pi E)v\cap\Lambda^n} b_{\lambda,\mu}\Theta(t)^{\Pi E}_{\lambda,\mu}$, and hence $\|P_1\pi^{\mathcal{E}}_t(\Phi^\gamma(a))\| = \|\pi^{\mathcal{E}}_t(\Phi^\gamma(a))\|$. Furthermore,

$$\begin{aligned} P_2(P_1\pi^{\mathcal{E}}_t(\Phi^\gamma(a)))P_2 &= P_2\Big(\sum_{\lambda,\mu\in(\Pi E)v\cap\Lambda^n} b_{\lambda,\mu}\Theta(t)^{\Pi E}_{\lambda,\mu}\Big)P_2 \\ &= \sum_{\lambda,\mu\in(\Pi E)v\cap\Lambda^n} b_{\lambda,\mu}P_2 t_\lambda\Big(\prod_{\nu\in T^{\Pi E}(n,v)}(t_v - t_\nu t^*_\nu)\Big)t^*_\mu P_2 \end{aligned} \tag{4.5.3}$$

by Lemma 3.5.4. For $\sigma \in \Lambda^n$, we have $P_2 t_\sigma = \sum_{\tau\in(\Pi E)v\cap\Lambda^n}(t_{\tau x(0,N)} t_{x(0,N)} t^*_\tau t_\sigma) = t_{\sigma x(0,N)} t^*_{x(0,N)}$ by Lemma 3.1.2(3). Taking adjoints gives $t^*_\sigma P_2 = t_{x(0,N)} t^*_{\sigma x(0,N)}$ as well. Applying these equalities on either side of (4.5.3), we obtain

$$\begin{aligned} &P_2(P_1\pi^{\mathcal{E}}_t(\Phi^\gamma(a)))P_2 \\ &\quad= \sum_{\lambda,\mu\in(\Pi E)v\cap\Lambda^n} \Big(b_{\lambda,\mu} t_{\lambda x(0,N)} t^*_{x(0,n)}\Big(\prod_{\nu\in T^{\Pi E}(n,v)}(t_v - t_\nu t^*_\nu)\Big)t_{x(0,N)} t^*_{\mu x(0,N)}\Big). \end{aligned}$$

Corollary 4.5.5 then shows that

$$\begin{aligned} &P_2(P_1\pi^{\mathcal{E}}_t(\Phi^\gamma(a)))P_2 \\ &\quad= \sum_{\lambda,\mu\in(\Pi E)v\cap\Lambda^n} \Big(b_{\lambda,\mu} t_{\lambda x(0,N)}\Big(\prod_{\beta\in\mathrm{Ext}(x(0,N);T^{\Pi E}(n,v))} t_{x(n)} - t_\beta t^*_\beta\Big)t^*_{\mu x(0,N)}\Big). \end{aligned}$$

For $\lambda \in (\Pi E)v \cap \Lambda^n$, we have $t_{\lambda x(0,N)} \in \Lambda^{n+N}$, and it follows from Lemma 3.1.2(3) that for $\lambda, \mu \in (\Pi E)v \cap \Lambda^n$, we have $t^*_{\lambda x(0,N)} t_{\mu x(0,N)} = \delta_{\lambda,\mu} t_{x(N)}$. Hence

$$\Big\{t_{\lambda x(0,N)}\Big(\prod_{\beta\in\mathrm{Ext}(x(0,N);T^{\Pi E}(n,v))} t_{x(n)} - t_\beta t^*_\beta\Big)t^*_{\mu x(0,N)} : \lambda, \mu \in (\Pi E)v \cap \Lambda^n\Big\}$$

is a collection of matrix units which are all nonzero by Corollary 4.5.5. Compression by P_2 is therefore an isomorphism of $M^t_{\Pi E}(n,v)$. Hence $\|P_2(P_1\pi^{\mathcal{E}}_t(\Phi^\gamma(a)))P_2\| = \|P_1\pi^{\mathcal{E}}_t(\Phi^\gamma(a))\| = \|\pi^{\mathcal{E}}_t(\Phi^\gamma(a))\|$.

On the other hand, we have $\|P_2(P_1\pi^{\mathcal{E}}_t(a))P_2)\| \le \|\pi^{\mathcal{E}}_t(a)\|$ because P_1 and P_2 are projections. Thus, the proof of Lemma 4.5.6 will be complete if we can establish that $P_2(P_1\pi^{\mathcal{E}}_t(a))P_2 = P_2(P_1\pi^{\mathcal{E}}_t(\Phi^\gamma(a)))P_2$. To do this, it suffices to show that if $\lambda,\mu \in \Pi E$ with $d(\lambda) \neq d(\mu)$ and $s(\lambda) = s(\mu)$, we have $P_2(P_1t_\lambda t^*_\mu)P_2 = 0$. So fix $\lambda,\mu \in \Pi E$ with $d(\lambda) \neq d(\mu)$ and $s(\lambda) = s(\mu)$. We consider two cases.

Case 1. Suppose that $d(\lambda) \not\le n$. Then for $\sigma \in (\Pi E)v \cap \Lambda^n$, we have that $(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)$ implies $d(\beta) > 0$ and $\sigma\beta \in \Pi E$ by Remark 3.4.3(1). Hence $\Theta(t)^{\Pi E}_{\sigma,\sigma}t_\lambda = 0$ for all $\sigma \in (\Pi E)v\cap\Lambda^n$, giving $P_1t_\lambda = 0$, and hence $P_2(P_1t_\lambda t^*_\mu)P_2 = 0$ as required. □ Case 1

Case 2. Suppose that $d(\lambda) \le n$. Then

$$\begin{aligned}
P_2P_1t_\lambda t^*_\mu P_2 &= P_2\Big(\sum_{\sigma\in(\Pi E)v\cap\Lambda^n} t_\sigma\Big(\prod_{\substack{\sigma\sigma'\in\Pi E\\ d(\sigma')>0}}(t_{s(\sigma)} - t_{\sigma'}t^*_{\sigma'})\Big)t^*_\sigma t_\lambda t^*_\mu\Big)P_2\\
&= P_2\Big(\sum_{\lambda\nu\in(\Pi E)v\cap\Lambda^n} t_{\lambda\nu}\Big(\prod_{\substack{\lambda\nu\sigma'\in\Pi E\\ d(\sigma')>0}}(t_{s(\nu)} - t_{\sigma'}t^*_{\sigma'})\Big)t^*_{\mu\nu}\Big)P_2\\
&= P_2\Big(\sum_{\lambda\nu\in(\Pi E)v\cap\Lambda^n} \Big(\prod_{\substack{\lambda\nu\sigma'\in\Pi E\\ d(\sigma')>0}}(t_{\lambda\nu}t^*_{\lambda\nu} - t_{\lambda\nu\sigma'}t^*_{\lambda\nu\sigma'})\Big)t_{\lambda\nu}t^*_{\mu\nu}\Big)P_2 \qquad (4.5.4)
\end{aligned}$$

by Lemma 3.5.4. Lemma 3.1.2(2) shows that P_2 commutes with each factor $t_{\lambda\nu}t^*_{\lambda\nu} - t_{\lambda\nu\sigma'}t^*_{\lambda\nu\sigma'}$ in (4.5.4), so we can rewrite (4.5.4) as

$$P_2P_1t_\lambda t^*_\mu P_2 = \sum_{\lambda\nu\in(\Pi E)v\cap\Lambda^n}\Big(\Big(\prod_{\substack{\lambda\nu\sigma'\in\Pi E\\ d(\sigma')>0}}(t_{\lambda\nu}t^*_{\lambda\nu} - t_{\lambda\nu\sigma'}t^*_{\lambda\nu\sigma'})\Big)P_2t_{\lambda\nu}t^*_{\mu\nu}P_2\Big). \qquad (4.5.5)$$

It therefore suffices to establish that for all ν such that $\lambda\nu \in (\Pi E)v\cap\Lambda^n$, we have $P_2t_{\lambda\nu}t^*_{\mu\nu}P_2 = 0$. So fix ν such that $\lambda\nu \in (\Pi E)v\cap\Lambda^n$. Lemma 3.1.2(3) to shows that for $\tau \in (\Pi E)v\cap\Lambda^n$, we have $t_{\tau x(0,N)}t^*_{\tau x(0,N)}t_{\lambda\nu} = \delta_{\tau,\lambda\nu}t_{\lambda\nu x(0,N)}t^*_{x(0,N)}$. Hence $P_2t_{\lambda\nu} = t_{\lambda\nu x(0,N)}t^*_{x(0,N)}$. But now

$$\begin{aligned}
P_2t_{\lambda\nu}t^*_{\mu\nu}P_2 &= t_{\lambda\nu x(0,N)}t^*_{\mu\nu x(0,N)}\sum_{\tau\in(\Pi E)v\cap\Lambda^n}(t_{\tau x(0,N)}t^*_{\tau x(0,N)})\\
&= t_{\lambda\nu x(0,N)}\sum_{\tau\in(\Pi E)v\cap\Lambda^n}(t^*_{\mu\nu x(0,N)}t_{\tau x(0,N)}t^*_{\tau x(0,N)})\\
&= t_{\lambda\nu x(0,N)}\sum_{\substack{\tau\in(\Pi E)v\cap\Lambda^n\\ (\alpha,\beta)\in\Lambda^{\min}(\mu\nu x(0,N),\tau x(0,N))}}(t_\alpha t^*_\beta t^*_{\tau x(0,N)}). \qquad (4.5.6)
\end{aligned}$$

But $d(\mu) \neq d(\lambda)$, so $d(\mu\nu) \neq d(\lambda\nu) = n$, and hence $\mu\nu \neq \tau$ for each $\tau \in (\Pi E)v \cap \Lambda^n$. It follows that $\Lambda^{\min}(\mu\nu x(0,N), \tau x(0,N)) = \emptyset$ for all $\tau \in (\Pi E)v \cap \Lambda^n$ by definition of x and N. Hence (4.5.6) is equal to zero. □ Case 2

Since $P_2(P_1 t_\lambda t_\mu^*)P_2 = 0$ in both cases, the proof of the lemma is complete. □

Proof of Theorem 4.5.2. Lemma 4.5.6 shows that the formula

$$t_\lambda t_\mu^* \mapsto \delta_{d(\lambda),d(\mu)} t_\lambda t_\mu^*$$

extends to a well-defined norm-decreasing linear map Φ^t on $\pi_t^{\mathcal{E}}(C^*(\Lambda;\mathcal{E}))$. Precisely as in the proof of Corollary 4.1.8, we now suppose that $\pi_t^{\mathcal{E}}(a) = 0$. Then $\pi_t^{\mathcal{E}}(a^*a) = 0$ and hence $\pi_t^{\mathcal{E}}(\Phi^\gamma(a^*a)) = \Phi^t(\pi_t^{\mathcal{E}}(a^*a)) = 0$. Since $\pi_t^{\mathcal{E}}$ is faithful on $C^*(\Lambda;\mathcal{E})^\gamma$ by Theorem 3.5.8 and Lemma 4.2.2, it follows that $\Phi^\gamma(a^*a) = 0$. But Φ^γ is faithful on positive elements by Proposition 4.1.7 so we can deduce that $a^*a = 0$ and hence that $a = 0$ by the C^*-identity. □

4.6. The Cuntz-Krieger algebra

As mentioned in Remark 4.1.2, the special case $\mathcal{E} = \mathrm{FE}(\Lambda)$ allows us to regard the Cuntz-Krieger algebra $C^*(\Lambda)$ associated to a finitely aligned k-graph Λ as in [**30**] as a relative Cuntz-Krieger algebra. In this section, we use the results proved so far in this chapter to recover the gauge-invariant uniqueness theorem and prove a new version of the Cuntz-Krieger uniqueness theorem for $C^*(\Lambda)$.

Definition 4.6.1. Let (Λ, d) be a finitely aligned k-graph. By a *Cuntz-Krieger Λ-family*, we mean a relative Cuntz-Krieger $(\Lambda;\mathrm{FE}(\Lambda))$-family. Define $C^*(\Lambda) := C^*(\Lambda;\mathrm{FE}(\Lambda))$, and $s_\lambda := s_{\mathrm{FE}(\Lambda)}(\lambda)$ for all $\lambda \in \Lambda$. Define $\partial\Lambda := \partial(\Lambda;\mathrm{FE}(\Lambda))$, and for each $\lambda \in \Lambda$, define $S_\lambda := S_{\mathrm{FE}(\Lambda)}(\lambda) \in \mathcal{B}(\ell^2(\partial\Lambda))$; we call $\partial\Lambda$ the *boundary-path space* and we call $\{S_\lambda : \lambda \in \Lambda\}$ the *boundary-path representation*.

Theorem 4.6.2 ([**30**, Corollary 4.3]). *Let (Λ, d) be a finitely aligned k-graph. And let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz-Krieger Λ-family. The homomorphism π_t of $C^*(\Lambda)$ such that $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$ is injective on $C^*(\Lambda)^\gamma$ if and only if*

(1) *t_v is nonzero for all $v \in \Lambda^0$.*

Moreover, π_t is injective on all of $C^*(\Lambda)$ if and only if (1) holds and in addition

(2) *there is a strongly continuous action $\theta : \mathbb{T}^k \to \mathrm{Aut}(C^*(\{t_\lambda : \lambda \in \Lambda\}))$ satisfying $\theta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for all $\lambda \in \Lambda$.*

Proof. We have that $\mathrm{FE}(\Lambda) \setminus \overline{\mathrm{FE}(\Lambda)} = \emptyset$ by definition, so condition (2) of Theorem 4.3.12 is vacuous, and the result is a special case of Theorem 4.3.12. □

NOTATION 4.6.3. Let (Λ, d) be a finitely aligned k-graph, let $E \subset \Lambda$ be finite. We define $(d \times s)^+(\Pi E) := (d \times s)^+_{\mathrm{FE}(\Lambda)}(\Pi E)$ where $(d \times s)^+_{\mathrm{FE}(\Lambda)}(\Pi E)$ is as defined in Notation 4.3.14.

COROLLARY 4.6.4. *Let (Λ, d) be a finitely aligned k-graph, and let $E \subset \Lambda$ be finite. Then*

$$M^s_{\Pi E} \cong \bigoplus_{(n,v) \in (d \times s)^+(\Pi E)} M_{(\Pi E)v \cap \Lambda^n}(\mathbb{C}).$$

PROOF. This is a special case of Corollary 4.3.15. □

THEOREM 4.6.5. *Let (Λ, d) be a finitely aligned k-graph which satisfies*

$$\text{for all } v \in \Lambda^0 \text{ there exists } x \in v\partial\Lambda \text{ satisfying (4.5.1).} \tag{4.6.1}$$

There exists a Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ such that $t_v \neq 0$ for all $v \in \Lambda^0$. Furthermore, any two such Cuntz-Krieger Λ-families generate canonically isomorphic C^*-algebras.

PROOF. We have that $\mathrm{FE}(\Lambda) \setminus \overline{\mathrm{FE}(\Lambda)} = \emptyset$ by definition. Condition (C) therefore reduces to Condition (4.6.1), so the result is a special case of Theorem 4.5.2. □

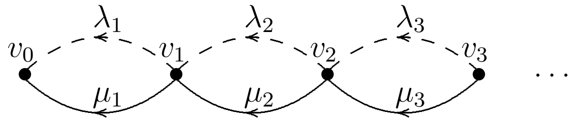
COROLLARY 4.6.6. *Let (Λ, d) be a finitely aligned k-graph. The boundary-path representation π_S is faithful on $C^*(\Lambda)^\gamma$. If Λ satisfies Condition (4.6.1) then π_S is faithful.*

REMARK 4.6.7. Let $x \in \Lambda^*$ belong to the collection of boundary paths $\Lambda^{\leq\infty}$ of [**30**, Definition 2.8]; that is, there exists $n_x \in \mathbb{N}^k$ with $n_x \leq d(x)$ such that if $m \in \mathbb{N}^k$ with $n_x \leq m \leq d(x)$, and if $m_i = d(x)_i$ is finite, then $x(m)\Lambda^{e_i}$ is empty. It is easy to check from this that if $\lambda, \mu \in \Lambda r(x)$, then $\mathrm{MCE}(\lambda x, \mu x)$ is nonempty if and only if $\lambda x = \mu x$. It follows that Theorem 4.6.5 is formally stronger than [**30**, Theorem 4.5]. However, we do not have any examples in which Theorem 4.6.5 applies but [**30**, Theorem 4.5] does not, and it seems likely that the two theorems are equivalent.

4.7. A faithful representation of the relative Cuntz-Krieger algebra

We have shown that the $\mathcal{E}$-relative boundary-path representation is faithful on the core of $C^*(\Lambda; \mathcal{E})$. It would be nice to know that the $\mathcal{E}$-relative boundary-path representation is faithful on all of $C^*(\Lambda; \mathcal{E})$, but the following example shows that this is not necessarily the case.

EXAMPLE 4.7.1. Consider the unique 2-graph Ξ_2 with 1-skeleton



The set $v_i\partial\Xi_2$ is a singleton for every vertex v_i. Namely, $v_i\partial\Xi_2 = \{x_i\}$ where $x_i : \Omega_{2,(\infty,\infty)} \to \Xi_2$ is the unique graph morphism satisfying

$$x_i(0, n) = \lambda_{i+1}\lambda_{i+2} \dots \lambda_{i+n_1}\mu_{i+n_1+1}\mu_{i+n_1+2} \dots \mu_{i+n_1+n_2} \quad \text{for all } n \in \mathbb{N}^2$$

provided by Lemma 3.7.2.

The factorisation property ensures that $\lambda_1 x_1 = x_0 = \mu_1 x_1$. Since $s(\lambda_1)\partial\Xi_2 = \{x_1\} = s(\mu_1)\partial\Xi_2$, it follows that the partial isometries S_{λ_1} and S_{μ_1} in the boundary path representation are both equal to the rank-one operator $e_{x_1} \otimes \overline{e}_{x_0}$.

To see why this implies that the boundary-path representation is not faithful, set $z = (-1, 1) \in \mathbb{T}^2$. Since $\gamma_z \in \operatorname{Aut}(C^*(\Xi_2))$ is isometric, we have

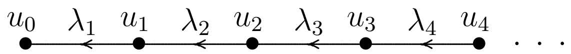
$$\|s_{\lambda_1} - s_{\mu_1}\| = \|\gamma_z(s_{\lambda_1} + s_{\mu_1})\| = \|s_{\lambda_1} + s_{\mu_1}\| \geq \|S_{\lambda_1} + S_{\mu_1}\| = \|2e_{x_1} \otimes \overline{e}_{x_0}\| = 2.$$

However,

$$\pi_S(s_{\lambda_1} - s_{\mu_1}) = S_{\lambda_1} - S_{\mu_1} = e_{x_1} \otimes \overline{e}_{x_0} - e_{x_1} \otimes \overline{e}_{x_0} = 0.$$

It follows that π_S is not faithful.

We can describe both the universal Cuntz-Krieger algebra of Ξ_2 and the C^*-algebra generated by the boundary-path representation for Ξ_2 as follows. Let $\Xi_1 \cong \Omega_{1,\infty}$ be the 1-graph which consists of a one-way infinite path:



Let $f : \mathbb{N}^2 \to \mathbb{N}$ be given by $f(i, j) := i + j$. It is straightforward to check that Ξ_2 is isomorphic to the 2-graph $f^*(\Xi_1)$ obtained from [**18**, Definition 1.9]. Since f is surjective, it follows from [**18**, Corollary 3.5(iii)] that $C^*(\Xi_2) \cong C^*(\Xi_1) \otimes C(\mathbb{T}^{2-1})$; since it is well-known that $C^*(\Xi_1)$ is isomorphic to $\mathcal{K}$, it follows that $C^*(\Xi_2) \cong \mathcal{K} \otimes C(\mathbb{T})$.

By contrast, since for each $i \in \mathbb{N}$ we have that x_i is the unique boundary path with range v_i in Ξ_2, it follows that for $\tau \in \Xi_2$, the partial isometry S_τ is precisely

$e_{x_i} \otimes \overline{e}_{x_j}$ where $v_i = r(\tau)$ and $v_j = s(\tau)$. That is, $C^*(\{S_\lambda : \lambda \in \Xi_2\})$ is generated by $\{e_{x_i} \otimes \overline{e}_{x_j} : i, j \in \mathbb{N}\}$, and hence $C^*(\{S_\lambda : \lambda \in \Xi_2\}) \cong \mathcal{K}$.

The problem in Example 4.7.1 is that the formula $\theta_z(S_\sigma) = z^{d(\sigma)} S_\sigma$ is not well-defined in $C^*(\{S_\sigma : \sigma \in \Lambda\})$; for $z = (-1, 1)$, it simultaneously specifies

$$\theta_z(e_{x_1} \otimes \overline{e}_{x_0}) = \theta_z(S_{\lambda_1}) = (-1)^0 \cdot (1)^1 S_{\lambda_1} = S_{\lambda_1} = e_{x_1} \otimes \overline{e}_{x_0} \qquad \text{and}$$
$$\theta_z(e_{x_1} \otimes \overline{e}_{x_0}) = \theta_z(S_{\mu_1}) = (-1)^1 \cdot (1)^0 S_{\mu_1} = -S_{\mu_1} = -e_{x_1} \otimes \overline{e}_{x_0}.$$

Thus there is no action θ of $\mathbb{T}^k$ on $C^*(\{S_\lambda : \lambda \in \Xi_2\})$ such that π_S is equivariant in θ and γ. Theorem 4.6.2 therefore implies that π_S is not faithful. In order to avoid this problem we need to augment the boundary path representation in such a way that partial isometries associated to paths of different degrees are always distinct.

Remark 4.7.2. The 2-graph Ξ_2 of Example 4.7.1 is the prototypical 2-graph which fails Condition (4.6.1): x_1 is the unique element of $v_1 \partial\Lambda$ and yet we have $x_0 \in \operatorname{MCE}(\lambda_1 x_1, \mu_1 x_1)$. Thus this example sheds some light on the significance of the aperiodicity condition.

Definition 4.7.3. Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \operatorname{FE}(\Lambda)$. For $n \in \mathbb{N}^k$, let $L_n \in \mathcal{B}(\ell^2(\mathbb{Z}^k))$ be the shift by n. So $L_n e_m = e_{m+n}$ for all $m \in \mathbb{Z}^k$. For $\lambda \in \Lambda$, define $(S_{\mathcal{E}} \otimes L)_\lambda := S_{\mathcal{E}}(\lambda) \otimes L_{d(\lambda)} \in \mathcal{B}(\ell^2(\partial(\Lambda; \mathcal{E}))) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^k))$. That is, $(S_{\mathcal{E}} \otimes L)_\lambda(e_x \otimes e_m) = \delta_{s(\lambda), r(x)} e_{\lambda x} \otimes e_{m+d(\lambda)}$.

