# RELATIVE CUNTZ-KRIEGER ALGEBRAS OF FINITELY ALIGNED HIGHER-RANK GRAPHS

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Abstract. We define the relative Cuntz-Krieger algebras associated to finitely aligned higher-rank graphs. We prove versions of the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem for relative Cuntz-Krieger algebras.

## 1. Introduction

Cuntz-Krieger algebras associated to directed graphs and their analogues have been of significant interest recently, due in large part to the explicit relationship between the loop-structure of a graph and the ideal structure of its Cuntz-Krieger algebra.

A directed graph $E$ consists of a collection $E^0$ of vertices, a collection $E^1$ of edges joining the vertices, and maps $r, s : E^1 \to E^0$ which indicate the ranges and sources of the edges. The Cuntz-Krieger algebra of $E$, denoted $C^*(E)$, is the universal algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and by partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal range projections such that for $e \in E^1$, we have $s_e^* s_e = p_{s(e)}$ and such that whenever $v \in E^0$ satisfies $0 < |r^{-1}(v)| < \infty$, we have

$$s_v = \sum_{e \in r^{-1}(v)} s_e s_e^*.$$

The universal property of $C^*(E)$ ensures that it carries a strongly continuous *gauge action* $\gamma$ of $\mathbb{T}^k$ satisfying $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z \cdot s_e$ for all $e \in E^1$ and $v \in E^0$. The gauge-invariant ideal structure of $C^*(E)$ was studied in [2]. Here Bates *et al.* identified the *saturated, hereditary* subsets $H$ of $E^0$, and showed that a large class of gauge-invariant ideals in $C^*(E)$ correspond to subgraphs $r^{-1}(H) \subset E$ where $H$ is saturated and hereditary; the ideal associated to $r^{-1}(H)$ is denoted $I_H$, and contains the Cunz-Krieger algebra $C^*(r^{-1}(H))$ as a full corner. Ideally, the quotient $C^*(E)/I_H$ would be isomorphic to the Cuntz-Krieger algebra $C^*(s^{-1}(E^0 \setminus H))$ of the complementary subgraph. In fact, to realise $C^*(E)/I_H$ as a Cuntz-Krieger algebra, one needs to append sources (that is, vertices $v$ such that $r^{-1}(v)$ is empty) to $s^{-1}(E^0 \setminus H)$ to obtain what is referred to in [2] as the *quotient graph* $E/H$. Using the uniqueness theorems for Cuntz-Krieger algebras, Bates *et al.* show that $C^*(E)/I_H$ is canonically isomorphic to the Cuntz-Krieger algebra $C^*(E/H)$ [2, Proposition 3.4], and thereby identify the remainder of the gauge-invariant ideals

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in $C^*(E)$ [2, Theorem 3.6]. They also produce a condition on $E$ under which all ideals of $C^*(E)$ are gauge-invariant [2, Corollary 3.8].

In recent work [6, Section 3], Muhly and Tomforde study the *relative graph algebras* $C^*(