Lemma 4.7.4. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \operatorname{FE}(\Lambda)$. The collection $\{(S_{\mathcal{E}} \otimes L)_\lambda : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family which we call the* augmented *$\mathcal{E}$-relative boundary path representation.*

Proof. For (TCK1), notice that for $v \in \Lambda^0$, we have $(S_{\mathcal{E}} \otimes L)_v = S_{\mathcal{E}}(v) \otimes \operatorname{id}_{\ell^2(\mathbb{Z}^k)}$, and since the $S_{\mathcal{E}}(v)$ are mutually orthogonal projections, it follows that the $(S_{\mathcal{E}} \otimes L)_v$ are also mutually orthogonal projections.

For (TCK2), we calculate

$$\begin{aligned}(S_{\mathcal{E}} \otimes L)_\lambda (S_{\mathcal{E}} \otimes L)_\mu &= S_{\mathcal{E}}(\lambda) S_{\mathcal{E}}(\mu) \otimes L_{d(\lambda)} L_{d(\mu)} \\ &= \delta_{s(\lambda), r(\mu)} S_{\mathcal{E}}(\lambda\mu) \otimes L_{d(\lambda)+d(\mu)} \\ &= \delta_{s(\lambda), r(\mu)} (S_{\mathcal{E}} \otimes L)_{\lambda\mu}.\end{aligned}$$

Now we must check (TCK3). For this, fix $\lambda, \mu \in \Lambda$. Then

$$\begin{aligned}(S_{\mathcal{E}} \otimes L)^*_\lambda (S_{\mathcal{E}} \otimes L)_\mu &= S_{\mathcal{E}}(\lambda)^* S_{\mathcal{E}}(\mu) \otimes L^*_{d(\lambda)} L_{d(\mu)} \\ &= \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} S_{\mathcal{E}}(\alpha) S_{\mathcal{E}}(\beta)^* \otimes L_{d(\mu)-d(\lambda)}. \qquad (4.7.1)\end{aligned}$$

But for $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$, we have $d(\lambda\alpha) = d(\mu\beta)$ and hence $d(\mu) - d(\lambda) = d(\alpha) - d(\beta)$. Substituting this in (4.7.1), we obtain

$$\begin{aligned}(S_{\mathcal{E}} \otimes L)^*_\lambda (S_{\mathcal{E}} \otimes L)_\mu &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} (S_{\mathcal{E}}(\alpha) S_{\mathcal{E}}(\beta)^* \otimes L_{d(\alpha)-d(\beta)}) \\ &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} (S_{\mathcal{E}}(\alpha) S_{\mathcal{E}}(\beta)^* \otimes L_{d(\alpha)} L^*_{d(\beta)}) \\ &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} (S_{\mathcal{E}} \otimes L)_\alpha (S_{\mathcal{E}} \otimes L)^*_\beta,\end{aligned}$$

verifying (TCK3).

Finally, to check (CK), let $E \in v\mathcal{E}$. Since $L_n L_n^* = \mathrm{id}_{\ell^2(\mathbb{Z}^k)}$, and since tensor products are bilinear,

$$\begin{aligned}&\prod_{\lambda\in E} \big((S_{\mathcal{E}} \otimes L)_v - (S_{\mathcal{E}} \otimes L)_\lambda (S_{\mathcal{E}} \otimes L)^*_\lambda\big) \\ &\qquad = \prod_{\lambda\in E} \big(S_{\mathcal{E}}(v) - S_{\mathcal{E}}(\lambda) S_{\mathcal{E}}(\lambda)^*\big) \otimes \mathrm{id}_{\ell^2(\mathbb{Z}^k)} = 0 \otimes \mathrm{id}_{\ell^2(\mathbb{Z}^k)}\end{aligned}$$

because $\{S_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. □

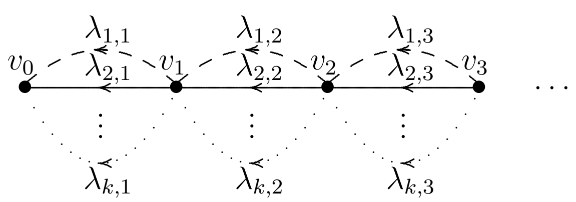
Example 4.7.5 (Example 4.7.1 continued). Recall the 2-graph Ξ_2 of Example 4.7.1, and recall that $C^*(\Xi_2) \cong \mathcal{K} \otimes C(\mathbb{T})$, whereas $C^*(\{S_\lambda : \lambda \in \Xi_2\}) \cong \mathcal{K}$.

Suppose $r(\lambda) = r(\mu) = v_i$ and $s(\lambda) = s(\mu)$. Then $S_\lambda = S_\mu = e_{x_i} \otimes \overline{e}_{x_{i+|d(\lambda)|}}$, and hence $S_\lambda S_\mu^* = e_{x_i} \otimes \overline{e}_{x_i} = S_{v_i}$. On the other hand $L_{d(\lambda)} L^*_{d(\mu)} = L_{d(\lambda)-d(\mu)}$. Now $|d(\lambda)| = |d(\mu)|$; that is, $d(\lambda)_1 + d(\lambda)_2 = d(\mu)_1 + d(\mu)_2$, and hence $(d(\lambda) - d(\mu))_1 = -(d(\lambda) - d(\mu))_2$. It follows that $L_{d(\lambda)} L^*_{d(\mu))} = L_{n,-n}$ where $n = d(\lambda)_1 - d(\mu)_1 \in \mathbb{Z}$.

By varying our choices of λ and μ in the preceding paragraph, we have that $S_{v_i} \otimes L_{n,-n}$ belongs to $C^*(\{(S \otimes L)_\lambda : \lambda \in \Xi_2\})$ for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$. Consequently, for all $\sigma \in \Xi_2$ and all $n \in \mathbb{Z}$, we have that $S_\sigma \otimes L_{d(\sigma)+(n,-n)} = (S \otimes L)_\sigma (S_{s(\sigma)} \otimes L_{(n,-n)})$ also belongs to $C^*(\{(S \otimes L)_\lambda : \lambda \in \Xi_2\})$.

Identifying $C^*(\{L_{(n,-n)} : n \in \mathbb{Z}\})$ with $C^*(\mathbb{Z})$, we have that $C^*(\{(S \otimes L)_\lambda : \lambda \in \Xi_2\})$ generates a copy of $C^*(\{S_\lambda : \lambda \in \Xi_2\}) \otimes C^*(\mathbb{Z}) \cong \mathcal{K} \otimes C(\mathbb{T}) \cong C^*(\Xi_2)$.

Indeed, let Ξ_k be the unique k-graph with 1-skeleton



where, in this 1-skeleton, the dotted edges have degree e_k. So Ξ_k is the k-dimensional analogue of Ξ_2. Algebraically, Ξ is described as follows:

$$(\Xi_k)^0 = \{v_i : i \in \mathbb{N}\}, \quad (\Xi_k)^{e_i} = \{\lambda_{i,j} : j \in \mathbb{N}\},$$
$$r(\lambda_{i,j}) = v_j \text{ and } s(\lambda_{i,j}) = v_{j+1} \quad \text{for all } i, j,$$
$$\lambda_{i,j}\lambda_{i',j+1} = \lambda_{i',j}\lambda_{i,j+1} \quad \text{for all } i, i', j.$$

With $f : \mathbb{N}^k \to \mathbb{N}$ given by $f(n) = \sum_{i=1}^k n_i$, we have $\Xi_k \cong f^*(\Xi_1)$ as in [**18**, Definition 1.9], and [**18**, Corollary 3.5(iii)] shows that $C^*(\Xi_k) \cong \mathcal{K} \otimes C(\mathbb{T}^{k-1})$. On the other hand, each $v_i\partial\Xi_k$ consists of just a single morphism x_i^k so the boundary-path representation generates a copy of $\mathcal{K}$ just as in Example 4.7.1. As for Ξ_2, the augmented-boundary path representation contains all elements of the form $S_{\mathcal{E}}(\sigma) \otimes L_{d(\sigma)+n}$ where $\sigma \in \Xi_k$ and $n \in \mathbb{Z}^k$ satisfies $\sum_{i=1}^k n_i = 0$. Identifying $C^*(\{L_{\left(n_1,\dots,n_{k-1},\sum_{i=1}^{k-1}(-n_i)\right)} : n \in \mathbb{Z}^{k-1}\})$ with $C^*(\mathbb{Z}^{k-1})$, we have that the augmented $\mathrm{FE}(\Xi_k)$-relative boundary-path representation generates a copy of $\mathcal{K} \otimes C^*(\mathbb{Z}^{k-1}) \cong C^*(\Xi_k)$.

In fact, for any k-graph Λ and any $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, the augmented $\mathcal{E}$-relative boundary-path representation is faithful on $C^*(\Lambda; \mathcal{E})$.

THEOREM 4.7.6. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. The representation $\pi^{\mathcal{E}}_{S_{\mathcal{E}}\otimes L} : C^*(\Lambda; \mathcal{E}) \to \mathcal{B}(\ell^2(\partial(\Lambda; \mathcal{E})) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^k))$ determined by the augmented $\mathcal{E}$-relative boundary-path representation is faithful.*

PROOF. By Corollary 4.3.10, we have that $S_{\mathcal{E}}(v) \neq 0$ for all $v \in \Lambda^0$, and hence for $v \in \Lambda^0$, $(S_{\mathcal{E}} \otimes L)_v = S_{\mathcal{E}}(v) \otimes \mathrm{id}_{\ell^2(\mathbb{Z}^k)}$ is also nonzero. Similarly, for $E \in \mathrm{FE}(\Lambda) \setminus \overline{\mathcal{E}}$, we have

$$\prod_{\lambda \in E} \big((S_{\mathcal{E}} \otimes L)_{r(E)} - (S_{\mathcal{E}} \otimes L)_\lambda (S_{\mathcal{E}} \otimes L)^*_\lambda\big) = \Big(\prod_{\lambda \in E} \big(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\lambda) S_{\mathcal{E}}(\lambda)^*\big)\Big) \otimes \mathrm{id}_{\ell^2(\mathbb{Z}^k)}$$

which is nonzero because $\prod_{\lambda \in E} \big(S_{\mathcal{E}}(r(E)) - S_{\mathcal{E}}(\lambda) S^*_{\mathcal{E}}(\lambda)\big)$ is nonzero by Corollary 4.3.10.

Let $\mathcal{D}$ denote the diagonal action $\mathcal{D}_z(e_m \otimes \overline{e}_n) := z^{m-n} e_m \otimes \overline{e}_n$ of $\mathbb{T}^k$ on $\mathcal{B}(\ell^2(\mathbb{Z}^k))$. Since each L_n is equal to the strong operator sum

$$L_n = \textstyle\sum_{m \in \mathbb{N}^k} e_{m+n} \otimes \overline{e}_m,$$

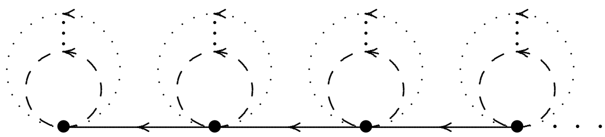
we have $\mathcal{D}_z(L_n) = z^n L_n$ for all $z \in \mathbb{T}^k$ and $n \in \mathbb{N}^k$. Define an action θ of $\mathbb{T}^k$ on $\mathcal{B}(\ell^2(\partial(\Lambda; \mathcal{E}))) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^k))$ by $\theta_z := \mathrm{id} \otimes \mathcal{D}_z$. That is $\theta_z(A \otimes B) := (A \otimes \mathcal{D}_z(B))$ for all $A \in \mathcal{B}(\ell^2(\partial(\Lambda; \mathcal{E})))$ and $B \in \mathcal{B}(\ell^2(\mathbb{Z}^k))$. Then for $z \in \mathbb{T}^k$,

$$\theta_z((S_{\mathcal{E}} \otimes L)_\lambda) = S_{\mathcal{E}}(\lambda) \otimes \mathcal{D}_z(L_{d(\lambda)}) = S_{\mathcal{E}}(\lambda) \otimes z^{d(\lambda)} L_{d(\lambda)} = z^{d(\lambda)} (S_{\mathcal{E}} \otimes L)_\lambda.$$

It follows that $\theta_z \circ \pi^{\mathcal{E}}_{S_{\mathcal{E}} \otimes L} = \pi^{\mathcal{E}}_{S_{\mathcal{E}} \otimes L} \circ \gamma_z$ for all $z \in \mathbb{T}^k$. It is clear that $\mathcal{D}$ is strongly continuous, and hence θ is strongly continuous as well. Hence Theorem 4.3.12 implies that $\pi^{\mathcal{E}}_{S_{\mathcal{E}} \otimes L}$ is faithful. □

REMARK 4.7.7. Periodic boundary paths in k-graphs take the place of loops in 1-graphs in the higher-rank versions of the Cuntz-Krieger uniqueness theorem (see for example [**18**, Theorem 4.6], [**30**, Theorem 4.5]). The k-graph Ξ_k is one of the prototypical examples of a k-dimensional infinite periodic path. So how do the graphs Ξ_k resemble rank-k loops?

Let $\mathcal{L}$ be the graph with a single vertex w and a single edge μ with $r(\mu) = s(\mu) = w$. The Cuntz-Krieger algebra of $\mathcal{L}$ is the universal algebra generated by a unitary, namely $C(\mathbb{T})$. It is therefore easy to see using [**30**, Corollary 4.4] that $C^*(\Xi_k)$ is canonically isomorphic to the C^*-algebra $C^*(\mathcal{L} \times \cdots \times \mathcal{L} \times \Xi_1)$ of the cartesian product of $k-1$ copies of $\mathcal{L}$ with a copy of Ξ_1; that is, the k-graph with 1-skeleton



where dotted edges have degree e_k. To write this isomorphism down explicitly, consider $\sigma, \tau \in \Xi_k$ with $|d(\sigma)| \geq |d(\tau)|$ and $r(\sigma) = v_i$. Then $s_\sigma s_\tau^* \in C^*(\Xi_k)$ is mapped to the element

$$s_{(\mu^{(d(\tau)_1 - d(\sigma)_1)}, w, \ldots, w, v_i)} \cdots s_{(w, \ldots, \mu^{(d(\tau)_{k-1} - d(\sigma)_{k-1})}, v_i)} s_{(w, \ldots, w, \lambda_{i+1} \cdots \lambda_{i+|d(\sigma)|-|d(\tau)|})}$$

where if $n < 0$ then $s_\mu^n = (s_\mu^{-n})^*$. If $|d(\sigma)| < |d(\tau)|$, then $s_\sigma s_\tau^* = (s_\tau s_\sigma^*)^*$, so the above formula completely determines a map from $C^*(\Xi_k)$ onto $C^*(\mathcal{L} \times \cdots \times \mathcal{L} \times \Xi_1)$. It is not hard to check that this map is an isomorphism.

4.8. Augmented representations

The trick employed in the previous section to jack up the boundary-path representation of $C^*(\Lambda; \mathcal{E})$ to a faithful representation works for arbitrary homomorphisms of relative Cuntz-Krieger algebras which are injective on the core.

DEFINITION 4.8.1. Let (Λ, d) be a finitely aligned k-graph, let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. Define a family $\{(t \otimes \chi)_\lambda : \lambda \in \Lambda\} \subset C^*(\{t_\lambda : \lambda \in \Lambda\}) \otimes C^*(\mathbb{Z}^k)$ by $(t \otimes \chi)_\lambda := t_\lambda \otimes \chi_{d(\lambda)}$ where $\chi : \mathbb{Z}^k \to C^*(\mathbb{Z}^k)$ is the canonical inclusion that takes n to the characteristic function of n. We call this collection the *augmented t-family*.

THEOREM 4.8.2. *Let (Λ, d) be a finitely aligned k-graph, let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. The augmented t-family is also a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. Furthermore, the homomorphism $\pi^{\mathcal{E}}_{(t\otimes\chi)}$ that takes $s_{\mathcal{E}}(\lambda)$ to $(t \otimes \chi)_\lambda$ is injective if and only if $\pi^{\mathcal{E}}_t$ is injective on $C^*(\Lambda)^\gamma$.*

PROOF. Calculations identical to those in the proof of Lemma 4.7.4 show that the augmented t-family is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. Let $\theta : \mathbb{T}^k \to \operatorname{Aut}\big(C^*(\{t_\lambda : \lambda \in \Lambda\}) \otimes C^*(\mathbb{Z}^k)\big)$ be the action determined by $\theta_z := \mathrm{id}_{C^*(\{t_\lambda : \lambda\in\Lambda\})} \otimes \hat{\alpha}_z$ where $\hat{\alpha}$ is the dual action of $\mathbb{T}^k$ on $C^*(\mathbb{Z}^k)$. As in the proof of Theorem 4.7.6, we have $\theta_z \circ \pi^{\mathcal{E}}_{(t\otimes\chi)} = \pi^{\mathcal{E}}_{(t\otimes\chi)} \circ \gamma$ because $\hat{\alpha}_z(\chi_n) = z^n \chi_n$ for all $n \in \mathbb{Z}^k$. Moreover, θ is strongly continuous because $\hat{\alpha}$ is strongly continuous. Since

$$\pi^{\mathcal{E}}_{(t\otimes\chi)}(C^*(\Lambda)^\gamma) = \pi^{\mathcal{E}}_t(C^*(\Lambda)^\gamma) \otimes 1_{C^*(\mathbb{Z}^k)} \underset{\text{can}}{\cong} \pi^{\mathcal{E}}_t(C^*(\Lambda)^\gamma),$$

the result now follows from Theorem 4.3.12.

4.9. The Cuntz-Krieger relation: some examples

In this section we explain why relation (CK) is the right relation for defining an appropriate analogue of a Cuntz-Krieger algebra for finitely aligned k-graphs. In particular, we explain why it has to be expressed in terms of products, rather than sums, of range projections. This section is taken almost verbatim from [**30**, Appendix A].

The objective of the Cuntz-Krieger relations is to associate to each finitely aligned k-graph Λ a universal C^*-algebra $C^*(\Lambda)$ generated by a collection $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries in such a way that

(a) The partial isometries s_λ are all nonzero;
(b) Connectivity in Λ is modelled by multiplication in $C^*(\Lambda)$;
(c) $C^*(\Lambda)$ is spanned by the elements $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$;
(d) The core, $\overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda, d(\lambda) = d(\mu)\}$, is AF; and
(e) A representation π of $C^*(\Lambda)$ is faithful on the core if and only if $\pi(s_v) \neq 0$ for $v \in \Lambda^0$.

Relations (TCK1) and (TCK2) address property (b). Relation (TCK3) ensures that property (c) is satisfied. Proposition 3.4.1 shows that relations (TCK1)–(TCK3) also guarantee property (d).

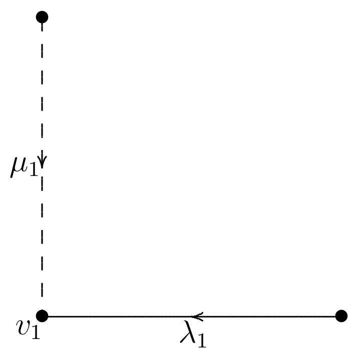
We must now produce a relation (CK) which guarantees that $C^*(\Lambda)$ satisfies (a) and (e); in the following discussion, therefore, we assume (TCK1)–(TCK3) hold.

Recall from [**29**] that given a k-graph Λ and an element n of $\mathbb{N}^k$, we write $\Lambda^{\leq n}$ for the collection of paths $\lambda \in \Lambda$ such that $d(\lambda) \leq n$ and such that for all $\lambda\nu \in \Lambda$ with $d(\nu) > 0$, we have $d(\lambda\nu) \not\leq n$. The analyses of [**12**] and [**29**] suggest that a suitable relation might be

$$t_v = \sum_{\lambda \in v\Lambda^{\leq n}} t_\lambda t_\lambda^* \text{ whenever } v\Lambda^{\leq n} \text{ is finite.} \tag{4.9.1}$$

However, this relation fails to guarantee (a), even for row-finite k-graphs, as can be seen from the following example:

EXAMPLE 4.9.1. Consider the row-finite 2-graph Λ_1 with 1-skeleton

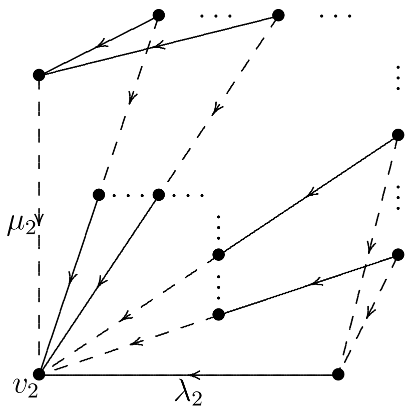


If $\{t_\lambda : \lambda \in \Lambda\}$ satisfy (TCK1)–(TCK3) and (4.9.1), then the range projections $s_{\lambda_1} s_{\lambda_1}^*$ and $s_{\mu_1} s_{\mu_1}^*$ are orthogonal by (4.9.1) for $n = (1, 1)$, but must both be equal to s_{v_1} by (4.9.1) with $n = (0, 1)$ and $n = (1, 0)$. Consequently $s_{v_1} = 0$, so (4.9.1) fails to ensure condition (a) for $C^*(\Lambda_1)$.

For the *row-finite* k-graphs of [**29**] in which $v\Lambda^{e_i}$ is always finite, the problem illustrated by this example was avoided by imposing the condition that the k-graphs in question be *locally convex*. The k-graph (Λ, d) is locally convex if whenever $v \in \Lambda^0$, $i \neq j$, $\lambda \in v\Lambda^{e_i}$ and $\mu \in v\Lambda^{e_j}$, we have both $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ nonempty [**29**, Definition 3.9].

For locally convex row-finite k-graphs, the Cuntz-Krieger relations used in [**29**] are equivalent to (TCK1)–(TCK3) and (4.9.2). It is shown in [**29**, Theorem 3.15] that these relations imply (a), and the discussion of [**29**, page 109] shows that they imply (e). However, Example 4.9.2 demonstrates that for non-row-finite k-graphs, local convexity is not enough to ensure that (4.9.1) implies (e).

Example 4.9.2. Consider the locally convex finitely aligned 2-graph Λ_2 with 1-skeleton



Relation (4.9.1) does not impose any equalities at v_2 because $v_2\Lambda_2^{\leq n}$ is infinite for all $n \neq 0$. The Cuntz-Krieger family $\{S_\lambda : \lambda \in \Lambda_2\}$ provided by the boundary-path representation satisfies $S_{v_2} - (S_{\lambda_2}S_{\lambda_2}^* + S_{\mu_2}S_{\mu_2}^*) = 0$. However, for any nontrivial projection P, taking $T_{v_2} := S_{v_2} \oplus P$ and $T_\sigma = S_\sigma \oplus 0$ for $\sigma \in \Lambda_2 \setminus \{v_2\}$ gives a Cuntz-Krieger Λ_2-family satisfying (TCK1)–(TCK3) and (4.9.1) in which $T_{v_2} - (T_{\lambda_2}T_{\lambda_2}^* + T_{\mu_2}T_{\mu_2}^*) \neq 0$. In particular, $\{S_\lambda : \lambda \in \Lambda^2\}$ satisfies (TCK1)–(TCK3) and (4.9.1), but the representation determined by $\{S_\lambda : \lambda \in \Lambda^2\}$ is not faithful on the core, even though $S_v \neq 0$ for all $v \in \Lambda_2^0$.

Relation (4.9.1) does not work for Λ_2 because there exists a finite subset E of $v_2\Lambda_2$ (namely $E = \{\lambda_2, \mu_2\}$) with the property that the sum of the range projections of paths in E dominates all range projections from in $v_2\Lambda_2 \setminus \{v_2\}$, but there is no such subset of the form $v_2\Lambda_2^{\leq n}$. For a finitely aligned k-graph Λ and $v \in \Lambda^0$, we can use (TCK3) to characterise the finite subsets of E of $v\Lambda$ with this property: they are precisely the finite exhaustive sets of Definition 2.4.3.

Example 4.9.2 therefore suggests that relation (CK) should be

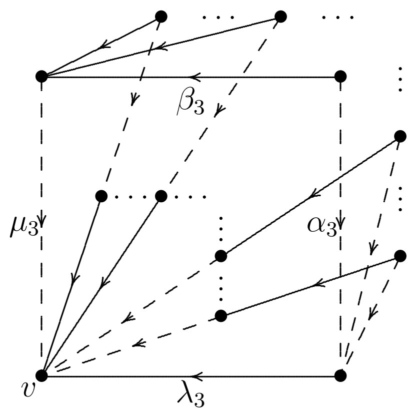
(4.9.2) $\quad t_v = \sum_{\lambda \in E} t_\lambda t_\lambda^* \quad$ for every $v \in \Lambda^0$ and finite exhaustive $E \subset v\Lambda \setminus \{v\}$.

Example 4.9.3 (Example 4.9.1 continued). The only finite exhaustive subset of $v_1\Lambda_1$ which does not contain v_1 is the set $\{\lambda_1, \mu_1\}$. In particular, (4.9.2) does not insist that either $t_{\lambda_1}t_{\lambda_1}^*$ or $t_{\mu_1}t_{\mu_1}^*$ is equal to t_{v_1}. Hence replacing (4.9.1) with (4.9.2) eliminates the pathology associated to the non-local-convexity of Λ_1.

EXAMPLE 4.9.4 (Example 4.9.2 continued). Every family $\{t_\lambda : \lambda \in \Lambda_2\}$ satisfying (TCK1)–(TCK3) and (4.9.2) satisfies $t_{v_2} = t_{\lambda_2} t^*_{\lambda_2} + t_{\mu_2} t^*_{\mu_2}$. Since the partial isometries $\{S_\sigma : \sigma \in \Lambda_2\}$ satisfy (TCK1)–(TCK3) and (4.9.2) and since S_v is nonzero for each $v \in \Lambda_2^0$, (4.9.2) ensures that (a) holds for $C^*(\Lambda_2)$. In fact, for Λ_2, relation 4.9.2 is equivalent to relation (CK), so Theorem 4.6.2 shows that for the 2-graph Λ_2, (4.9.2) guarantees both (a) and (e). It follows that (4.9.2) is exactly the right relation for Λ_2.

Since the left-hand side of (4.9.2) is a projection, the relation only makes sense if the projections in the sum on the right-hand side are mutually orthogonal. Hence in general, (4.9.2) is predicated on the notion that the range projections associated to paths in a finite exhaustive subset of $v\Lambda \setminus \{v\}$ are mutually orthogonal. However, Λ_2 shows that we cannot expect finite exhaustive sets always to consist of paths of a single degree, so there is no reason to think that the range projections associated to paths in an arbitrary finite exhausitve set are mutually orthogonal. The following example shows that indeed it is necessary to formulate a relation which takes into account the possibility that they are not.

EXAMPLE 4.9.5. Consider the locally convex 2-graph Λ_3 with 1-skeleton



As in Example 4.9.2, relation (CK) must insist that the range projections associated to λ_3 and μ_3 together fill up t_{v_3}, or else (e) will fail because $\{\lambda_3, \mu_3\}$ is finite and exhaustive. However, the range projections $t_{\lambda_3} t^*_{\lambda_3}$ and $t_{\mu_3} t^*_{\mu_3}$ can only be orthogonal if (a) is not satisfied: Lemma 3.1.2(1) ensures that $t_{\lambda_3} t^*_{\lambda_3} t_{\mu_3} t^*_{\mu_3} = t_{\lambda_3\alpha_3} t^*_{\lambda_3\alpha_3}$, so if $t_\lambda t^*_\lambda$ and $t_\mu t^*_\mu$ are mutually orthogonal, then we must have $t_{\lambda\alpha} = 0$, and hence $t_{s(\alpha)} = 0$. Indeed there is no finite exhaustive subset of $v\Lambda$ whose range projections are orthogonal. It follows that (4.9.2) fails to ensure condition (a) for the k-graph Λ_3 because in any family $\{t_\sigma : \sigma \in \Lambda_3\}$ satisfying (TCK1)–(TCK3) and (4.9.2), we have $t_{s(\alpha)} = 0$.

The solution to the problem illustrated in Example 4.9.5 is to use products rather than sums to express relation (CK).

EXAMPLE 4.9.6 (Example 4.9.5 continued). Lemma 3.1.2(2) says that in any family satisfying (TCK1)–(TCK3), the projections $t_{\lambda_3} t^*_{\lambda_3}$ and $t_{\mu_3} t^*_{\mu_3}$ commute. Consequently, the relation

$$(t_{v_3} - t_{\lambda_3} t^*_{\lambda_3})(t_{v_3} - t_{\mu_3} t^*_{\mu_3}) = 0. \tag{4.9.3}$$

is a well-defined replacement for (4.9.2). Relation 4.9.3 insists that the range projections associated to λ_3 and μ_3 fill up t_{v_3} without the undesirable side-effect of forcing $t_{s(\alpha)} = 0$.

Relation (CK), namely

$$\prod_{\lambda \in E} (t_v - t_\lambda t^*_\lambda) = 0 \quad \text{for every } v \in \Lambda^0 \text{ and finite exhaustive } E \subset v\Lambda,$$

is the generalisation of (4.9.3) to arbitrary finite exhaustive sets in an arbitrary finitely aligned k-graph. Note that (CK) reduces to (4.9.2) when the range projections associated to paths in E are mutually orthogonal (as in Λ_2). Theorem 4.6.2 shows that (CK) ensures (a) and (e).

CHAPTER 5

Gauge-invariant ideals in Cuntz-Krieger algebras

In this chapter, we analyse the gauge-invariant ideal structure of $C^*(\Lambda)$. We begin by defining saturated hereditary sets of vertices. Our definitions and techniques for this draw on the corresponding ideas in [**29**] and [**2**], though our definition of a saturated set looks different from the definitions in previous work because of the new relation (CK). We are able to show using methods similar to those of [**2**] that the collection of vertex projections associated to a saturated hereditary set $H \subset \Lambda^0$ generates a gauge-invariant ideal I_H in $C^*(\Lambda)$, and that I_H contains $C^*(\Lambda H)$ as a full corner. However, to answer the question of what the quotient $C^*(\Lambda)/I_H$ looks like, we have to use a different approach than in [**2**] because it is not clear how to carry the construction in [**2**] of the quotient graph of a 1-graph E to the setting of k-graphs. Our approach is to show that $C^*(\Lambda)/I_H$ is isomorphic to a relative Cuntz-Krieger algebra associated to $\Lambda \setminus \Lambda H$. We then use Theorem 4.3.12 to characterise all the gauge-invariant ideals of $C^*(\Lambda)$ and the associated quotients.

Muhly and Tomforde arrive indirectly at the same identification of quotients of the Cuntz-Krieger algebra with relative graph algebras of associated subgraphs for directed graphs in [**23**]. Given a directed graph E and a subset V of E^0, Muhly and Tomforde identify the relative Cuntz-Krieger algebra $C^*(E, V)$ with the Cuntz-Krieger algebra of a modified graph E_V. Let E be a directed graph, let $H \subset E^0$ be saturated and hereditary, let $F = E \setminus s^{-1}(H)$, and let $V = H^{\mathrm{fin}}_\infty$. Then the modified graph F_V of [**23**] is identical to the quotient graph E/H of [**2**] showing that $C^*(E)/I_H \cong C^*(F, V)$. This method is attractive since it allows results from the theory of Cuntz-Krieger algebras to be applied to relative Cuntz-Krieger algebras. The question of whether there exists a similar construction of a modified k-graph $\Lambda_{\mathcal{E}}$ from a k-graph Λ and a subset $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ so that $C^*(\Lambda_{\mathcal{E}}) \cong C^*(\Lambda; \mathcal{E})$ remains unanswered. The results in this chapter do indicate, though, that such a construction would be significantly more complicated for k-graphs than it is for 1-graphs.

Throughout this chapter, when we talk about an ideal in a C^*-algebra, we always mean a closed two-sided ideal.

5.1. Hereditary subsets and associated ideals

In this section, we define saturated hereditary sets of vertices. We show that given a finitely aligned k-graph Λ, and a saturated hereditary set $H \subset \Lambda^0$, the ideal in $C^*(\Lambda)$ generated by the vertex projections associated to $H \subset \Lambda^0$ is strongly Morita equivalent to the Cuntz-Krieger algebra of the subgraph $H\Lambda$.

DEFINITION 5.1.1. Let (Λ, d) be a finitely aligned k-graph. Define a relation $\leq$ on Λ^0 by $v \leq w$ if and only if $v\Lambda w \neq \emptyset$.

(1) A set $H \subset \Lambda^0$ is *hereditary* if $v \in H$ and $v \leq w$ imply $w \in H$.
(2) A set $H \subset \Lambda^0$ is *saturated* if for every finite exhaustive set F such that $s(F) \subset H$, we have $r(F) \in H$.

Given $H \subset \Lambda^0$, we write ΣH for the smallest saturated set containing H. We call ΣH the *saturation* of H.

We want to show that the saturation of a hereditary set H is also hereditary.

LEMMA 5.1.2. *Let (Λ, d) be a finitely aligned k-graph and let $G \subset \Lambda^0$. Then*

(1) *Let $S_G := \{v \in \Lambda^0 :$ there exists $F \subset v\Lambda G$ with F finite exhaustive$\}$. Then $S_G = \Sigma G$.*
(2) *If G is hereditary, then ΣG is hereditary.*

PROOF. For claim (1), we show first that S_G is saturated. To see this, let $v \in \Lambda^0$, and suppose that $E \subset v\Lambda$ is finite exhaustive with $s(E) \subset S_G$. If $v \in E$, then $v = s(v) \in s(E) \subset S_G$, so there is nothing to do. Hence we may assume that $E \in v\,\mathrm{FE}(\Lambda)$. By definition of S_G, for each $\lambda \in E$ there exists a finite exhaustive set $F_\lambda \subset s(\lambda)\Lambda$ such that $s(F_\lambda) \subset G$. Let $F := \{\lambda \in E : s(\lambda) \notin F_\lambda\}$. For $\lambda \in E\backslash F$, we have $s(\lambda) \in s(F_\lambda) \subset G$, so $s(E \setminus F) \subset G$. On the other hand, for $\lambda \in F$, we have $s(\lambda) = r(F_\lambda) \notin F_\lambda$, so $F_\lambda \in \mathrm{FE}(\Lambda)$. Let $E' := (E \setminus F) \cup \big(\bigcup_{\lambda \in F} \lambda F_\lambda\big)$. Then Lemma 4.2.9 applied to the set $\mathcal{E} := \{E, F_\lambda : \lambda \in F\} \subset \mathrm{FE}(\Lambda)$ shows that $E' \in \mathrm{FE}(\Lambda)$, and we have $s(E') \subset G$ by definition. Since $r(E') = r(E) = v$, it follows that $v \in S_G$ by definition of S_G. Since v and F were arbitrary, it follows that S_G is saturated.

We have $G \subset S_G$ by definition, and S_G is saturated by the previous paragraph. It therefore follows from the definition of ΣG that $\Sigma G \subset S_G$. On the other hand, every element of S_G must belong to ΣG by Definition 5.1.1(2), giving $S_G \subset \Sigma G$. It follows that $\Sigma G = S_G$, establishing (1).

To prove claim (2), let $v \in \Sigma G$ and $v \leq w$; say $\lambda \in v\Lambda w$. If $v \in G$ then $w \in G$ because G is hereditary, so suppose that $v \in \Sigma G \setminus G$. Then there exists

$F \in v\,\mathrm{FE}(\Lambda)$ such that $s(F) \subset G$. If $\lambda \in F\Lambda$, then $\lambda = \mu\mu'$ for some $\mu \in F$. But then $s(\mu) \in G$ by definition of F, and it follows that $s(\lambda) = s(\mu') \in G$ because G is hereditary. On the other hand, if $\lambda \notin F\Lambda$, then Lemma 4.2.7 shows that, $\mathrm{Ext}(\lambda; F) \in w\,\mathrm{FE}(\Lambda)$. We claim that $s(\mathrm{Ext}(\lambda; F)) \subset G$; it will then follow that $w \in \Sigma G$, completing the proof.

To see that $s(\mathrm{Ext}(\lambda; F)) \subset G$, note that $\alpha \in \mathrm{Ext}(\lambda; F)$ implies that there exist $\mu \in F$ and $\beta \in \Lambda$ such that $\lambda\alpha = \mu\beta$. But then $s(\alpha) = s(\lambda\alpha) = s(\mu\beta)$ which belongs to G because $s(\mu) \in G$ and G is hereditary. □

Lemma 5.1.3. *Let (Λ, d) be a finitely aligned k-graph, and let I be an ideal of $C^*(\Lambda)$. Then $H_I := \{v \in \Lambda^0 : s_v \in I\}$ is saturated and hereditary.*

Proof. Suppose $v \in H_I$ and $w \in \Lambda^0$ with $v \leq w$. So there exists $\lambda \in \Lambda$ with $r(\Lambda) = v$ and $s(\Lambda) = w$. Then $s_v \in I$, so $s_w = s_\lambda^* s_v s_\lambda \in I$, and then $w \in H_I$; consequently H_I is hereditary. Now suppose that $v \in \Lambda^0$ and there exists a finite exhaustive set $F \subset v\Lambda$ such that $s(F) \subset H_I$. If $v \in F$, then $v = s(v) \in s(F) \subset H_I$ and we are done, so we may assume that $F \in v\,\mathrm{FE}(\Lambda)$. Proposition 3.3.3 shows that $s_v = \Big(\prod_{\lambda \in \vee F}(s_v - s_\lambda s_\lambda^*)\Big) + \Big(\sum_{\lambda \in \vee F} Q(s)_\lambda^{\vee F}\Big)$. Since $F \in \mathrm{FE}(\Lambda)$, we have $\vee F \in \mathrm{FE}(\Lambda)$ as well, and since $\{s_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family, it follows that $\prod_{\lambda \in \vee F}(s_v - s_\lambda s_\lambda^*) = 0$, giving $s_v = \sum_{\lambda \in \vee F} Q(s)_\lambda^{\vee F}$, and we need only show that $\lambda \in \vee F$ implies $Q(s)_\lambda^{\vee F} \in I$.

If $\lambda \in \vee F$, then $\lambda = \mu\mu'$ for some $\mu \in F$. Since H_I is hereditary and $s(F) \subset H_I$, it follows that $\lambda \in \vee F$ implies $s(\lambda) \in H_I$. Hence for $\lambda \in \vee F$, we have $s_{s(\lambda)} \in I$, and then Definition 3.3.1 gives

$$Q(s)_\lambda^{\vee F} = s_\lambda s_{s(\lambda)} s_\lambda^* \Big(\prod_{\lambda\lambda' \in \vee F, d(\lambda) > 0} (s_\lambda s_\lambda^* - s_{\lambda\lambda'} s_{\lambda\lambda'}^*) \Big),$$

which belongs to I because $s_{s(\lambda)} \in I$ and I is an ideal. □

For $H \subset \Lambda^0$, let I_H be the ideal in $C^*(\Lambda)$ generated by $\{s_v : v \in H\}$.

Lemma 5.1.4. *Let (Λ, d) be a finitely aligned k-graph, and let $H \subset \Lambda^0$ be saturated and hereditary. Then $I_H = \overline{\mathrm{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda H\}$.*

Proof. For convenience, we denote $\overline{\mathrm{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda H\}$ by X_H throughout this proof. If $\lambda, \mu \in \Lambda H$ and $s_\lambda s_\mu^* \neq 0$, then $s(\lambda) = s(\mu) \in H$, and then $s_\lambda s_\mu^* = s_\lambda s_{s(\lambda)} s_\mu^* \in I_H$ because I_H is an ideal. It follows that $X_H \subset I_H$.

For the reverse containment, we just need to show that X_H is an ideal and contains the projections $\{s_v : v \in H\}$. We have that X_H is norm-closed, that X_H is closed under taking adjoints, and that the projections $\{s_v : v \in H\}$ belong to

X_H by definition, so we only need to show that X_H is closed under multiplication, and that for $a, b \in C^*(\Lambda)$, and $x \in X_H$, we have $axb \in X_H$.

To see that for $a, b \in C^*(\Lambda)$ and $x \in X_H$, we have $axb \in X_H$, note that by Lemma 3.1.2(5) and the definition of X_H, it suffices to show that for $\lambda, \mu \in \Lambda$ and $\sigma, \tau \in \Lambda H$, we have $s_\lambda s_\mu^* s_\sigma s_\tau^* \in X_H$ and $s_\sigma s_\tau^* s_\lambda s_\mu^* \in X_H$. Indeed since X_H is closed under taking adjoints, it suffices to show that

$$\lambda, \mu \in \Lambda \text{ and } \sigma, \tau \in \Lambda H \text{ imply } s_\lambda s_\mu^* s_\sigma s_\tau^* \in X_H. \tag{5.1.1}$$

Taking $\lambda, \mu \in \Lambda H$, (5.1.1) also establishes that X_H is closed under multiplication. Hence the proof will be complete once we have established (5.1.1). So let $\lambda, \mu \in \Lambda$ and $\sigma, \tau \in \Lambda H$. Then $s_\lambda s_\mu^* s_\sigma s_\tau^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\sigma)} s_{\lambda\alpha} s_{\tau\beta}^*$. But since H is hereditary and $s(\sigma)$ belongs to H, we have that $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$ implies that $s(\alpha) \in H$, and it follows that $s_\lambda s_\mu^* s_\sigma s_\tau^* \in X_H$ as required. □

Given a finitely aligned k-graph Λ and a saturated hereditary subset $H \subset \Lambda^0$, consider the subcategory $H\Lambda = \{\lambda \in \Lambda : r(\lambda) \in H\} \subset \Lambda$.

LEMMA 5.1.5. *Let (Λ, d) be a finitely aligned k-graph, and suppose that $H \subset \Lambda^0$ is saturated and hereditary. Then $(H\Lambda, d|_{H\Lambda})$ is also a finitely aligned k-graph, and $C^*(H\Lambda) \cong C^*(\{s_\lambda : \lambda \in H\Lambda\}) \subset C^*(\Lambda)$. Moreover, this subalgebra is a full corner in I_H.*

PROOF. First we show that $(H\Lambda, d|_{H\Lambda})$ is a finitely aligned k-graph. We begin by establishing the factorisation property; this shows in particular that $r|_{H\Lambda}$ and $s|_{H\Lambda}$ both have range in $(H\Lambda)^0$, so $H\Lambda$ is a k-graph. Let $\lambda \in H\Lambda$, and suppose that $d(\lambda) = m + n$ for $m, n \in \mathbb{N}^k$. Since Λ is a k-graph, there exist unique $\mu, \nu \in \Lambda$ with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. Since $r(\mu) = r(\lambda) \in H$, we have $\mu \in H\Lambda$. Since H is hereditary and $r(\mu) \in H$, we then have $s(\mu) \in H$. Since $s(\mu) = r(\nu)$ it follows that $\nu \in H\Lambda$, and so $H\Lambda$ satisfies the factorisation property as required. To see that $H\Lambda$ is finitely aligned, let $\lambda, \mu \in H\Lambda$. If $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$, then $r(\alpha) = s(\lambda)$ which belongs to H since H is hereditary; likewise, $r(\beta) \in H$. That is to say, $\Lambda^{\min}(\lambda, \mu) \subset (H\Lambda)^{\min}(\lambda, \mu)$. Since the reverse inclusion is trivial, we therefore have

$$(H\Lambda)^{\min}(\lambda, \mu) = \Lambda^{\min}(\lambda, \mu). \tag{5.1.2}$$

Since Λ is finitely aligned, $\Lambda^{\min}(\lambda, \mu)$ is finite, and it follows that $H\Lambda$ is also finitely aligned.

Next we show that $\{s_\lambda : \lambda \in H\Lambda\}$ is a Cuntz-Krieger $H\Lambda$-family. To see this, notice that (TCK1) and (TCK2) follow directly from (TCK1) and (TCK2) for the

Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$, and (TCK3) follows from the combination of (5.1.2) and (TCK3) for the Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$. For (CK), let $E \in \mathrm{FE}(H\Lambda)$. Then $r(E) \in H$, and hence $r(E)\Lambda \subset H\Lambda$. Now pick an arbitrary $\lambda \in r(E)\Lambda$. Then there exists $\mu \in E$ such that $(H\Lambda)^{\min}(\lambda, \mu) \neq \emptyset$ because $\lambda \in H\Lambda$ and $E \in \mathrm{FE}(H\Lambda)$. Consequently $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. Since $\lambda \in r(E)\Lambda$ was arbitrary, it follows that $E \in \mathrm{FE}(\Lambda)$. But then (CK) for the Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ implies that $\prod_{\mu \in E}(s_{r(E)} - s_\mu s_\mu^*) = 0$, establishing (CK) for $\{s_\lambda : \lambda \in H\Lambda\}$.

The universal property of $C^*(H\Lambda)$ now gives a homomorphism $\pi : C^*(H\Lambda) \to C^*(\{s_\lambda : r(\lambda) \in H\})$. Write γ_H for the gauge action on $C^*(H\Lambda)$ and $\gamma|$ for the restriction of the gauge action on $C^*(\Lambda)$ to $C^*(\{s_\lambda : r(\lambda) \in H\})$. Since the gauge acton on $C^*(\Lambda)$ is strongly continuous, so is $\gamma|$. We have $\pi \circ (\gamma_H)_z = (\gamma|)_z \circ \pi$ for all $z \in \mathbb{T}^k$ by definition, and since $\{s_v : v \in \Lambda^0\}$ are nonzero by Corollary 4.3.10, Theorem 4.6.2 implies that π is injective.

For the final statement, just use the argument of [**3**, Theorem 4.1(c)] to see that $C^*(\{s_\lambda : r(\lambda) \in H\})$ is the corner of I_H determined by the projection $P_H := \sum_{v \in H} s_v \in \mathcal{M}(I_H)$, and that this projection is full in I_H. □

5.2. Quotients of Cuntz-Krieger algebras

We want to identify the structure of $C^*(\Lambda)/I_H$ for a saturated hereditary $H \subset \Lambda^0$. Let (Λ, d) be a k-graph, and let $H \subset \Lambda^0$ be a saturated hereditary set. Consider the subcategory $\Lambda \setminus \Lambda H = \{\lambda \in \Lambda : s(\lambda) \notin H\}$.

Lemma 5.2.1. *Let (Λ, d) be a finitely aligned k-graph, and let $H \subset \Lambda^0$ be saturated and hereditary. Then $(\Lambda \setminus \Lambda H, d|_{\Lambda \setminus \Lambda H})$ is also a finitely aligned k-graph.*

Proof. We first check the factorisation property for $(\Lambda \setminus \Lambda H, d|_{\Lambda \setminus \Lambda H})$, and then that $(\Lambda \setminus \Lambda H, d|_{\Lambda \setminus \Lambda H})$ is finitely aligned. For the factorisation property, let $\lambda \in \Lambda \setminus \Lambda H$, and suppose that $m + n = d(\lambda)$. By the factorisation property for Λ, there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\mu\nu = \lambda$. Since $s(\nu) = s(\lambda) \notin H$, we have $\nu \in \Lambda \setminus \Lambda H$. Since, by definition of $\leq$, we have $r(\nu) \leq s(\nu)$ it follows that $r(\nu) \notin H$ because H is hereditary. But $r(\nu) = s(\mu)$ so it follows that $\mu \in \Lambda \setminus \Lambda H$. Finite alignedness of the k-graph $\Lambda \setminus \Lambda H$ is trivial since $(\Lambda \setminus \Lambda H)^{\min}(\lambda, \mu) \subset \Lambda^{\min}(\lambda, \mu)$ for all $\lambda, \mu \in \Lambda \setminus \Lambda H$. □

We now want to show that the quotient of $C^*(\Lambda)$ by an ideal I_H as in Section 5.1 is a relative Cuntz-Krieger algebra associated to the subgraph $\Lambda \setminus \Lambda H$.

Definition 5.2.2. Let (Λ, d) be a finitely aligned k-graph and let $H \subset \Lambda^0$ be saturated and hereditary. Define $\mathcal{E}_H := \{E \setminus EH : E \in \mathrm{FE}(\Lambda)\}$.

Lemma 5.2.3. *Let (Λ, d) be a finitely aligned k-graph, and suppose that $H \subset \Lambda^0$ is saturated and hereditary. Then $\mathcal{E}_H \subset \mathrm{FE}(\Lambda \setminus \Lambda H)$.*

Proof. Suppose that $E \in \mathcal{E}_H$ and that $\mu \in r(E)\Lambda \setminus \Lambda H$. Suppose for contradiction that $\mathrm{Ext}_{\Lambda\setminus\Lambda H}(\mu; E) = \emptyset$; that is, $\lambda \in E$ and $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ imply $s(\alpha) = s(\beta) \in H$. Since $E \in \mathcal{E}_H$, there exists $F \in \mathrm{FE}(\Lambda)$ such that $E = F \setminus FH$. To prove the lemma we establish two claims.

Claim 1: $\mathrm{Ext}_\Lambda(\mu; F) \in \mathrm{FE}(\Lambda)$. To see this, note that by Lemma 4.2.7, we need only show that $\mu \notin F\Lambda$. Since H is hereditary, we automatically have $\mu \notin (FH)\Lambda$, so it suffices to show that $\mu \notin E\Lambda$. So suppose for contradiction that $\mu \in E\Lambda$. Then $s(\mu) \in \mathrm{Ext}_\Lambda(\mu; E)$. Since μ was chosen from $r(E)\Lambda \setminus \Lambda H$, we have that $s(\mu) \notin H$, giving $s(\mu) \in \mathrm{Ext}_\Lambda(\mu; E) \cap (\Lambda \setminus \Lambda H) = \mathrm{Ext}_{\Lambda\setminus\Lambda H}(\mu; E)$, contradicting $\mathrm{Ext}_{\Lambda\setminus\Lambda H}(\mu; E) = \emptyset$. □ Claim 1

Claim 2: $\mathrm{Ext}_\Lambda(\mu; F) \subset \Lambda H$. To see this, note that we have

$$\mathrm{Ext}_\Lambda(\mu; F) = \mathrm{Ext}_\Lambda(\mu; E \cup FH) = \mathrm{Ext}_\Lambda(\mu; E) \cup \mathrm{Ext}_\Lambda(\mu; FH). \tag{5.2.1}$$

Since $\mathrm{Ext}_{\Lambda\setminus\Lambda H}(\mu; E) = \emptyset$ by assumption, we have $\mathrm{Ext}_\Lambda(\mu; E) \subset \Lambda H$. Moreover, we have $\mathrm{Ext}_\Lambda(\mu; FH) \subset \Lambda H$ because H is hereditary. Hence (5.2.1) ensures that $\mathrm{Ext}_\Lambda(\mu; F) \subset \Lambda H$ as required. □ Claim 2

Combining Claim 1 and Claim 2, we have that $s(\mu) = r(\mathrm{Ext}_\Lambda(\mu; F))$ belongs to H by Definition 5.1.1(2) because H is saturated. But this contradicts $\mu \in r(E)\Lambda \setminus \Lambda H$. □

Theorem 5.2.4. *Let (Λ, d) be a finitely aligned k-graph, and let $H \subset \Lambda^0$ be saturated and hereditary. Then $\{s_\lambda + I_H : \lambda \in \Lambda \setminus \Lambda H\}$ is a relative Cuntz-Krieger $(\Lambda\setminus\Lambda H; \mathcal{E}_H)$-family in $C^*(\Lambda)/I_H$, and the canonical homomorphism $\pi_{s+I_H}^{\mathcal{E}_H} : C^*(\Lambda\setminus \Lambda H; \mathcal{E}_H) \to C^*(\Lambda)/I_H$ is an isomorphism.*

To prove Theorem 5.2.4, we need to collect some additional results.

Lemma 5.2.5. *Let (Λ, d) be a finitely aligned k-graph, and let $H \subset \Lambda^0$ be saturated and hereditary. Then $\mathcal{E}_H$ is satiated.*

Proof. For (S1), let $E \in \mathcal{E}_H$ and suppose that $F \subset \Lambda \setminus \Lambda H$ is finite with $E \subset F$ and $F \cap \Lambda^0 = \emptyset$. By definition of $\mathcal{E}_H$ there exists $E' \in \mathrm{FE}(\Lambda)$ such that $E' \setminus E'H = E$. But then $F' := F \cup E'H$ contains E', and hence is exhaustive, and is clearly finite because both F and E' are. We have $F \cap \Lambda^0 = \emptyset$ by assumption, and $E'H \cap \Lambda^0 = \emptyset$ because $r(E) \in (\Lambda \setminus \Lambda H)^0$, and $s(E'H) \subset H$. Since $F = F' \setminus F'H$, it follows that $F \in \mathcal{E}_H$.

For (S2), let $E \in \mathcal{E}_H$, and suppose that $\mu \in r(E)(\Lambda \setminus \Lambda H)$ with $\mu \notin E\Lambda$. Since $E \in \mathcal{E}_H$ there exists $E' \in \mathrm{FE}(\Lambda)$ such that $E' \setminus E'H = E$. We have $\mathrm{Ext}_\Lambda(\mu; E') \in \mathrm{FE}(\Lambda)$ by Lemma 4.2.7, and we also have

$$\begin{aligned}\mathrm{Ext}_\Lambda(\mu; E') &= \mathrm{Ext}_\Lambda(\mu; E) \cup \mathrm{Ext}_\Lambda(\mu; E'H)\\ &= \mathrm{Ext}_{\Lambda\setminus\Lambda H}(\mu; E) \cup \mathrm{Ext}_\Lambda(\mu; E)H \cup \mathrm{Ext}_\Lambda(\mu; E'H).\end{aligned}$$

Since both $\mathrm{Ext}\,\Lambda(\mu; E)H$ and $\mathrm{Ext}_\Lambda(\mu; E'H)$ are subsets of ΛH, it follows that

$$\begin{aligned}\mathrm{Ext}_{\Lambda\setminus\Lambda H}(\mu; E) &= \mathrm{Ext}_\Lambda(\mu; E') \setminus \big(\mathrm{Ext}_\Lambda(\mu; E)H \cup \mathrm{Ext}_\Lambda(\mu; E'H)\big)\\ &= \mathrm{Ext}_\Lambda(\mu; E') \setminus \big(\mathrm{Ext}_\Lambda(\mu; E')H\big),\end{aligned}$$

and hence belongs to $\mathcal{E}_H$.

For (S3), let $E \in \mathcal{E}_H$, suppose that $0 < n_\lambda \le d(\lambda)$ for all $\lambda \in E$, and let $F := \{\lambda(0, n_\lambda) : \lambda \in E\}$. Since $E \in \mathcal{E}_H$, there exists $E' \in \mathrm{FE}(\Lambda)$ such that $E = E' \setminus E'H$. For $\mu \in E'H$, let $n_\mu := d(\mu)$. Since $E' \in \mathrm{FE}(\Lambda)$, Lemma 4.2.8 applied to the set $\mathcal{E} = \{E'\}$ shows that $\{\mu(0, n_\mu) : \mu \in E'\} \in \mathrm{FE}(\Lambda)$ also. Moreover, $F \cap E'H = \emptyset$ because $E \subset \Lambda \setminus \Lambda H$ and H is hereditary. Hence

$$F = \{\mu(0, n_\mu) : \mu \in E'\} \setminus (\{\mu(0, n_\mu) : \mu \in E'\}H).$$

It follows that $F \in \mathcal{E}_H$.

Finally, for (S4), suppose that $E \in \mathcal{E}_H$; say $E' \in \mathrm{FE}(\Lambda)$ and $E = E' \setminus E'H$. Let $G \subset E$, and for each $\lambda \in G$, suppose that $G_\lambda \in \mathcal{E}_H$ with $r(G_\lambda) = s(\lambda)$. We must show that $F := (E \setminus G) \cup \big(\bigcup_{\lambda\in G} \lambda G_\lambda\big) \in \mathcal{E}_H$. Since each $G_\lambda \in \mathcal{E}_H$, for each $\lambda \in G$, there exists $G'_\lambda \in \mathrm{FE}(\Lambda)$ with $G_\lambda = G'_\lambda \setminus G'_\lambda H$. Let $F' := (E' \setminus G) \cup \big(\bigcup_{\lambda\in G} \lambda G'_\lambda\big)$. We will show that $F = F' \setminus F'H$, and that $F' \in \mathrm{FE}(\Lambda)$; it follows from the definition of $\mathcal{E}_H$ that $F \in \mathcal{E}_H$, proving the result.

Lemma 4.2.9 applied to the collection $\mathcal{E} := \{E', G'_\lambda : \lambda \in G\} \subset \mathrm{FE}(\Lambda)$ shows that $F' \in \mathrm{FE}(\Lambda)$. Hence it remains only to show that $F = F' \setminus F'H$. But since H is hereditary, we have

$$\begin{aligned}F'H = \Big((E' \setminus G) \cup \big(\textstyle\bigcup_{\lambda\in G} \lambda G'_\lambda\big)\Big)H &= (E' \setminus G)H \cup \big(\textstyle\bigcup_{\lambda\in G} \lambda(G'_\lambda H)\big)\\ &= E'H \cup \big(\textstyle\bigcup_{\lambda\in G} \lambda G'_\lambda\big)H\end{aligned}$$

because $G \subset E \subset \Lambda \setminus \Lambda H$. Consequently

$$F' \setminus F'H = \big((E' \setminus G) \cup \big(\textstyle\bigcup_{\lambda\in G} \lambda G'_\lambda\big)\big) \setminus \big(E'H \cup \big(\textstyle\bigcup_{\lambda\in G} \lambda G'_\lambda H\big)\big) = F$$

as required. □

Lemma 5.2.6. *Let (Λ, d) be a finitely aligned k-graph, and let $H \subset \Lambda^0$ be saturated and hereditary. Let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz-Krieger Λ-family, and let I_H^t be the ideal in $C^*(\{t_\lambda : \lambda \in \Lambda\})$ generated by $\{t_v : v \in H\}$. Then $\{t_\lambda + I_H^t : \lambda \in \Lambda \setminus \Lambda H\}$ is a relative Cuntz-Krieger $(\Lambda \setminus \Lambda H; \mathcal{E}_H)$-family.*

Proof. Relations (TCK1) and (TCK2) hold for $\{t_\lambda + I_H^t : \lambda \in \Lambda \setminus \Lambda H\}$ because the quotient map is a homomorphism. For (TCK3), let $\lambda, \mu \in \Lambda \setminus \Lambda H$ and notice that since $\{t_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family, we have

$$(t_\lambda + I_H^t)^*(t_\mu + I_H^t) = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^* + I_H^t.$$

To show that this is equal to $\sum_{(\alpha,\beta)\in(\Lambda\setminus\Lambda H)^{\min}(\lambda,\mu)} t_\alpha t_\beta^* + I_H^t$, we need to show that

$$(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu) \setminus (\Lambda \setminus \Lambda H)^{\min}(\lambda, \mu) \text{ implies } t_\alpha t_\beta^* \in I_H^t.$$

So fix $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu) \setminus (\Lambda \setminus \Lambda H)^{\min}(\lambda, \mu)$. Then $s(\alpha) = s(\beta) \in H$, and it follows that $s_\alpha s_\beta^* = s_\alpha s_{s(\alpha)} s_\beta^*$ belongs to I_H^t because I_H^t is an ideal.

It remains to check (CK). Let $E \in \mathcal{E}_H$, say $F \in \mathrm{FE}(\Lambda)$ and $E = F \setminus FH$, and let $v := r(E)$. We must show that $\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*)$ belongs to I_H^t. We know that $\prod_{\lambda\in F}(t_v - t_\lambda t_\lambda^*) = 0$, and it follows that

$$\Big(\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*)\Big)\Big(\prod_{\mu\in FH}(t_v - t_\mu t_\mu^*)\Big) = 0. \tag{5.2.2}$$

Next notice that $\prod_{\mu\in\vee(FH)}(t_v - t_\mu t_\mu^*) \le \prod_{\mu\in FH}(t_v - t_\mu t_\mu^*)$ by definition of $\vee(FH)$. Combining this with (5.2.2) gives

$$\Big(\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*)\Big)\Big(\prod_{\mu\in\vee(FH)}(t_v - t_\mu t_\mu^*)\Big) = 0. \tag{5.2.3}$$

Furthermore, by Proposition 3.3.3, we have

$$t_v = \prod_{\mu\in\vee(FH)}(t_v - t_\mu t_\mu^*) + \sum_{\mu\in\vee(FH)} Q(t)_\mu^{\vee(FH)}.$$

Hence we can calculate

$$\begin{aligned}\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*) &= \Big(\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*)\Big)t_v \\ &= \Big(\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*)\Big)\Big(\prod_{\mu\in\vee(FH)}(t_v - t_\mu t_\mu^*) + \sum_{\mu\in\vee(FH)} Q(t)_\mu^{\vee(FH)}\Big).\end{aligned}$$

By (5.2.3), this collapses to

$$\begin{aligned}\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*) &= \Big(\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*)\Big)\Big(\sum_{\mu\in\vee(FH)} Q(t)_\mu^{\vee(FH)}\Big) \\ &= \sum_{\mu\in\vee(FH)}\Big(\Big(\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*)\Big)Q(t)_\mu^{\vee(FH)}\Big). \qquad (5.2.4)\end{aligned}$$

Since H is hereditary, we have $\vee(FH) \subset \Lambda H$, and hence for $\mu \in \vee(FH)$, we have $Q(t)_\mu^{\vee(FH)} = t_\mu t_{s(\mu)} t_\mu^* Q(t)_\mu^{\vee(FH)} \in I_H^t$ because $t_{s(\mu)} \in I_H^t$ by definition, and I_H^t is an ideal. It follows that each summand in (5.2.4) belongs to I_H^t, and hence $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) \in I_H^t$ as required. □

Proof of Theorem 5.2.4. The first claim of the theorem is an immediate consequence of Lemma 5.2.6. For the second statement, note that since $I_H \subset C^*(\Lambda)$ is fixed under the gauge action, γ descends to a strongly continuous action θ of $\mathbb{T}^k$ on $C^*(\Lambda)/I_H$ such that $\pi_{s+I_H}^{\mathcal{E}_H}$ is equivariant in γ and θ. Hence we need only show that $s_v + I_H$ and $\prod_{\lambda \in E}(s_{r(E)} - s_\lambda s_\lambda^*) + I_H$ are nonzero in $C^*(\Lambda)/I_H$ for each $v \in \Lambda^0 \setminus H$ and each $E \in \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \overline{\mathcal{E}}_H$. By Lemma 5.2.5, we have $\overline{\mathcal{E}}_H = \mathcal{E}_H$, so we must show that for all $v \in \Lambda^0 \setminus H$ and for all $E \in \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$, we have that $s_v \notin I_H$ and $\prod_{\lambda \in E}(s_{r(E)} - s_\lambda s_\lambda^*) \notin I_H$. To see this, fix $v \in \Lambda^0 \setminus \Lambda H$ and fix $E \in \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$.

Claim 1: For all $a \in \operatorname{span}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda H\}$, we have

(1) $\|s_v - a\| \geq 1$; and
(2) $\big\|\big(\prod_{\lambda \in E}(s_{r(E)} - s_\lambda s_\lambda^*)\big) - a\big\| \geq 1$.

Proof of Claim 1. Express $a = \sum_{\lambda \in F} a_{\lambda,\mu} s_\lambda s_\mu^*$ where F is a finite subset of ΛH, and $\{a_{\lambda,\mu} : \lambda, \mu \in F\} \subset \mathbb{C}$. Let π_S be the boundary-path representation of $C^*(\Lambda)$ and let $A := \pi_S(a) = \sum_{\lambda,\mu \in F} a_{\lambda,\mu} S_\lambda S_\mu^*$.

To check (1), note that since $v \notin H$ and since H is saturated, we have that $vF \cap \Lambda^0 = \emptyset$ and that $vF \notin \mathrm{FE}(\Lambda)$. Hence there exists $\tau \in v\Lambda$ such that $\Lambda^{\min}(\tau, \lambda) = \emptyset$ for all $\lambda \in F$. By Lemma 4.3.9, we can choose a boundary path x in $s(\tau)\partial\Lambda$. By choice of τ, we have that $\tau x \in v\partial\Lambda \setminus F\partial\Lambda$. But now

$$\|S_v - A\| \geq \|(S_v - A)e_{\tau x}\| = \|S_v e_{\tau x} - \sum_{\lambda,\mu \in F}(a_{\lambda,\mu} S_\lambda S_\mu^* e_{\tau x})\|. \tag{5.2.5}$$

Since $S_\mu^* e_{\tau x} = \delta_{\mu,(\tau x)(0,d(\mu)} e_{(\tau x)|_{d(\mu)}^{d(\tau x)}}$, and since $\tau x \notin F\partial\Lambda$ by choice, we have $S_\mu^* e_{\tau x} = 0$ for all $\mu \in F$, and hence (5.2.5) gives $\|\pi_S(s_v - A)\| = \|S_v - A\| \geq \|S_v e_{\tau x}\| = \|e_{\tau x}\| = 1$. Since π_S is a C^*-homomorphism, and hence norm-decreasing, this establishes (1).

For (2), note that $E \notin \mathcal{E}_H$, and $F \subset \Lambda H$ is finite, so we know that $E \cup F \notin \mathrm{FE}(\Lambda)$. Hence there exists $\tau \in \Lambda$ such that $\Lambda^{\min}(\sigma, \tau) = \emptyset$ for all $\sigma \in E \cup F$. By Lemma 4.3.9, there exists $x \in \partial\Lambda$ such that $r(x) = s(\tau)$. Set $y := \tau x \in \partial\Lambda$. By choice of τ, we have that $y(0, d(\sigma)) \neq \sigma$ for all $\sigma \in E \cup F$. Hence $S_\sigma^* e_y = 0$ for all

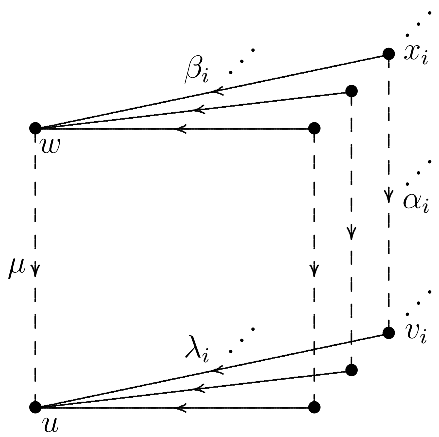
$\sigma \in E \cup G$. In particular, $\sigma \in F$ implies $S_\sigma^* e_y = 0$, so $A e_y = 0$, and $\lambda \in E$ implies $S_\lambda^* e_y = 0$. It follows that $\big(\prod_{\lambda\in E}(S_{r(E)} - S_\lambda S_\lambda^*)\big)e_y = S_{r(E)} e_y = e_y$. Hence

$$\big\|\big(\textstyle\prod_{\lambda\in E}(S_{r(E)} - S_\lambda S_\lambda^*) - A\big)\big\| \geq \big\|\big(\textstyle\prod_{\lambda\in E}(S_{r(E)} - S_\lambda S_\lambda^*) - A\big)e_y\big\| = \|e_y\| = 1.$$

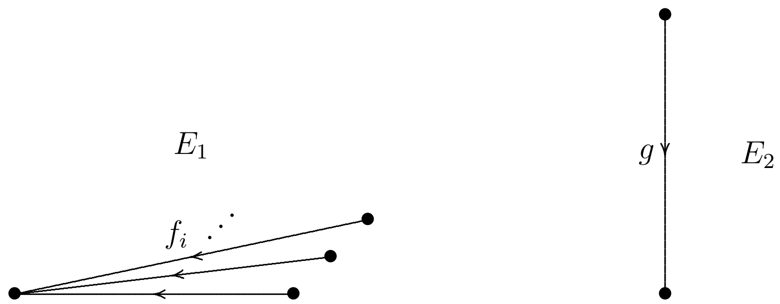
It follows that $\big\|\prod_{\lambda\in E}(S_{r(E)} - S_\lambda S_\lambda^*) - A\big\| \geq 1$. Again since π_S is automatically norm-decreasing, this establishes (2). □ Claim 1

Since $\operatorname{span}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda H\}$ is a dense subset of I_H by Lemma 5.1.4, Claim 1 shows that neither s_v nor $\prod_{\lambda\in E}(s_{r(E)} - s_\lambda s_\lambda^*)$ belongs to I_H. Since $v \in \Lambda^0 \setminus H$ and $E \in \operatorname{FE}(\Lambda \setminus \Lambda H) \setminus \overline{\mathcal{E}}$ were arbitrary, this completes the proof of Theorem 5.2.4. □

EXAMPLE 5.2.7. Let (Λ, d) be the unique 2-graph with 1-skeleton



Note that Λ is isomorphic to the cartesian product graph $E_1 \times E_2$ where E_1 and E_2 are the graphs drawn below:



Hence [**30**, Corollary 4.4] shows that $C^*(\Lambda) \cong C^*(E_1) \otimes C^*(E_2)$. Calculating by hand, one can see that $C^*(E_2) = M_2(\mathbb{C})$, while $C^*(E_1)$ is equal to $(D \otimes M_2(\mathbb{C})) + \mathbb{C}P \subset \mathcal{B}(\ell^2(\mathbb{N})) \otimes M_2(\mathbb{C})$ where D is the diagonal subalgebra $\overline{\operatorname{span}}\{e_i \otimes \overline{e}_i : i \in \mathbb{N}\} \subset \mathcal{K}(\ell^2(\mathbb{N}))$, and P is the projection

$$\operatorname{id}_{\ell^2(\mathbb{N})} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

onto the first coordinate in each copy of $\mathbb{C}^2$ in $\ell^2(\mathbb{N}) \otimes \mathbb{C}^2$. For the remainder of this example, we write M_n for $M_n(\mathbb{C})$, and $\mathcal{K}$ for $\mathcal{K}(\ell^2(\mathbb{N}))$. By the above, we have

$$C^*(\Lambda) \cong \big((D \otimes M_2) + \mathbb{C}P\big) \otimes M_2.$$

The Cuntz-Krieger Λ-family which implements this isomorphism is:

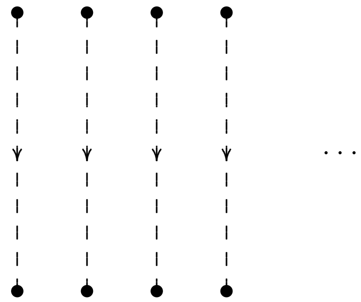
$$\begin{aligned}
t_{\lambda_i} &:= \left((e_i \otimes \overline{e}_i) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0P\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
t_{\beta_i} &:= \left((e_i \otimes \overline{e}_i) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0P\right) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
t_{\mu} &:= \left(0_{\mathcal{K}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1P\right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
t_{\alpha_i} &:= \left((e_i \otimes \overline{e}_i) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 0P\right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

We have not listed the vertex projections explicitly here because they can be recovered from the partial isometries listed above using relations (TCK3) and (CK). Specifically (TCK3) gives $t_w = t_\mu^* t_\mu$, $t_{v_i} = t_{\lambda_i}^* t_{\lambda_i}$ for all $i \in \mathbb{N}$, and $t_{x_i} = t_{\beta_i}^* t_{\beta_i}$ for all $i \in \mathbb{N}$, while (CK) applied to the set $\{\mu\} \in u\,\mathrm{FE}(\Lambda)$ gives $t_u = t_\mu t_\mu^*$.

To make sense of what each of the component parts in this Cuntz-Krieger family is for, regard them as follows:

- The factor of D keeps track of the infinitely many λ_i–β_i pairs.
- The first copy of M_2 corresponds to the first coordinate in $\mathbb{N}^2$, and hence to horizontal position in the 1-skeleton as drawn above.
- The second copy of M_2 corresponds to the second coordinate in $\mathbb{N}^2$, and hence to vertical position in the 1-skeleton as drawn above.
- The projection P models t_u and t_w which contain infinitely many mutually orthogonal range projections.

Let α be the action of $\mathbb{Z}$ on M_2 such that $\theta_z(e_1 \otimes \overline{e}_2) = ze_1 \otimes \overline{e}_2$. Then setting $\theta_{(z_1,z_2)} := (\mathrm{id}_{\mathcal{B}(\ell^2(\mathbb{N}))} \otimes \alpha_{z_1}) \otimes \alpha_{z_2}$ gives a strongly continuous action of $\mathbb{T}^2$ which implements the gauge action. Since all the vertex projections are nonzero, it follows from Theorem 4.6.2 that $\pi_t : s_\lambda \mapsto t_\lambda$ is a faithful representation of $C^*(\Lambda)$. Now let $H := \{v_i, x_i : i \geq 2\} \subset \Lambda^0$. It is easy to check that H is saturated and hereditary. Lemma 5.1.5 shows that I is Morita equivalent to the C^*-algebra of the 2-graph



which is just $D \otimes M_2$. Indeed, using the fact that I_H is spanned by the partial isometries $s_\lambda s_\mu^*$ with $\lambda, \mu \in \Lambda H$, one can see that $I_H \cong (D \otimes M_2) \otimes M_2$. To see how I_H fits inside $C^*(\Lambda)$, notice that it does not contain t_{λ_1}, t_{α_1}, t_{β_1} or t_μ, and that these elements generate the subalgebra

$$\big(((e_1 \otimes \overline{e}_1) \otimes M_2) + \mathbb{C}P\big) \otimes M_2$$

of $\pi_t(C^*(\Lambda))$. Writing $(1 - P_{e_1})$ for the orthogonal projection $\big(\mathrm{id}_{\ell^2(\mathbb{N})} - (e_1 \otimes \overline{e}_1)\big)$ onto $\overline{\operatorname{span}}\{e_i : i \geq 2\}$ in $\ell^2(\mathbb{N})$, we have that $(1 - P_{e_1})$ belongs to the commutant of the diagonal algebra $D \subset \mathcal{K}$. Hence the image of I_H in the representation of $C^*(\Lambda)$ is

$$\pi_t(I_H) = \big(((1 - P_{e_1}) \otimes \mathrm{id}_{M_2}) \otimes \mathrm{id}_{M_2}\big)\pi_t(C^*(\Lambda)).$$

Theorem 5.2.4 implies that $C^*(\Lambda)/I_H$ is canonically isomorphic to $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H)$, where $\mathcal{E}_H = \big\{\{\mu\}, \{\alpha_1\}\big\}$, and $\Lambda \setminus \Lambda H$ is the 2-graph whose 1-skeleton consists of the vertices $\{u, v_1, w, x_1\} \subset \Lambda^0$ and the edges $\{\lambda_1, \mu, \alpha_1, \beta_1\} \subset \Lambda$.

To finish off the example, we describe the quotient algebra directly. Using the above analysis, it is easy to see that $C^*(\Lambda)/I_H$ is canonically isomorphic to $((e_1 \otimes \overline{e}_1) \otimes M_2 + \mathbb{C}P) \otimes M_2 \cong M_4 \oplus M_2$.

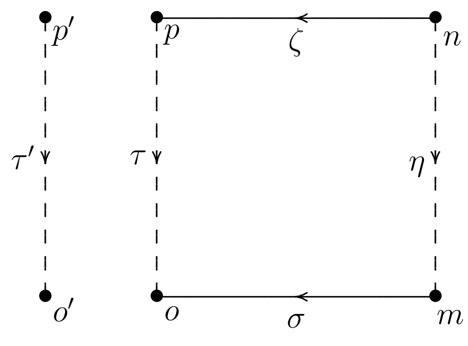
REMARK 5.2.8. Let E be a directed graph. Then the relative graph algebras $C^*(E, V)$ associated to E are determined by subsets V of E^0 rather than collections $\mathcal{E}$ of finite exhaustive sets [**23**]. The idea is that the relative graph algebra $C^*(E, V)$ is the universal algebra generated by a Toeplitz-Cuntz-Krieger E-family $\{s_e, p_v : e \in E^1, v \in E^0\}$ in which $p_v = \sum_{e \in r^{-1}(v)} s_e s_e^*$ whenever $v \in V$ and $r^{-1}(v)$ is finite [**23**]. In this thesis we have specified relative Cuntz-Krieger algebras in terms of subsets $\mathcal{E}$ of $\mathrm{FE}(\Lambda)$, but we have not yet demonstrated that this extra complexity is necessary. That is, we have not yet showed that not every relative Cuntz-Krieger algebra is of the form $C^*(\Lambda; V\,\mathrm{FE}(\Lambda))$ for some $V \subset \Lambda^0$.

For the 2-graph Λ and the saturated hereditary set $H = \{v_i, x_i : i \geq 2\} \subset \Lambda^0$ of Example 5.2.7, we have that $\mu \in u\mathcal{E}_H$, but $\{\lambda_1\} \in \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$. Hence

$\mathcal{E}_H \neq V\,\mathrm{FE}(\Lambda \setminus \Lambda H)$ for any subset V of $(\Lambda \setminus \Lambda H)^0$. Hence relative Cuntz-Krieger algebras of the form $C^*(\Lambda, V) := C^*(\Lambda; V\,\mathrm{FE}(\Lambda))$ where V is a subset of Λ^0 are not general enough even to capture all the quotients of $C^*(\Lambda)$ by its gauge-invariant ideals as relative algebras associated to subgraphs. In particular, relative Cuntz-Krieger algebras of this form do not embody all relative Cuntz-Krieger algebras. In other words, we really do need to associate a relative Cuntz-Krieger algebra to each subset $\mathcal{E}$ of $\mathrm{FE}(\Lambda)$, and not just to each subset V of Λ^0.

It is important to notice, however, that if Λ is a 1-graph, then every relative Cuntz-Krieger algebra *is* of the form $C^*(\Lambda, V)$ for an appropriate choice of $V \subset \Lambda^0$. Specifically, if $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ is satiated, we define $V_{\mathcal{E}} := \{v \in \Lambda^0 : v = r(E) \text{ for some } E \in \mathcal{E}\}$. It is straightforward to check that for this choice of $V_{\mathcal{E}} \subset \Lambda^0$, the relative Cuntz-Krieger algebra $C^*(\Lambda; \mathcal{E})$ defined in this thesis is canonically isomorphic to the relative graph algebra $C^*(\Lambda, V_{\mathcal{E}})$ defined in [**23**].

Remark 5.2.9. It is worth noticing that for the 2-graph Λ and the saturated hereditary set I_H of Example 5.2.7, $C^*(\Lambda)/I_H$ and hence the relative Cuntz-Krieger algebra $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H)$ can be viewed as the Cuntz-Krieger algebra of a modified graph reminiscent of the quotient graph in [**3**] and the adjusted graph used to study relative graph algebras in [**23**]. Let Γ be the unique 2-graph with 1-skeleton



Then the formulas

$$\begin{array}{lll} s_o & \mapsto & s_{\mathcal{E}}(\lambda_1)s_{\mathcal{E}}(\lambda_1)^* \\ s_{o'} & \mapsto & s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\lambda_1)s_{\mathcal{E}}(\lambda_1)^* \\ s_p & \mapsto & s_{\mathcal{E}}(\mu)^* s_{\mathcal{E}}(\lambda_1)s_{\mathcal{E}}(\lambda_1)^* s_{\mathcal{E}}(\mu) \\ s_{p'} & \mapsto & s_{\mathcal{E}}(\mu)^*\big(s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\lambda_1)s_{\mathcal{E}}(\lambda_1)^*\big)s_{\mathcal{E}}(\mu) \\ s_\tau & \mapsto & s_{\mathcal{E}}(\lambda_1)s_{\mathcal{E}}(\lambda_1)^* s_{\mathcal{E}}(\mu) \\ s_{\tau'} & \mapsto & \big(s_{\mathcal{E}}(u) - s_{\mathcal{E}}(\lambda_1)s_{\mathcal{E}}(\lambda_1)^*\big)s_{\mathcal{E}}(\mu) \end{array} \qquad \begin{array}{lll} s_m & \mapsto & s_{\mathcal{E}}(v_1) \\ s_n & \mapsto & s_{\mathcal{E}}(x_1) \\ s_\zeta & \mapsto & s_{\mathcal{E}}(\beta_1) \\ s_\eta & \mapsto & s_{\mathcal{E}}(\alpha_1) \\ s_\sigma & \mapsto & s_{\mathcal{E}}(\lambda_1) \end{array}$$

determine a canonical isomorphism of $C^*(\Gamma)$ onto $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H) \cong C^*(\Lambda)/I_H$.

5.3. The gauge-invariant ideal structure of the Cuntz-Krieger algebra

Theorem 4.3.12 shows that every nontrivial gauge-invariant ideal in $C^*(\Lambda;\mathcal{E}_H)$ which contains no vertex projection $s_{\mathcal{E}_H}(v)$ must contain some collection of projections

$$\Big\{\prod_{\lambda\in E}\big(s_{\mathcal{E}_H}(r(E)) - s_{\mathcal{E}_H}(\lambda)s_{\mathcal{E}_H}(\lambda)^*\big) : E \in B\Big\}$$

where B is a subset of $\mathrm{FE}(\Lambda\setminus\Lambda H)\setminus\mathcal{E}_H$. In this section, we show that the gauge-invariant ideals of $C^*(\Lambda)$ are indexed by pairs (H, B) where H is a saturated hereditary subset of Λ^0 and $B \subset \mathrm{FE}(\Lambda\setminus\Lambda H)\setminus\mathcal{E}_H$ is such that $B\cup\mathcal{E}_H$ is satiated.

Definition 5.3.1. Let (Λ, d) be a finitely aligned k-graph and let $H \subset \Lambda^0$ be saturated and hereditary. Let B be a subset of $\mathrm{FE}(\Lambda\setminus\Lambda H)\setminus\mathcal{E}_H$. We define $J_{H,B}$ to be the ideal of $C^*(\Lambda)$ generated by the projections

$$\big\{s_v : v\in H\big\}\cup\Big\{\prod_{\lambda\in E}\big(s_{r(E)} - s_\lambda s_\lambda^*\big) : E\in B\Big\}.$$

We denote the ideal of $C^*(\Lambda\setminus\Lambda H;\mathcal{E}_H)$ generated by the projections

$$\Big\{\prod_{\lambda\in E}\big(s_{\mathcal{E}_H}(r(E)) - s_{\mathcal{E}_H}(\lambda)s_{\mathcal{E}_H}(\lambda)^*\big) : E \in B\Big\}.$$

by I_B.

Suppose that $H \subset \Lambda^0$ is saturated and hereditary and that $B\subset \mathrm{FE}(\Lambda\setminus\Lambda H)\setminus\mathcal{E}_H$ is such that $\mathcal{E}_H\cup B$ is satiated. Then $I_B = q(J_{H,B})$ where q is the quotient map from $C^*(\Lambda)$ to $C^*(\Lambda)/I_H \cong C^*(\Lambda\setminus\Lambda H;\mathcal{E}_H)$.

Lemma 5.3.2. *Let (Λ, d) be a finitely aligned k-graph and let $H \subset \Lambda^0$ be saturated and hereditary. Let B be a subset of* $\mathrm{FE}(\Lambda\setminus\Lambda H)\setminus\mathcal{E}_H$, *and suppose that $\mathcal{E}_H\cup B$ is satiated in $\Lambda\setminus\Lambda H$. Then $J_{H,B}$ is equal to*

$$\overline{\mathrm{span}}\Big\{s_\lambda s_\mu^*, s_\sigma\Big(\prod_{\nu\in E}\big(s_{r(E)} - s_\nu s_\nu^*\big)\Big)s_\tau^* : \lambda,\mu\in\Lambda H, E\in\mathcal{E}_H\cup B, \sigma,\tau\in\Lambda r(E)\Big\}.$$

Proof. For convenience, we will denote

$$\overline{\mathrm{span}}\Big\{s_\lambda s_\mu^*, s_\sigma\Big(\prod_{\nu\in E}\big(s_{r(E)} - s_\nu s_\nu^*\big)\Big)s_\tau^* : \lambda,\mu\in\Lambda H, E\in\mathcal{E}_H\cup B, \sigma,\tau\in\Lambda r(E)\Big\}$$

by $X_{H,B}$ for the duration of this proof.

Since $\{s_v : v\in H\}$ and $\{\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*) : E\in\mathcal{E}_H\cup B\}$ are subsets of $J_{H,B}$ by definition, and since $J_{H,B}$ is an ideal, it is clear that each $s_\lambda s_\mu^*$ where $\lambda,\mu\in\Lambda H$, and each $s_\sigma\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\tau^*$ where $E\in\mathcal{E}_H\cup B$ and $\sigma,\tau\in\Lambda r(E)$, belongs to $J_{H,B}$. It follows immediately that $X_{H,B}\subset J_{H,B}$.

On the other hand, since $C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$, we have that $J_{H,B}$ is generated as a C^*-algebra by

$$\big\{s_\lambda s_\mu^* s_v s_\sigma s_\tau^* : \lambda, \mu, \sigma, \tau \in \Lambda, v \in H\big\} \cup \Big\{s_\lambda s_\mu^*\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\sigma s_\tau^* : \lambda, \mu, \sigma, \tau \in \Lambda, E \in \mathcal{E}_H \cup B\Big\}.$$

Since $X_{H,B}$ is manifestly closed under involution, we can establish that $J_{H,B} \subset X_{H,B}$ and thereby complete the proof of the lemma by showing that:

(1) $X_{H,B}$ is closed under multiplication; and
(2) each $s_\lambda s_\mu^* s_v s_\sigma s_\tau^*$ where $v \in H$ and each $s_\lambda s_\mu^*\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\sigma s_\tau^*$ where $E \in \mathcal{E}_H \cup B$ belongs to $X_{H,B}$.

For (1) we note that if $\lambda, \mu, \sigma, \tau \in \Lambda H$ then $s_\lambda s_\mu^* s_\sigma s_\tau^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\sigma)} s_{\lambda\alpha} s_{\tau\sigma}^*$. Since H is hereditary, each α and β belong to ΛH, and it follows that $s_\lambda s_\mu^* s_\sigma s_\tau^* \in X_{H,B}$. If $\lambda, \mu \in \Lambda H$, $E \in \mathcal{E}_H \cup B$, and $\sigma, \tau \in r(E)\Lambda$, then

$$s_\lambda s_\mu^* s_\sigma\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\tau^* = \sum_{\alpha,\beta\in\Lambda^{\min}(\mu,\sigma)} s_{\lambda\alpha} s_\beta^*\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\tau^*. \tag{5.3.1}$$

Let β be any path in Λ, and let $E \subset r(\beta)\Lambda$ be finite. Since Lemma 3.1.2(2) ensures that $s_\beta s_\beta^*$ commutes with $s_{r(E)} - s_\nu s_\nu^*$ for each $\nu \in E$, we have

$$s_\beta^* \prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*) = s_\beta^* \prod_{\nu\in E}(s_\beta s_\beta^*(s_{r(E)} - s_\nu s_\nu^*)). \tag{5.3.2}$$

If $\nu \in E$ and $(\eta_1, \zeta_1), (\eta_2, \zeta_2) \in \Lambda^{\min}(\beta, \nu)$, then $d(\eta_1) = d(\eta_2)$ by definition, and it follows from Lemma 3.1.2(3) that $t_{\beta\eta_1} t_{\beta\eta_1}^*$ and $t_{\beta\eta_2} t_{\beta\eta_2}^*$ are orthogonal so that

$$(t_{r(\beta)} - t_{\beta\eta_1} t_{\beta\eta_1}^*)(t_{r(\beta)} - t_{\beta\eta_2} t_{\beta\eta_2}^*) = t_{r(\beta)} - (t_{\beta\eta_1} t_{\beta\eta_1}^* + t_{\beta\eta_2} t_{\beta\eta_2}^*). \tag{5.3.3}$$

Applying (TCK3) to (5.3.2), and then using (5.3.3), we can therefore calculate

$$\begin{aligned} s_\beta^* \prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*) &= s_\beta^* \prod_{\nu\in E}(s_\beta s_\beta^* - \sum_{(\eta,\zeta)\in\Lambda^{\min}(\beta,\nu)} s_{\beta\eta} s_{\beta\eta}^*) \\ &= s_\beta^* \prod_{\eta\in\operatorname{Ext}(\beta;E)} (s_\beta s_\beta^* - s_{\beta\eta} s_{\beta\eta}^*) \\ &= s_\beta^* s_\beta \prod_{\eta\in\operatorname{Ext}(\beta;E)} (s_{s(\beta)} - s_\eta s_\eta^*) s_\beta^* \\ &= \prod_{\eta\in\operatorname{Ext}(\beta;E)} (s_{s(\beta)} - s_\eta s_\eta^*) s_\beta^*. \end{aligned} \tag{5.3.4}$$

Combining (5.3.4) with (5.3.1) gives

$$s_\lambda s_\mu^* s_\sigma\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\tau^* = \sum_{\alpha,\beta\in\Lambda^{\min}(\mu,\sigma)} s_{\lambda\alpha} \prod_{\eta\in\operatorname{Ext}(\beta;E)} (s_{s(\beta)} - s_\eta s_\eta^*) s_{\tau\beta}^*. \tag{5.3.5}$$

The terms in the right-hand side of (5.3.5) for which β belongs to $E\Lambda$ are equal to zero because for these terms, $s(\beta) \in \operatorname{Ext}(\beta; E)$. For all the other terms in the right-hand side of (5.3.5) $\operatorname{Ext}(\beta; E) \in \mathcal{E}_H \cup B$ by (S2), so $s_\lambda s_\mu^* s_\sigma\Big(\prod_{\nu \in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\tau^* \in X_{H,B}$ as required. Since $X_{H,B}$ is closed under involution, it follows that $s_\sigma\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\tau^* s_\lambda s_\mu^*$ also belongs to $X_{H,B}$. Finally, suppose that $E, E' \in \mathcal{E}_H \cup B$ and that $\lambda, \mu \in r(E)\Lambda$ and $\sigma, \tau \in r(E')\Lambda$. Then

$$\begin{aligned}
s_\lambda&\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\mu^* s_\sigma\Big(\prod_{\nu'\in E'}(s_{r(E')} - s_{\nu'} s_{\nu'}^*)\Big)s_\tau^* \\
&= \sum_{\alpha,\beta\in\Lambda^{\min}(\mu,\sigma)} \Big(s_\lambda\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\alpha s_\beta^*\Big(\prod_{\nu'\in E'}(s_{r(E')} - s_{\nu'} s_{\nu'}^*)\Big)s_\tau^*\Big) \\
&= \sum_{\alpha,\beta\in\Lambda^{\min}(\mu,\sigma)} \Big(s_{\lambda\alpha}\Big(\prod_{\xi\in\operatorname{Ext}(\alpha;E)\cup\operatorname{Ext}(\beta;E')}(s_{r(E)} - s_\xi s_\xi^*)\Big)s_{\tau\beta}^*\Big)
\end{aligned}$$

by two applications of (5.3.4). The terms in the sum on the previous line for which α belongs to $E\Lambda$ or β belongs to $E'\Lambda$ are equal to zero because for these terms, $s(\alpha) = s(\beta)$ belongs to either $\operatorname{Ext}(\alpha; E)$ or $\operatorname{Ext}(\beta; E')$. For the remaining terms, we have $\operatorname{Ext}(\alpha; E)$ and $\operatorname{Ext}(\beta; E')$ in $\mathcal{E}_H \cup B$ by (S2). Hence $\operatorname{Ext}(\alpha; E) \cup \operatorname{Ext}(\beta; E') \in \mathcal{E}_H \cup B$ by (S1), so the product belongs to $X_{H,B}$. We have now established (1).

To establish (2), note that if $v \in H$ and $s_\lambda s_\mu^* s_v s_\sigma s_\tau^* \neq 0$, then $r(\mu) = r(\sigma) = v$, and since H is hereditary, it follows that for $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$, we have $s(\alpha) = s(\beta) \in H$. Consequently, $s_\lambda s_\mu^* s_v s_\sigma s_\tau^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\sigma)} s_{\lambda\alpha} s_{\tau\beta}^*$ belongs to $X_{H,B}$ as required. Finally, if $E \in \mathcal{E}_H \cup B$, and $s_\lambda s_\mu^*\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\sigma s_\tau^* \neq 0$, then $\mu, \sigma \in r(E)\Lambda \setminus E\Lambda$, so two applications of (5.3.4), separated by an application of (TCK3) give

$$\begin{aligned}
s_\lambda s_\mu^*&\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\sigma s_\tau^* \\
&= s_\lambda\Big(\prod_{\nu'\in\operatorname{Ext}(\mu;E)}(s_{s(\mu)} - s_{\nu'} s_{\nu'}^*)\Big)s_\mu^* s_\sigma s_\tau^* \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\sigma)} \Big(s_\lambda\Big(\prod_{\nu'\in\operatorname{Ext}(\mu;E)}(s_{s(\mu)} - s_{\nu'} s_{\nu'}^*)\Big)s_\alpha s_\beta^* s_\tau^*\Big) \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\sigma)} \Big(s_{\lambda\alpha}\Big(\prod_{\xi\in\operatorname{Ext}(\alpha;\operatorname{Ext}(\mu;E))}(s_{s(\alpha)} - s_\xi s_\xi^*)\Big)s_{\tau\beta}^*\Big).
\end{aligned}$$

Once again, for the nonzero terms in this sum we have $\alpha \notin \operatorname{Ext}(\mu; E)\Lambda$, so two applications of (S2) give $\operatorname{Ext}(\alpha; \operatorname{Ext}(\mu; E)) \in \mathcal{E}_H \cup B$. Hence $s_\lambda s_\mu^*\Big(\prod_{\nu\in E}(s_{r(E)} - s_\nu s_\nu^*)\Big)s_\sigma s_\tau^*$ belongs to $X_{H,B}$, establishing (2). □

We now investigate the structure of $C^*(\Lambda)/J_{H,B}$.

LEMMA 5.3.3. *Let (Λ, d) be a finitely aligned k-graph and let $H \subset \Lambda^0$ be saturated and hereditary. Let B be a subset of* $\mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$*. Then*

$$C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H)/I_B = C^*(\Lambda \setminus \Lambda H; (\mathcal{E}_H \cup B)).$$

PROOF. Let $q_B : C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H) \to C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H)/I_B$ be the quotient map. Since every relative $(\Lambda \setminus \Lambda H; \mathcal{E}_H \cup B)$-family is in particular a relative Cuntz-Krieger $(\Lambda \setminus \Lambda H; \mathcal{E}_H)$-family, Theorem 4.1.4 implies that there exists a homomorphism $\pi^{\mathcal{E}_H}_{S_{\mathcal{E}_H \cup B}} : C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H) \to C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H \cup B)$ determined by $\pi^{\mathcal{E}_H}_{S_{\mathcal{E}_H \cup B}}(s_{\mathcal{E}_H}(\lambda)) = s_{\mathcal{E}_H \cup B}(\lambda)$ for all $\lambda \in \Lambda \setminus \Lambda H$. Since (CK) for $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H \cup B)$ ensures that I_B belongs to the kernel of $\pi^{\mathcal{E}_H}_{S_{\mathcal{E}_H \cup B}}$, it follows that $\pi^{\mathcal{E}_H}_{S_{\mathcal{E}_H \cup B}}$ descends to a homomorphism $\widetilde{\pi}^{\mathcal{E}_H}_{S_{\mathcal{E}_H \cup B}}$ from $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H)/I_B$ to $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H \cup B)$ such that

$$\widetilde{\pi}^{\mathcal{E}_H}_{S_{\mathcal{E}_H \cup B}}(q_B(s_{\mathcal{E}_H}(\lambda))) = s_{\mathcal{E}_H \cup B}(\lambda) \text{ for all } \lambda \in \Lambda \setminus \Lambda H. \tag{5.3.6}$$

We claim that $\{q_B(s_{\mathcal{E}_H}(\lambda)) : \lambda \in \Lambda \setminus \Lambda H\}$ is a relative Cuntz-Krieger $(\Lambda \setminus \Lambda H; \mathcal{E}_H \cup B)$-family. To see this, observe first that (TCK1)–(TCK3) hold because q_B is a homomorphism. For (CK) note that if $E \in \mathcal{E}_H \cup B$, then either $E \in \mathcal{E}_H$ or $E \in B$. If $E \in \mathcal{E}_H$, then $q_B\Big(\prod_{\lambda \in E}\big(s_{\mathcal{E}_H}(r(E)) - s_{\mathcal{E}_H}(\lambda)s_{\mathcal{E}_H}(\lambda)^*\big)\Big) = q_B(0) = 0$ by (CK) for $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H)$. On the other hand, if $E \in B$ then $\prod_{\lambda \in E}\big(s_{\mathcal{E}_H}(r(E)) - s_{\mathcal{E}_H}(\lambda)s_{\mathcal{E}_H}(\lambda)^*\big)$ belongs to I_B by definition, and since $I_B = \ker q_B$, it follows that $q_B\big(\prod_{\lambda \in E}\big(s_{\mathcal{E}_H}(r(E)) - s_{\mathcal{E}_H}(\lambda)s_{\mathcal{E}_H}(\lambda)^*\big)\big) = 0$. This establishes (CK), so $\{q_B(s_{\mathcal{E}_H}(\lambda)) : \lambda \in \Lambda \setminus \Lambda H\}$ is a relative Cuntz-Krieger $(\Lambda \setminus \Lambda H; \mathcal{E}_H \cup B)$-family as claimed.

It follows from the previous paragraph and Theorem 4.1.4 that there exists a homomorphism $\pi^{\mathcal{E}_H \cup B}_{q_B(s_{\mathcal{E}_H})} : C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H \cup B) \to C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H)/I_B$ such that

$$\pi^{\mathcal{E}_H \cup B}_{q_B(S_{\mathcal{E}_H})}(s_{\mathcal{E}_H \cup B}(\lambda)) = q_B(s_{\mathcal{E}_H}(\lambda)) \text{ for all } \lambda \in \Lambda \setminus \Lambda H. \tag{5.3.7}$$

Equations (5.3.6) and (5.3.7) show that $\widetilde{\pi}^{\mathcal{E}_H}_{S_{\mathcal{E}_H \cup B}}$ and $\pi^{\mathcal{E}_H \cup B}_{q_B(S_{\mathcal{E}_H})}$ are mutually inverse, completing the proof. □

COROLLARY 5.3.4. *Let (Λ, d) be a finitely aligned k-graph, let $H \subset \Lambda^0$ be saturated and hereditary, and let B be a subset of* $\mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$*. Then there is an isomorphism $\phi : C^*(\Lambda)/J_{H,B} \to C^*(\Lambda \setminus \Lambda H; (\mathcal{E}_H \cup B))$ determined by $\phi(s_\lambda + J_{H,B}) := s_{\mathcal{E}_H \cup B}(\lambda)$ for all $\lambda \in \Lambda \setminus \Lambda H$.*

PROOF. The result will follow from Lemma 5.3.3 if we can demonstrate that $C^*(\Lambda)/J_{H,B}$ is canonically isomorphic to $(C^*(\Lambda)/I_H)/I_B$. Let

$$\begin{aligned} q_{H,B} &: C^*(\Lambda) \to C^*(\Lambda)/J_{H,B}, \\ q_H &: C^*(\Lambda) \to C^*(\Lambda)/I_H, \\ q_B &: C^*(\Lambda)/I_H \to (C^*(\Lambda)/I_H)/I_B \end{aligned}$$

be the quotient maps. Lemma 5.3.2 and the definitions of I_H and I_B show that $J_{H,B}$ is contained in the kernel of $q_B \circ q_H$, giving a canonical homomorphism π_1 of $C^*(\Lambda)/J_{H,B}$ onto $(C^*(\Lambda)/I_H)/I_B$. On the other hand, since $I_H \subset J_{H,B}$, there is a canonical homomorphism π_2 of $C^*(\Lambda)/I_H$ onto $C^*(\Lambda)/J_{H,B}$ whose kernel contains I_B by definition. It follows that π_2 descends to a canonical homomorphism $\tilde{\pi}_2$ of $(C^*(\Lambda)/I_H)/I_B$ onto $C^*(\Lambda)/J_{H,B}$ which is inverse to π_1. $\square$

DEFINITION 5.3.5. Let (Λ, d) be a finitely aligned k-graph. For each gauge-invariant ideal I in $C^*(\Lambda)$, let

$$H_I := \{v \in \Lambda^0 : s_v \in I\} \quad \text{and}$$
$$B_I := \Big\{E \in \mathrm{FE}(\Lambda \setminus \Lambda H_I) \setminus \mathcal{E}_{H_I} : \prod_{\lambda \in E} \big(s_{\mathcal{E}_{H_I}}(r(E)) - s_{\mathcal{E}_{H_I}}(\lambda) s_{\mathcal{E}_{H_I}}(\lambda)^*\big) \in q_{H_I}(I)\Big\},$$

where q_{H_I} is the quotient map from $C^*(\Lambda)$ to $C^*(\Lambda)/I_{H_I}$.

THEOREM 5.3.6. *Let (Λ, d) be a finitely aligned k-graph.*

(1) *Let I be a gauge-invariant ideal of $C^*(\Lambda)$. Then H_I is nonempty, saturated and hereditary, $\mathcal{E}_{H_I} \cup B_I$ is satiated in $\Lambda \setminus \Lambda H_I$, and $I = J_{H_I, B_I}$.*
(2) *Let $H \subset \Lambda^0$ be nonempty, saturated and hereditary, and let $B \subset \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$ be such that $\mathcal{E}_H \cup B$ is satiated in $\Lambda \setminus \Lambda H$. Then $H_{J_{H,B}} = H$ and $B_{J_{H,B}} = B$.*

PROOF OF PART (1) OF THEOREM 5.3.6. Theorem 4.6.2 shows that H_I is nonempty, and Lemma 5.1.3 shows that H_I is saturated and hereditary. We have that $\mathcal{E}_{H_I} \cup B_I$ is satiated by Lemma 4.2.2.

We have $J_{H_I, B_I} \subset I$ by definition, so there is a canonical homomorphism π of $C^*(\Lambda)/J_{H_I, B_I}$ onto $C^*(\Lambda)/I$. By Corollary 5.3.4, this gives us a homomorphism, also denoted π, of $C^*(\Lambda \setminus \Lambda H_I; \mathcal{E}_{H_I} \cup B_I)$ onto $C^*(\Lambda)/I$. Since I is gauge-invariant, the gauge action on $C^*(\Lambda)$ descends to a strongly continuous action θ of $\mathbb{T}^k$ on $C^*(\Lambda)/I$ such that $\theta_z \circ \pi = \pi \circ \gamma_z$ where γ is the gauge action on $C^*(\Lambda \setminus \Lambda H_I; \mathcal{E}_{H_I} \cup B_I)$.

If $\pi(s_{\mathcal{E}_{H_I}\cup B_I}(v))$ is equal to 0 in $C^*(\Lambda)/I$, then $s_v \in I$ by definition, so $v \in H_I$. Hence $\pi(s_{\mathcal{E}_{H_I}\cup B_I}(v)) \neq 0_{C^*(\Lambda)/I}$ for all $v \in \Lambda^0 \setminus H_I$.

Now suppose that $E \in \mathrm{FE}(\Lambda \setminus \Lambda H_I)$ satisfies

$$\pi\Big(\prod_{\lambda\in E}\big(s_{\mathcal{E}_{H_I}\cup B_I}(r(E)) - s_{\mathcal{E}_{H_I}\cup B_I}(\lambda)s_{\mathcal{E}_{H_I}\cup B_I}(\lambda)^*\big)\Big) = 0_{C^*(\Lambda)/I}.$$

Then either $E \in \mathcal{E}_{H_I}$, or else $E \in B_I$ by the definition of B_I. But then $\prod_{\lambda\in E}(s_{r(E)} - s_\lambda s_\lambda^*) \in J_{H_I,B_I}$, so that

$$\prod_{\lambda\in E}(s_{\mathcal{E}_{H_I}\cup B_I}(r(E)) - s_{\mathcal{E}_{H_I}\cup B_I}(\lambda)s_{\mathcal{E}_{H_I}\cup B_I}(\lambda)^*) = 0_{C^*(\Lambda\setminus\Lambda H_I;\mathcal{E}_{H_I}\cup B_I)}.$$

Since $\mathcal{E}_{H_I}\cup B_I$ is satiated, it follows that π is nonzero on gap projections associated to sets in $\mathrm{FE}(\Lambda \setminus \Lambda H_I) \setminus \overline{(\mathcal{E}_{H_I} \cup B_I)}$.

Theorem 4.3.12 now shows that π is faithful, and hence that $I = J_{H_I,B_I}$ as required. □

Proof of Part (2) of Theorem 5.3.6. We have $H \subset H_{J_{H,B}}$ and $B \subset B_{J_{H,B}}$ by definition. If $v \in H_{J_{H,B}}$, then $s_v \in J_{H,B}$ and hence its image in $C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H \cup B)$ is trivial. It follows that either $v \in H$ or $s_{\mathcal{E}_H\cup B}(v) = 0$. But $s_{\mathcal{E}_H\cup B}(v) \neq 0$ for all $v \in (\Lambda \setminus \Lambda H)^0$ by Corollary 4.3.10, giving $v \in H$.

If $E \in B_{J_{H,B}}$, then we have

$$\prod_{\lambda\in E}\big(s_{\mathcal{E}_H}(v) - s_{\mathcal{E}_H}(\lambda)s_{\mathcal{E}_H}(\lambda)^*\big) \in I_B \subset C^*(\Lambda \setminus \Lambda H; \mathcal{E}_H).$$

Hence $\prod_{\lambda\in E}\big(s_{\mathcal{E}_H\cup B}(v) - s_{\mathcal{E}_H\cup B}(\lambda)s_{\mathcal{E}_H\cup B}(\lambda)^*\big)$ is trivial in $C^*(\Lambda\setminus\Lambda H; \mathcal{E}_H\cup B)$. Since $\mathcal{E}_H \cup B$ is satiated, it follows that either $E \in \mathcal{E}_H$ or $E \in B$. But $B_{J_{H,B}} \cap \mathcal{E}_H = \emptyset$ by definition, and it follows that $E \in B$ as required. □

We can use Theorem 5.3.6 to describe the lattice of gauge-invariant ideals in terms of pairs (H, B); to do this, however, we need to define a partial order on pairs (H, B) which matches up with the partial order amongst gauge-invariant ideals given by inclusion.

Definition 5.3.7. Let (Λ, d) be a finitely aligned k-graph. Define

$$\begin{aligned}\mathrm{SH{\times}S}(\Lambda) := \{(H, B) : \emptyset \neq H \subset \Lambda^0, H \text{ saturated and hereditary},\\ B \subset \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H,\ \mathcal{E}_H \cup B \text{ satiated}\}.\end{aligned}$$

Define a relation $\preceq$ on $\mathrm{SH{\times}S}(\Lambda)$ by $(H_1, B_1) \preceq (H_2, B_2)$ if and only if

(1) $H_1 \subset H_2$; and
(2) if $E \in B_1$ and $r(E) \notin H_2$, then $E \setminus EH_2$ belongs to $\mathcal{E}_{H_2} \cup B_2$.

The notation SH×S(Λ) is supposed to suggest ordered pairs consisting of a saturated hereditary set followed by a satiated one. Whilst this notation is admittedly clumsy, the author could not think of a more elegant alternative that was equally suggestive.

THEOREM 5.3.8. *Let* (Λ, d) *be a finitely aligned* k*-graph. The map* $(H, B) \mapsto J_{H,B}$ *is a bijection between* SH×S(Λ) *and the gauge-invariant ideals of* $C^*(\Lambda)$*. Moreover, for* $(H_1, B_1), (H_2, B_2) \in$ SH×S(Λ)*, we have* $J_{H_1,B_1} \subset J_{H_2,B_2}$ *if and only if* $(H_1, B_1) \preceq (H_2, B_2)$*. Hence the map* $(H, B) \mapsto J_{H,B}$ *is a lattice isomorphism from* SH×S(Λ) *ordered by* $\preceq$ *to the collection of gauge-invariant ideals of* $C^*(\Lambda)$ *ordered by inclusion.*

PROOF. For the first statement of the Proposition, we use Theorem 5.3.6 to see that the maps $I \mapsto (H_I, B_I)$ and $(H, B) \mapsto J_{H,B}$ have the appropriate domains and ranges and are mutually inverse.

Hence, we need only establish that for $(H_1, B_1), (H_2, B_2) \in$ SH×S(Λ), we have $J_{H_1,B_1} \subset J_{H_2,B_2}$ if and only if $(H_1, B_1) \preceq (H_2, B_2)$.

First suppose that $J_{H_1,B_1} \subset J_{H_2,B_2}$. Theorem 5.3.6 shows immediately that $H_1 \subset H_2$, so if we can show that $F \in B_1$ with $r(F) \notin H_2$ implies $F \setminus FH_2 \in \mathcal{E}_{H_2} \cup B_2$, it will follow that $(H_1, B_1) \preceq (H_2, B_2)$. Suppose that $E = F \setminus FH_2$ for some $F \in B_1$ with $r(F) \notin H_2$. Suppose further for contradiction that $E \notin \mathcal{E}_{H_2} \cup B_2$. Let $q_i : C^*(\Lambda) \to C^*(\Lambda)/J_{H_i,B_i}$ where $i = 1, 2$ denote the quotient maps; by Corollary 5.3.4, we can regard q_i as a homomorphism of $C^*(\Lambda)$ onto $C^*(\Lambda \setminus \Lambda H_i; \mathcal{E}_{H_i} \cup B_i)$ for $i = 1, 2$. Since $J_{H_1,B_1} \subset J_{H_2,B_2}$, we have that q_2 factors through q_1; that is, there is a homomorphism $\pi : C^*(\Lambda \setminus \Lambda H_1; \mathcal{E}_{H_1} \cup B_1) \to C^*(\Lambda \setminus \Lambda H_2; \mathcal{E}_{H_2} \cup B_2)$ such that $\pi \circ q_1 = q_2$. Since $F \in B_1$, we have $q_1\big(\prod_{\lambda \in F}(s_{r(F)} - s_\lambda s_\lambda^*)\big) = \prod_{\lambda\in F}(s_{\mathcal{E}_{H_2}\cup B_2}(r(F)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*) = 0$ by (CK). But we have $\pi \circ q_1 = q_2$, and hence we have the desired contradiction if we can establish that $q_2\big(\prod_{\lambda\in F}(s_{r(F)} - s_\lambda s_\lambda^*)\big) \neq 0$. Since $s(\lambda) \in H_2$ implies $q_2(s_\lambda s_\lambda^*) = 0$ by definition, we have that

$$q_2\Big(\prod_{\lambda\in F}(s_{r(F)} - s_\lambda s_\lambda^*)\Big) = \prod_{\lambda\in E}\big(s_{\mathcal{E}_{H_2}\cup B_2}(r(E)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*\big),$$

so it suffices to show that $\prod_{\lambda\in E}\big(s_{\mathcal{E}_{H_2}\cup B_2}(r(E)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*\big) \neq 0$. We consider two cases.

Case 1: E belongs to FE($\Lambda \setminus \Lambda H_2$). Then Lemma 4.3.9(2) ensures that $\prod_{\lambda\in E}\big(s_{\mathcal{E}_{H_2}\cup B_2}(r(E)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*\big)$ is nonzero as required. □ Case 1

Case 2: $E \notin \mathrm{FE}(\Lambda\backslash\Lambda H_2)$. Then there exists $\mu \in r(E)\Lambda\backslash\Lambda H_2$ with $\mathrm{Ext}(\mu; E) = \emptyset$; we then have

$$\textstyle\prod_{\lambda\in E}\big(s_{\mathcal{E}_{H_2}\cup B_2}(r(E)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*\big)s_{\mathcal{E}_{H_2}\cup B_2}(\mu)s_{\mathcal{E}_{H_2}\cup B_2}(\mu)^* \\ = s_{\mathcal{E}_{H_2}\cup B_2}(\mu)s_{\mathcal{E}_{H_2}\cup B_2}(\mu)^*$$

by (TCK3). Since $\|s_{\mathcal{E}_{H_2}\cup B_2}(\mu)s_{\mathcal{E}_{H_2}\cup B_2}(\mu)^*\| = \|s_{\mathcal{E}_{H_2}\cup B_2}(\mu)^*s_{\mathcal{E}_{H_2}\cup B_2}(\mu)\| = 1$ by Lemma 4.3.9(1), it follows that

$$\prod_{\lambda\in E}\big(s_{\mathcal{E}_{H_2}\cup B_2}(r(E)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*\big)s_{\mathcal{E}_{H_2}\cup B_2}(\mu)s_{\mathcal{E}_{H_2}\cup B_2}(\mu)^* \neq 0$$

as required. □ Case 2

In either case, we have that $(H_1, B_1) \preceq (H_2, B_2)$ establishing the "only if" assertion of the theorem.

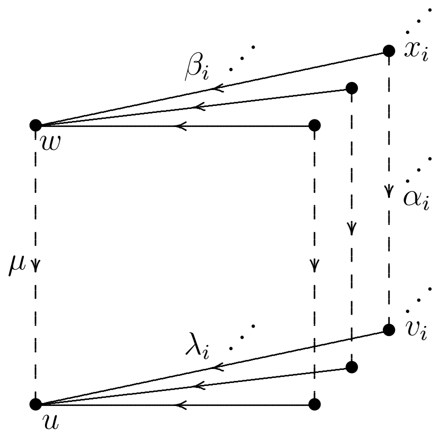
Now suppose that $(H_1, B_1) \preceq (H_2, B_2) \in \mathrm{SH{\times}S}(\Lambda)$. We must show that $J_{H_1,B_1} \subset J_{H_2,B_2}$. To do so, it suffices to show that the projections which generate J_{H_1,B_1} all belong to J_{H_2,B_2}. That is, we must show that $v \in H_1$ implies $s_v \in J_{H_2,B_2}$, and that $E \in B_1$ implies $\prod_{\lambda\in E}(s_v - s_\lambda s_\lambda^*) \in J_{H_2,B_2}$. First suppose that $v \in H_1$. Since $(H_1, B_1) \preceq (H_2, B_2)$, we have that $H_1 \subset H_2$, and hence $v \in H_2$ giving $s_v \in J_{H_2,B_2}$ by definition. Now suppose that $E \in B_1$. If $r(E) \in H_2$, then since H_2 is hereditary we have $E \subset \Lambda H_2$ and it follows immediately that $\prod_{\lambda\in E}(s_{r(E)} - s_\lambda s_\lambda^*) \in J_{H_2,B_2}$ as required. On the other hand, if $r(E) \notin H_2$, then since $(H_1, B_1) \preceq (H_2, B_2)$, we have that $E \setminus EH_2 \in \mathcal{E}_{H_2} \cup B_2$. Notice that if $\lambda \in \Lambda H_2$, then $s_\lambda s_\lambda^* \in J_{H_2,B_2}$ automatically, and hence

$$q_2\Big(\prod_{\lambda\in E}(s_{r(E)} - s_\lambda s_\lambda^*)\Big) = \prod_{\lambda\in E\backslash EH_2}(s_{\mathcal{E}_{H_2}\cup B_2}(r(E)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*). \tag{5.3.8}$$

If $E \setminus EH_2 \in \mathcal{E}_{H_2}$, then Lemma 5.2.6 guarantees that $\prod_{\lambda\in E\backslash EH_2}(s_{\mathcal{E}_{H_2}\cup B_2}(r(E)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*) = 0$ and hence that $\prod_{\lambda\in E}(s_{r(E)} - s_\lambda s_\lambda^*) \in J_{H_2,B_2}$ by (5.3.8) combined with Corollary 5.3.4. Finally, if $E \setminus EH_2 \in B_2$, then Corollary 5.3.4 again shows that $\prod_{\lambda\in E\backslash EH_2}(s_{\mathcal{E}_{H_2}\cup B_2}(r(E)) - s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)s_{\mathcal{E}_{H_2}\cup B_2}(\lambda)^*) = 0$, and it follows once again from (5.3.8) and another application of Corollary 5.3.4 that $\prod_{\lambda\in E}(s_{r(E)} - s_\lambda s_\lambda^*) \in J_{H_2,B_2}$, completing the proof. □

REMARK 5.3.9. Given a k-graph Λ and a saturated hereditary $H \subset \Lambda^0$, the ideal I_H associated to H in Section 5.1 occurs in the above listing as $J_{H,\emptyset}$; Lemma 5.2.5 shows that $\mathcal{E}_H \cup \emptyset$ is satiated.

Example 5.3.10. We return to considering the 2-graph Λ of Example 5.2.7; that is, the unique 2-graph with 1-skeleton



We will list all the gauge-invariant ideals in $C^*(\Lambda)$. For this, we need only identify all the pairs (H, B) where $H \subset \Lambda^0$ is saturated and hereditary and $B \subset \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$ has the property that $\mathcal{E}_H \cup B$ is satiated. To do this, note that any saturated and hereditary $H \subset \Lambda^0$ which contains either u or w must be all of Λ^0. Moreover, if $H \subset \Lambda^0$ is saturated and hereditary then $x_i \in H$ if and only if $v_i \in H$. Hence the saturated hereditary sets are $\Lambda^0, \emptyset$ and all sets of the form $H_S := \{x_i, v_i : i \in S\}$ for some $S \subset \mathbb{N}$. For all $S \subset \mathbb{N}$, we have $\mathcal{E}_{H_S} = \{\{\mu\}, \{\alpha_i\} : i \in \mathbb{N} \setminus S\}$, and so $\mathrm{FE}(\Lambda \setminus \Lambda H_S) \setminus \mathcal{E}_{H_S}$ is empty if $\mathbb{N} \setminus S$ is infinite, and is equal to $\{\{\lambda_i : i \in \mathbb{N} \setminus S\}, \{\beta_i : i \in \mathbb{N} \setminus S\}\}$ if $\mathbb{N} \setminus S$ is finite. Moreover if $\mathbb{N} \setminus S$ is finite, and if $\mathcal{E}_{H_S} \cup B$ is satiated, then $\{\lambda_i : i \in \mathbb{N} \setminus S\} \in B$ if and only if $\{\beta_i : i \in \mathbb{N} \setminus S\} \in B$. For $S \subset \mathbb{N}$ with $\mathbb{N} \setminus S$ finite, we write B_S for $\{\{\lambda_i : i \in \mathbb{N} \setminus S\}, \{\beta_i : i \in \mathbb{N} \setminus S\}\}$. Hence the ideals

$$I_{H_S} = J_{H_S,\emptyset} \text{ where } \emptyset \neq S \subset \mathbb{N} \quad \text{and} \quad J_{H_S,B_S} \text{ where } \emptyset \neq S \subsetneq \mathbb{N} \text{ with } \mathbb{N} \setminus S \text{ finite}$$

are distinct, and this is a complete listing of the non-trivial gauge-invariant ideals in $C^*(\Lambda)$.

It is easy to identify the form of each of these ideals and the associated quotient using calculations similar to those employed in Example 5.2.7. There are three cases to consider:

Case 1: S is finite and $\mathbb{N} \setminus S$ is infinite. Then $J_{H_S,\emptyset} \cong M_2 \otimes (\oplus_{i \in S} M_2)$ and $C^*(\Lambda)/J_{H_S,\emptyset} \cong C^*(\Lambda)$.

Case 2: Both S and $\mathbb{N} \setminus S$ are infinite. Then $J_{H_S,\emptyset} \cong M_2 \otimes (D \otimes M_2)$ where D is the diagonal subalgebra of $\mathcal{K}$ as in Example 5.2.7, and $C^*(\Lambda)/J_{H_S,\emptyset} \cong C^*(\Lambda)$.

Case 3: S is infinite and $\mathbb{N} \setminus S$ is finite. Then we have

$$J_{H_S,\emptyset} \cong M_2 \otimes (D \otimes M_2) \quad \text{and} \quad C^*(\Lambda)/J_{H_S,\emptyset} \cong M_2 \oplus \Big(\bigoplus_{i\in\mathbb{N}\setminus S} M_4\Big),$$
$$J_{H_S,B_S} \cong M_2 \otimes (D \otimes M_2 + \mathbb{C}P) \quad \text{and} \quad C^*(\Lambda)/J_{H_S,B_S} \cong \bigoplus_{i\in\mathbb{N}\setminus S} M_4.$$

5.4. Higher-rank graphs for which all ideals are gauge-invariant

In this section we use the Cuntz-Krieger uniqueness theorem for relative Cuntz-Krieger algebras to identify a class of k-graphs Λ for which every ideal of $C^*(\Lambda)$ is of the form $J_{H,B}$ where $(H, B) \in \mathrm{SH{\times}S}(\Lambda)$.

Definition 5.4.1. Let (Λ, d) be a finitely aligned k-graph. We say that Λ satisfies *Condition (D)* if the pair $(\Lambda \setminus \Lambda H, \mathcal{E}_H \cup B)$ satisfies Condition (C) for every $(H, B) \in \mathrm{SH{\times}S}(\Lambda)$.

Theorem 5.4.2. *Let (Λ, d) be a finitely aligned k-graph which satisfies Condition (D).*

(1) *Let I be an ideal of $C^*(\Lambda)$. Then H_I is nonempty, saturated and hereditary, $B_I \cup \mathcal{E}_{H_I}$ is satiated in $\Lambda \setminus \Lambda H_I$, and $I = J_{H_I,B_I}$.*
(2) *Let $H \subset \Lambda^0$ be nonempty, saturated and hereditary, and let $B \subset \mathrm{FE}(\Lambda \setminus \Lambda H) \setminus \mathcal{E}_H$ be such that $B \cup \mathcal{E}_H$ is satiated in $\Lambda \setminus \Lambda H$. Then $H_{J_{H,B}} = H$ and $B_{J_{H,B}} = B$.*

Hence every ideal of $C^*(\Lambda)$ is gauge-invariant, and $(H, B) \to J_{H,B}$ is a lattice-isomorphism from $\mathrm{SH{\times}S}(\Lambda)$ ordered by $\preceq$ to the collection of closed two-sided ideals in $C^*(\Lambda)$ ordered by inclusion.

Proof. The proof of (1) is the same as the proof of part (1) of Theorem 5.3.6 except that, since we do not know that I is gauge-invariant, we do not automatically have a strongly continuous action θ on $C^*(\Lambda)/I$ such that $\theta_z \circ \pi = \pi \circ \gamma_z$. Consequently, we cannot apply Theorem 4.3.12 in the final line of the proof to deduce that π is faithful. However, since Λ satisfies Condition (D), the pair $(\Lambda\setminus\Lambda H, \mathcal{E}_H \cup B)$ satisfies Condition (C), so Theorem 4.5.2 shows that π is faithful.

Statement (2) is precisely statement (2) of Theorem 5.3.6.

Statements (1) and (2) now combine to show that every ideal of $C^*(\Lambda)$ is gauge-invariant. The final statement now follows from Theorem 5.3.8. □

5.5. Maximal tails

In this section we take a first step towards deciding precisely which $(H, B) \in$ SH$\times$S(Λ) correspond to gauge-invariant ideals which are primitive.

LEMMA 5.5.1. *Let (Λ, d) be a finitely aligned k-graph and suppose that $H_1, H_2 \subset \Lambda^0$ are hereditary. Then $\Sigma(H_1 \cap H_2) = \Sigma H_1 \cap \Sigma H_2$.*

PROOF. Since the intersection of two saturated sets is clearly saturated, we have $\Sigma H_1 \cap \Sigma H_2$ a saturated set which contains $H_1 \cap H_2$ by definition. Hence $\Sigma(H_1 \cap H_2) \subset \Sigma H_1 \cap \Sigma H_2$.

For the reverse direction, fix $v \in \Sigma H_1 \cap \Sigma H_2$. We must show that $v \in \Sigma(H_1 \cap H_2)$. To do this we consider three cases:

Case 1: $v \in H_1 \cap H_2$. Then v belongs to $\Sigma(H_1 \cap H_2)$ because $(H_1 \cap H_2) \subset \Sigma(H_1 \cap H_2)$ by definition. □ Case 1

Case 2: $v \in H_1 \cap (\Sigma_{H_2} \setminus H_2)$ or $v \in (\Sigma H_1 \setminus H_1) \cap H_2$. If $v \in H_1 \cap (\Sigma_{H_2} \setminus H_2)$, then there exists $E \in v\,\mathrm{FE}(\Lambda)$ with $s(E) \subset H_2$ and $r(E) = v \in H_1$. But H_1 is hereditary, so $s(E) \subset H_1$, giving $E \in v\,\mathrm{FE}(\Lambda)(H_1 \cap H_2)$, and hence $v \in \Sigma(H_1 \cap H_2)$. By symmetry, we have that $v \in (\Sigma H_1 \setminus H_1) \cap H_2$ implies $v \in \Sigma(H_1 \cap H_2)$ as well. □ Case 2

Case 3: $v \in (\Sigma H_1 \setminus H_1) \cap (\Sigma H_2 \setminus H_2)$. Then there exist $E_1, E_2 \in v\,\mathrm{FE}(\Lambda)$ such that $s(E_1) \subset H_1$ and $s(E_2) \subset H_2$. Let $E := \bigcup\{\mathrm{MCE}(\lambda, \mu) : \lambda \in E_1, \mu \in E_2\}$. Since Λ is finitely aligned and the E_i are finite, we have that E is finite. If $\sigma \in E$, then $\sigma = \lambda\alpha = \mu\beta$ for some $\lambda \in E_1$ and $\mu \in E_2$. Since $s(\lambda) \in s(E_1) \subset H_1$ and $s(\mu) \in s(E_2) \subset H_2$, and since H_1 and H_2 are hereditary, it follows that $s(E) \subset H_1 \cap H_2$. It therefore suffices to show that $E \in v\,\mathrm{FE}(\Lambda)$.

We have that $E \subset v\Lambda$ by definition. We have $E \cap \Lambda^0 = \emptyset$ because $r(E) = v \notin H_1 \cap H_2$ by assumption, and $s(E) \subset H_1 \cap H_2$. To see that E is exhaustive, fix $\sigma \in v\Lambda$. Since E_1 is exhaustive, there exists $\lambda \in E_1$ such that $\Lambda^{\min}(\lambda, \sigma) \neq \emptyset$, say $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \sigma)$. Furthermore, since $\mathrm{Ext}(\lambda; E_2)$ is exhaustive by Lemma 4.2.7, there exists $\mu \in E_2$ and $(\eta, \zeta) \in \Lambda^{\min}(\lambda, \mu)$ such that $\Lambda^{\min}(\alpha, \eta) \neq \emptyset$, say $(\tau, \rho) \in \Lambda^{\min}(\alpha, \eta)$. But we now have $\sigma\beta\tau = \lambda\alpha\tau = \lambda\eta\rho = \mu\zeta\rho$; in particular, $\lambda\eta = \mu\zeta$ belongs to E by definition, and σ and $\lambda\eta$ have at least one common extension, so that $\Lambda^{\min}(\sigma, \lambda\eta) \neq \emptyset$. Since $\sigma \in v\Lambda$ was arbitrary, it follows that E is exhaustive as required. □ Case 3

We therefore have $v \in \Sigma(H_1 \cap H_2)$. But $v \in \Sigma H_1 \cap \Sigma H_2$ was arbitrary, so it follows that $\Sigma H_1 \cap \Sigma H_2 \subset \Sigma(H_1 \cap H_2)$, completing the proof. □

LEMMA 5.5.2. *Let (Λ, d) be a finitely aligned k-graph, and let H_1 and H_2 be saturated hereditary subsets of Λ^0. Then $I_{H_1} \cap I_{H_2} = I_{H_1 \cap H_2}$.*

PROOF. Since $I_{H_1} \cap I_{H_2}$ is an ideal of $C^*(\Lambda)$ which contains the projections $\{s_v : v \in H_1 \cap H_2\}$, we automatically have that $I_{H_1 \cap H_2} \subset I_{H_1} \cap I_{H_2}$. For the reverse inclusion, first note that $I_{H_1} \cap I_{H_2} = I_{H_1} I_{H_2}$. By Lemma 5.1.4, it therefore suffices to show that if $\lambda, \mu \in \Lambda H_1$ and $\sigma, \tau \in \Lambda H_2$, then $s_\lambda s_\mu^* s_\sigma s_\tau^* \in I_{H_1 \cap H_2}$. So let $\lambda, \mu \in \Lambda H_1$ and $\sigma, \tau \in \Lambda H_2$. Then $s_\lambda s_\mu^* s_\sigma s_\tau^* = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\sigma)} s_{\lambda\alpha} s_{\mu\sigma}^*$. For $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$, we have $r(\alpha) = s(\mu) \in H_1$, so $s(\alpha)$ also belongs to H_1 because H_1 is hereditary. Likewise, $r(\beta) = s(\sigma) \in H_2$, and hence $s(\beta) \in H_2$ because H_2 is hereditary. So for each $(\alpha, \beta) \in \Lambda^{\min}(\mu, \sigma)$, we have that $s(\alpha) = s(\beta)$ belongs to both H_1 and H_2, and hence $s_\lambda s_\mu^* s_\sigma s_\tau^* \in \operatorname{span}\{s_\rho s_\xi^* : \rho, \xi \in \Lambda(H_1 \cap H_2)\}$ which is a subset of $I_{H_1 \cap H_2}$ by another application of Lemma 5.1.4. □

PROPOSITION 5.5.3. *Let (Λ, d) be a finitely aligned k-graph, and let $I \subset C^*(\Lambda)$ be an ideal. Then $M_I := \Lambda^0 \setminus H_I$ satisfies*

(MT1) *if $v \in \Lambda^0$, $w \in M_I$ and $v\Lambda w \neq \emptyset$, then $v \in M_I$; and*
(MT2) *if $v \in M_I$ and $E \in v\,\mathrm{FE}(\Lambda)$, then there exists $\lambda \in E$ with $s(\lambda) \in M_I$.*

If I is a primitive ideal of $C^*(\Lambda)$, then M_I satisfies the additional condition that

(MT3) *for all $v_1, v_2 \in M_I$, there exists $w \in M_I$ such that both $v_1\Lambda w$ and $v_2\Lambda w$ are nonempty.*

We call M_I the maximal tail *associated to I.*

PROOF. Condition (MT1) is equivalent to the statement that H_I is hereditary, and (MT2) is equivalent to the statement that H_I is saturated. This proves the first statement of the proposition.

For the second statement, let I be a primitive ideal in $C^*(\Lambda)$. Suppose for contradiction that there exist v_1 and v_2 in M_I such that there is no $w \in M_I$ with both $v_1\Lambda w$ and $v_2\Lambda w$ nonempty.

Let $H_1 := s(v_1\Lambda) = \{v' \in \Lambda^0 : v\Lambda v' \neq \emptyset\}$. Likewise, let $H_2 = s(v_2\Lambda)$. We have that H_1 and H_2 are hereditary by their definition, so Lemma 5.5.1 shows that $\Sigma H_1 \cap \Sigma H_2 = \Sigma(H_1 \cap H_2)$. The statement that $v_1\Lambda w$ and $v_2\Lambda w$ are never simultaneously nonempty for $w \in M_I$ is equivalent to the statement that $H_1 \cap H_2 \subset H_I$. Since H_I is saturated, it follows that $\Sigma(H_1 \cap H_2) \subset H_I$, so $I_{\Sigma(H_1 \cap H_2)} \subset I$.

Combining all this with Lemma 5.5.2, we therefore have

$$I_{\Sigma H_1} \cap I_{\Sigma H_2} = I_{\Sigma H_1 \cap \Sigma H_2} = I_{\Sigma(H_1 \cap H_2)} \subset I.$$

But both v_1 and v_2 belong to $M_I = \Lambda^0 \backslash H_I$, and hence $s_{v_1} \in I_{H_1} \backslash I$ and $s_{v_2} \in I_{H_2} \backslash I$. That is, neither I_{H_1} nor I_{H_2} is a subset of I, which is a contradiction because I is a primitive ideal in $C^*(\Lambda)$. $\square$

CHAPTER 6

Simple, purely infinite, nuclear Cuntz-Krieger algebras

In this chapter we investigate which finitely aligned k-graphs Λ have simple, purely infinite and nuclear C^*-algebras. We show that all relative Cuntz-Krieger algebras are nuclear. We show that if Λ satisfies Condition (C) and is *cofinal*, then $C^*(\Lambda)$ is simple. Finally, we show that if Λ satisfies Condition (C) and every vertex of Λ can be reached from a loop with an entrance in Λ, then every nontrivial hereditary subalgebra of $C^*(\Lambda)$ contains an infinite projection. We begin with a short section in which we establish a little notation and a technical lemma.

6.1. A technical lemma

The definitions in this section and the proof of Lemma 6.1.4 are based almost entirely on the definitions and techniques appearing between [**30**, Notation 3.12] and the proof of [**30**, Proposition 3.13] inclusive. They are presented separately here only because the conclusion of Lemma 6.1.4 is not stated explicitly in [**30**].

For the remainder of this section, fix a finitely aligned k-graph (Λ, d), a finite set $E \subset \Lambda$, and a linear combination $a = \sum_{\lambda,\mu \in E} a_{\lambda,\mu} s_\lambda s_\mu^*$ in $C^*(\Lambda)$. Express $\Phi^\gamma(a)$ as $\sum_{\lambda,\mu \in \Pi E} a'_{\lambda,\mu} \Theta(s)^{\Pi E}_{\lambda,\mu}$.

NOTATION 6.1.1. For $\lambda \in \Pi E$ we will write $T(\lambda)$ as short-hand notation for the set $T^{\Pi E}(d(\lambda), s(\lambda))$ introduced in Definition 3.6.1. We must bear in mind, however, that $T(\lambda)$ depends only on $d(\lambda)$ and $s(\lambda)$ so that for $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, we have $T(\lambda) = T(\mu)$.

DEFINITION 6.1.2. For n, v such that $T^{\Pi E}(n, v) \notin \mathrm{FE}(\Lambda)$, we fix a path $\xi^{\Pi E}(n, v) \in v\Lambda$ such that $\Lambda^{\min}(\xi^{\Pi E}(n, v), \sigma) = \emptyset$ for all $\sigma \in T^{\Pi E}(n, v)$. As with $T^{\Pi E}(n, v)$, for $\lambda \in \Pi E$ such that $T(\lambda) = T^{\Pi E}(d(\lambda), s(\lambda))$ is not exhaustive, we will write $\xi(\lambda)$ as short-hand notation for $\xi^{\Pi E}(d(\lambda), s(\lambda))$. Like $T(\lambda)$, the path $\xi(\lambda)$ depends only on $s(\lambda)$ and $d(\lambda)$, so that for $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, we have $\xi(\lambda) = \xi(\mu)$.

DEFINITION 6.1.3. Define projections

$$P_{n,v} := \sum_{\lambda \in (\Pi E)v \cap \Lambda^n} s_{\lambda\xi(\lambda)} s^*_{\lambda\xi(\lambda)}$$

for each n and v such that $(\Pi E)v \cap \Lambda^n$ is nonempty and $T^{\Pi E}(n, v) \notin \mathrm{FE}(\Lambda)$.

LEMMA 6.1.4. *For $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, let*

$$\mathcal{F}_{\Pi E}(n, v) := \overline{\operatorname{span}}\{s_{\lambda\xi(\lambda)} s^*_{\mu\xi(\lambda)} : \lambda, \mu \in (\Pi E)v \cap \Lambda^n\}.$$

For n, v such that $(\Pi E)v \cap \Lambda^n$ is nonempty, $P_{n,v}\Phi(a) \in \mathcal{F}_{\Pi E}(n, v)$. Furthermore, there exist n_0, v_0 such that $(\Pi E)v_0 \cap \Lambda^{n_0}$ is nonempty and $\|P_{n_0,v_0}\Phi(a)\| = \|\Phi(a)\|$. If $\|\Phi(a)\| > 0$, then $T^{\Pi E}(n_0, v_0)$ is non-exhaustive.

PROOF. By [**30**, Lemma 3.15], we have that each $s_{\lambda\xi(\lambda)} s^*_{\lambda\xi(\lambda)} \leq Q(s)^{\Pi E}_\lambda$. Since the $Q(s)^{\Pi E}_\lambda$ are mutually orthogonal projections, this gives $s_{\lambda\xi(\lambda)} s^*_{\lambda\xi(\lambda)} Q(s)^{\Pi E}_\mu = \delta_{\lambda,\mu} s_{\lambda\xi(\lambda)} s^*_{\lambda\xi(\lambda)}$. Hence, for $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, we have

$$P_{n,v}\Theta(s)^{\Pi E}_{\lambda,\mu} = P_{n,v} Q(s)^{\Pi E}_\lambda s_\lambda s^*_\mu = s_{\lambda\xi(\lambda)} s^*_{\lambda\xi(\lambda)} s_\lambda s^*_\mu = s_{\lambda\xi(\lambda)} s^*_{\mu\xi(\lambda)}, \tag{6.1.1}$$

and it follows from the definition of $\mathcal{F}_{\Pi E}(n, v)$ that $P_{n,v}\Phi(a) \in \mathcal{F}_{\Pi E}(n, v)$.

If $\|\Phi(a)\| = 0$, then we trivially have that $\|P_{n_0,v_0}\Phi(a)\| = \|\Phi(a)\|$ for any choice of n_0, v_0 for which the expression makes sense. On the other hand, if $\|\Phi(a)\| \neq 0$, then Corollary 4.3.15 shows that there exists a vertex v_0 and a degree n_0 such that

$$\|\Phi(a)\| = \Big\| \sum_{\lambda,\mu \in (\Pi E)v_0 \cap \Lambda^{n_0}} a_{\lambda,\mu} \Theta(s)^{\Pi E}_{\lambda,\mu} \Big\|.$$

Clearly for this n_0, v_0 we must have $(\Pi E)v_0 \cap \Lambda^{n_0}$ nonempty and $T^{\Pi E}(n_0, v_0)$ non-exhaustive. But $\{s_{\lambda\xi(\lambda)} s^*_{\mu\xi(\lambda)} : \lambda \in (\Pi E)v_0 \cap \Lambda^{n_0}\}$ is the collection of nonzero matrix units which spans $\mathcal{F}_{\Pi E}(n_0, v_0)$. Since for $\lambda, \mu \in (\Pi E)v_0 \cap \Lambda^{n_0}$, equation (6.1.1) shows that $P_{n_0,v_0}\Theta(s)^{\Pi E}_{\lambda,\mu} = s_{\lambda\xi(\lambda)} s^*_{\mu\xi(\lambda)}$, it follows that $b \mapsto P_{n_0,v_0} b$ maps nonzero matrix units in $M^s_{\Pi E}(n_0, v_0)$ to nonzero matrix units in $\mathcal{F}_{\Pi E}(n_0, v_0)$, and hence implements an isomorphism of $M^s_{\Pi E}(n_0, v_0)$ onto $\mathcal{F}_{\Pi E}(n_0, v_0)$, giving $\|\Phi(a)\| = \big\| \sum_{\lambda,\mu \in (\Pi E)v_0 \cap \Lambda^{n_0}} a_{\lambda,\mu} \Theta(s)^{\Pi E}_{\lambda,\mu} \big\| = \|P_{n_0,v_0}\Phi(a)\|$ as required. □

6.2. Nuclearity, simplicity, and pure infinity

We can use a general result of Quigg to show that $C^*(\Lambda; \mathcal{E})$ is always nuclear.

PROPOSITION 6.2.1. *Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{E}$ be a subset of* $\mathrm{FE}(\Lambda)$. *Then $C^*(\Lambda; \mathcal{E})^\gamma$ is AF and $C^*(\Lambda; \mathcal{E})$ is nuclear. In particular, both $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$ are nuclear.*

PROOF. Since $C^*(\Lambda;\mathcal{E})^\gamma$ is the closure of the increasing union of the finite-dimensional algebras $\{M^{s\mathcal{E}}_{\Pi E} : E \subset \Lambda \text{ is finite}\}$, we have that $C^*(\Lambda;\mathcal{E})^\gamma$ is AF. Since $\mathbb{T}^k$ is the dual of $\mathbb{Z}^k$, the triple $(C^*(\Lambda;\mathcal{E}), \mathbb{Z}^k, \gamma)$ is a discrete cosystem in the sense of [**27**]. Since $\mathbb{Z}^k$ is amenable and since $C^*(\Lambda;\mathcal{E})^\gamma$ is AF and hence nuclear, it now follows from [**27**, Corollary 2.17] that $C^*(\Lambda;\mathcal{E})$ is also nuclear. □

To give a simplicity condition for $C^*(\Lambda)$ we adapt the methods of [**3**, Proposition 5.1] to our situation.

DEFINITION 6.2.2. Let (Λ, d) be a finitely aligned k-graph. We say that Λ is *cofinal* if for all $v \in \Lambda^0$ and $x \in \partial\Lambda$, there exists $n \le d(x)$ such that $v\Lambda x(n)$ is nonempty.

PROPOSITION 6.2.3. *Let* (Λ, d) *be a finitely aligned* k*-graph, suppose that* Λ *satisfies Condition* (C)*. Then* $C^*(\Lambda)$ *is simple if and only if* Λ *is cofinal.*

PROOF. First suppose that Λ is cofinal and that I is an ideal in $C^*(\Lambda)$. If $s_v \in I$ for all $v \in \Lambda^0$, then $I = C^*(\Lambda)$ by (TCK2). Suppose that $v \in \Lambda^0$ with $s_v \notin I$. We must show that H_I is empty, for then Theorem 4.5.2 shows that I is trivial. Since H_I is saturated, we have that if $v' \notin H_I$ and $E \in v\,\mathrm{FE}(\Lambda)$, then there exists $\lambda \in E$ such that $s(\lambda) \notin H_I$. It follows that the λ_i in the proof of statement (1) of Lemma 4.3.9 can always be chosen so that $s(\lambda_i) \notin H_I$. Since H_I is hereditary, it follows that $\lambda_i(n) \notin H_I$ for all $i \in \mathbb{N}$ and $n \le d(\lambda_i)$. Hence there is an element x of $v\partial\Lambda$ such that $x(n) \notin H_I$ for all n. Now suppose for contradiction that there exists $w \in H_I$. By cofinality of Λ, there exists $n \in \mathbb{N}^k$ such that $w\Lambda x(n)$ is nonempty. But since H_I is hereditary, this implies that $x(n)$ belongs to H_I contradicting the construction of x.

Now suppose that $C^*(\Lambda)$ is simple. Let $x \in \partial\Lambda$, and let

$$H_x := \{w \in \Lambda^0 : w\Lambda x(n) = \emptyset \text{ for all } n\}.$$

We must show that H_x is empty. It is clear that H_x is hereditary. We claim that H_x is saturated: suppose that $E \in v\,\mathrm{FE}(\Lambda)$ with $s(E) \subset H_x$, and suppose for contradiction that $\lambda \in v\Lambda x(n)$. If $\lambda = \mu\mu'$ for $\mu \in E$, then $\mu' \in s(\mu)\Lambda x(n)$, contradicting $s(\mu) \in H_x$. On the other hand, if $\lambda \notin E\Lambda$, then $\mathrm{Ext}(\lambda; E) \in \mathrm{FE}(\Lambda)$ by Lemma 4.2.7. Since $x \in \partial(\Lambda;\mathcal{E})$, it follows that $x(n, n + d(\alpha)) = \alpha$ for some $\alpha \in \mathrm{Ext}(\lambda; E)$; say $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ where $\mu \in E$. Then $\beta \in s(\mu)\Lambda x(n + d(\alpha))$, again contradicting $s(\mu) \in H_x$. This proves our claim.

Now $H_x \neq \Lambda^0$ because, in particular, $r(x) \notin H_x$. It follows from Theorem 5.3.6 that if H_x is nonempty then $J_{H_x,\emptyset}$ is a nontrivial ideal in $C^*(\Lambda)$ which is impossible since $C^*(\Lambda)$ is simple by assumption. □

To give a condition under which $C^*(\Lambda)$ is purely infinite, we draw upon [**18**] for an appropriate graph-theoretic condition, and then upon [**3**] for our argument.

Definition 6.2.4. Let (Λ, d) be a k-graph and let $\mu \in \Lambda$. We call μ a *loop with an entrance* if $d(\mu) > 0$, $s(\mu) = r(\mu)$, and there exists $\alpha \in s(\mu)\Lambda$ with $d(\alpha) \leq d(\mu)$ and $\mu(0, d(\alpha)) \neq \alpha$.

Proposition 6.2.5. *Let (Λ, d) be a finitely aligned k-graph, and suppose that Λ satisfies Condition* (C). *Suppose also that for every $v \in \Lambda^0$, there exist $\lambda \in v\Lambda$ and $\mu \in s(\lambda)\Lambda s(\lambda)$ such that μ is a loop with an entrance. Then every hereditary subalgebra of $C^*(\Lambda)$ contains an infinite projection. In particular, if Λ is also cofinal, then $C^*(\Lambda)$ is purely infinite.*

Disclaimer 6.2.6. There is some contention regarding the use of the term "purely infinite" in relation to C^*-algebras which are not simple. The author has no axe to grind on the subject, and the wording of Proposition 6.2.5 merely reflects that for simple C^*-algebras the term is not under debate.

To prove the proposition we need two lemmas.

Lemma 6.2.7. *Let (Λ, d) be a finitely aligned k-graph, and suppose that Λ satisfies Condition* (C). *Suppose also that for every $v \in \Lambda^0$, there exist $\lambda \in v\Lambda$ and $\mu \in s(\lambda)\Lambda s(\lambda)$ such that μ is a loop with an entrance. Then for each $v \in \Lambda^0$, the projection s_v is infinite.*

Proof. Let $v \in \Lambda$ and let $\lambda \in v\Lambda$ and $\mu \in s(\lambda)\Lambda s(\lambda)$ such that μ is a loop with an entrance. Since s_λ is a partial isometry, we have $s_v \geq s_\lambda s_\lambda^* \sim s_\lambda^* s_\lambda = s_w$, so that s_v is infinite if s_w is infinite. By definition of a loop with an entrance, there exists $x \in w\partial\Lambda$ such that $x(0, d(\mu)) \neq \mu$. It follows that in the boundary path representation we have $S_\mu S_\mu^* e_x = 0$ so that $(S_w - S_\mu S_\mu^*)e_x = e_x \neq 0$. The universal property of $C^*(\Lambda)$ now implies that $s_w - s_\mu s_\mu^* \neq 0$ as well, so that $s_w > s_\mu s_\mu^*$. But we now have $s_w = s_\mu^* s_\mu \sim s_\mu s_\mu^* < s_w$, so that s_w is infinite, and it follows that s_v is infinite as required. □

Lemma 6.2.8. *Let $E \subset \Lambda^n$ be finite, let $w \in s(E)$, and let t be a positive element of $\mathcal{F}_E(w) := \operatorname{span}\{s_\lambda s_\mu^* : \lambda, \mu \in Ew\}$. Then there is a projection r in $C^*(t) \subset \mathcal{F}_E(w)$ such that $rtr = \|t\|r$.*

PROOF. The proof is formally identical to that of [**3**, Lemma 5.4]. □

PROOF OF PROPOSITION 6.2.5. Our proof follows that of [**3**, Proposition 5.3] very closely, though we need to use some results from Section 6.1 to circumvent the technical issues which arise because the k-graphs here are not row-finite whereas the directed graphs in [**3**] are.

Fix a nontrivial hereditary subalgebra A of $C^*(\Lambda)$, and a positive element $a \in A$ such that $\Phi^\gamma(a) \in C^*(\Lambda)^\gamma$ satisfies $\|\Phi^\gamma(a)\| = 1$. Let $b = \sum_{\lambda,\mu \in E} b_{\lambda,\mu} s_\lambda s_\mu^*$ be a finite linear combination such that $b > 0$ and $\|a - b\| \leq \frac{1}{4}$. Such a b exists because $\operatorname{span}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$ is a dense *-subalgebra of $C^*(\Lambda)$. Let $b_0 := \Phi(b)$. Since Φ is norm-decreasing and linear, we have

$$1 - \|b_0\| = \big|\|\Phi(a)\| - \|\Phi(b)\|\big| \leq \|\Phi(a - b)\| \leq \|a - b\| \leq \frac{1}{4},$$

and hence $\|b_0\| \geq \frac{3}{4}$. Furthermore, $b_0 \geq 0$ because Φ is positive. Applying Lemma 6.1.4, we obtain a projection P_{n_0,v_0} such that $b_1 := P_{n_0,v_0} b_0$ satisfies $b_1 \in \mathcal{F}_{\Pi E}(n_0, v_0)$ and $\|b_1\| = \|b_0\|$. Notice that $b_1 \geq 0$. By Lemma 6.2.8 there exists a projection $r \in C^*(b_1)$ with $r b_1 r = \|b_1\| r$. Recalling the definitions of $T(\lambda)$ from Notation 6.1.1 and of $\xi(\lambda)$ from Definition 6.1.2, let $v_1 := s(\xi(\lambda))$ for any $\lambda \in (\Pi E)v_0 \cap \Lambda^{n_0}$. Let

$$S := \{\lambda \xi(\lambda) : \lambda \in (\Pi E)v_0 \cap \Lambda^{n_0}\} \subset \Lambda v_1.$$

Since $b_1 \in \operatorname{span}\{s_\lambda s_\mu^* : \lambda, \mu \in S\}$, which is a matrix algebra indexed by S, we can express r as a finite sum $r = \sum_{\lambda,\mu \in S} r_{\lambda,\mu} s_\lambda s_\mu^*$, and the $S \times S$ matrix $(r_{\lambda,\mu})$ is a projection.

Since (Λ, d) satisfies Condition (C), there exists $x \in v_1 \partial\Lambda$ which satisfies (4.5.1). By Lemma 4.5.3, for distinct $\lambda, \mu \in S$, there exists $n^x_{\lambda,\mu}$ such that $\Lambda^{\min}(\lambda x(0, n^x_{\lambda,\mu}), \mu x(0, n^x_{\lambda,\mu})) = \emptyset$. Let

$$M := \bigvee\{n^x_{\lambda,\mu} : \lambda, \mu \in S, \lambda \neq \mu\},$$

and let $x_M := x(0, M)$. Let $q := \sum_{\lambda,\mu \in S} r_{\lambda,\mu} s_{\lambda x_M} s^*_{\mu x_M}$. Since the matrix $(r_{\lambda,\mu})$ is a projection in $M_S(\mathbb{C})$, we know that q is a projection in $\mathcal{F}_{N_E + d(x_M)}$, and since $s_{x_M} s^*_{x_M}$ is a subprojection of s_{v_1}, we have $q \leq r$. Using the defining property of x_M as in the proof of Lemma 4.5.6, we have that $q P_{n_0,v_0} b q = q P_{n_0,v_0} b_0 q = q b_1 q$. Combining this with our choice of r, and with the fact that $q \leq P_{n_0,v_0}$ by definition, we obtain

$$q b q = q b_1 q = q r b_1 r q = \|b_1\| r q = \|b_0\| q \geq \frac{3}{4} q.$$

Since $\|a-b\| \leq \frac{1}{4}$, we have $qaq \geq qbq - \frac{1}{4}q \geq \frac{3}{4}q - \frac{1}{4}q = \frac{1}{2}q$, and it follows that qaq is invertible in $qC^*(\Lambda)q$. Write c for the inverse of qaq in $qC^*(\Lambda)q$, and let

$$t := c^{1/2}qa^{1/2}.$$

Then $tt^* = c^{1/2}qaqc^{1/2} = 1_{qC^*(\Lambda)q} = q$, and $t^*t = a^{1/2}qcqa^{1/2} \leq \|c\|a$, and hence t^*t belongs to the hereditary subalgebra A of $C^*(\Lambda)$.

We now need only show that t^*t is an infinite projection. Let $w := s(x_M)$. We have that s_w is an infinite projection by Lemma 6.2.7. Furthermore, by the hypothesis of the proposition, there exist paths λ, μ such that $d(\mu) > 0$, $r(\lambda) = w$, and $s(\lambda) = r(\mu) = s(\mu)$. Since b_1 is nonzero, we know that S is nonempty. For $\sigma \in S$, we have $\sigma x_M \lambda \in \Lambda w$, so that

$$s_{\sigma x_M} s^*_{\sigma x_M} \geq s_{\sigma x_M \lambda} s^*_{\sigma x_M \lambda} \sim s^*_{\sigma x_M \lambda} s_{\sigma x_M \lambda} = s_w$$

is also infinite. Since $s_{\sigma x_M} s^*_{\sigma x_M}$ is a minimal projection in the finite-dimensional C^*-algebra $\operatorname{span}\{s_{\sigma x_M} s^*_{\tau x_M} : \sigma, \tau \in S\}$ to which q belongs, we have $s_{\sigma x_M} s^*_{\sigma x_M}$ equivalent to a subprojection of q so that q is infinite as well. But now $t^*t \sim tt^* = q$ is infinite, completing the proof. □

REMARK 6.2.9. We would like to know when $C^*(\Lambda)$ satisfies the Universal Coefficient Theorem of [**36**] as well as being simple purely infinite and nuclear, because such algebras are completely determined by their K-theory [**26**, Theorem 4.2.4]. It has been relatively straightforward to establish that the Cuntz-Krieger algebras studied in previous treatments satisfy the UCT under fairly mild conditions on the underlying graph, but the usual arguments (see for example [**24, 18, 31**]) do not carry over easily to non-row-finite higher rank graphs.

APPENDIX A

Proof of a lemma due to Farthing, Muhly, and Yeend

In this appendix we present a proof of [**10**, Lemma 1.5] which is stated as Lemma 4.3.8, but is not proved there. At the time of writing, [**10**] is in draft form, so is not yet available as a preprint. The proof here differs slightly in detail, but not at all in general form or idea, from that appearing in the current draft of [**10**]. We state the lemma again here for convenience.

LEMMA ([**10**, Lemma 1.5]). *Let* (Λ, d) *be a* k*-graph. For* $v \in \Lambda^0$*,* $E \subset v\Lambda$*,* $\lambda_1 \in v\Lambda$ *and* $\lambda_2 \in s(\lambda_1)\Lambda$*,*

$$\mathrm{Ext}(\lambda_2; \mathrm{Ext}(\lambda_1; E)) = \mathrm{Ext}(\lambda_1\lambda_2; E).$$

PROOF. For the inclusion $\mathrm{Ext}(\lambda_2; \mathrm{Ext}(\lambda_1; E)) \subset \mathrm{Ext}(\lambda_1\lambda_2; E)$ suppose that $\alpha \in \mathrm{Ext}(\lambda_2; \mathrm{Ext}(\lambda_1; E))$. Then there exist $\sigma \in E$ and $(\eta, \zeta) \in \Lambda^{\min}(\lambda_1, \sigma)$ such that there exists $\beta \in \Lambda$ with $(\alpha, \beta) \in \Lambda^{\min}(\lambda_2, \eta)$. We then have

$$\lambda_1\lambda_2\alpha = \lambda_1\eta\beta = \sigma\zeta\beta \tag{A.0.1}$$

and

$$\begin{aligned} d(\lambda_1\lambda_2\alpha)_i &= d(\lambda_1)_i + \max\{d(\lambda_2)_i, d(\eta)_i\} \quad \text{since } (\alpha, \beta) \in \Lambda^{\min}(\lambda_2, \eta) \\ &= \max\{d(\lambda_1)_i + d(\lambda_2)_i, d(\lambda_1)_i + d(\eta)_i\} \\ &= \max\{d(\lambda_1)_i + d(\lambda_2)_i, \max\{d(\lambda_1)_i, d(\sigma)_i\}\} \quad \text{since } (\eta, \zeta) \in \Lambda^{\min}(\lambda_1, \sigma) \\ &= \max\{d(\lambda_1\lambda_2)_i, d(\sigma)_i\} \quad \text{since } d(\lambda_1\lambda_2)_i \geq d(\lambda_1)_i \end{aligned}$$

so that $d(\lambda_1\lambda_2\alpha) = d(\lambda_1\lambda_2) \vee d(\sigma)$. Combining this with (A.0.1) gives $(\alpha, \zeta\beta) \in \Lambda^{\min}(\lambda_1\lambda_2, \sigma)$ and hence $\alpha \in \mathrm{Ext}(\lambda_1\lambda_2; E)$ as required.

For the reverse inclusion, suppose that $\alpha \in \mathrm{Ext}(\lambda_1\lambda_2; E)$. So there exists $\sigma \in E$ and $\beta \in s(\sigma)\Lambda$ such that $(\alpha, \beta) \in \Lambda^{\min}(\lambda_1\lambda_2, \sigma)$. By definition, we have $d(\lambda_1\lambda_2\alpha) = d(\lambda_1\lambda_2) \vee d(\sigma) \geq d(\lambda_1) \vee d(\sigma)$, so we may define

$$\begin{aligned} \eta &:= (\lambda_1\lambda_2\alpha)(d(\lambda_1), d(\lambda_1) \vee d(\sigma)) = (\lambda_2\alpha)(0, (d(\lambda_1) \vee d(\sigma)) - d(\lambda_1)) \\ \zeta &:= (\lambda_1\lambda_2\alpha)(d(\sigma), d(\lambda_1) \vee d(\sigma)) = \beta(0, (d(\lambda_1) \vee d(\sigma)) - d(\sigma)), \end{aligned}$$

giving $(\eta, \zeta) \in \Lambda^{\min}(\lambda_1, \sigma)$ and hence $\eta \in \mathrm{Ext}(\lambda_1; E)$ by definition. Let $\eta' := (\lambda_2\alpha)(d(\eta), d(\lambda_2\alpha))$; so $\eta\eta' = \lambda_2\alpha$. Since $\eta \in \mathrm{Ext}(\lambda_1; E)$ we will have $\alpha \in$

$\mathrm{Ext}(\lambda_2; \mathrm{Ext}(\lambda_1; E))$, completing the proof, if we can establish that $d(\eta\eta') = d(\lambda_2) \vee d(\eta)$. But for this, we calculate

$$\begin{aligned}
d(\eta\eta')_i &= d(\lambda_1\eta\eta')_i - d(\lambda_1)_i \\
&= d(\lambda_1\lambda_2\alpha)_i - d(\lambda_1)_i \quad \text{by definition of } \eta, \eta' \\
&= \max\{d(\lambda_1\lambda_2)_i, d(\sigma)_i\} - d(\lambda_1)_i \quad \text{since } (\alpha, \beta) \in \Lambda^{\min}(\lambda_1\lambda_2, \sigma) \\
&= \max\{d(\lambda_1\lambda_2)_i, \max\{d(\lambda_1)_i, d(\sigma)_i\}\} - d(\lambda_1)_i \quad \text{since } d(\lambda_1\lambda_2)_i \geq d(\lambda_1)_i \\
&= \max\{d(\lambda_1)_i + d(\lambda_2)_i, d(\lambda_1\eta)_i\} - d(\lambda)_i \quad \text{since } (\eta, \zeta) \in \Lambda^{\min}(\lambda_1, \sigma) \\
&= \max\{d(\lambda_2)_i, d(\lambda_1\eta)_i - d(\lambda_1)_i\} \\
&= \max\{d(\lambda_2)_i, d(\eta)_i\},
\end{aligned}$$

giving $d(\eta\eta') = d(\lambda_2) \vee d(\eta)$ as required. □

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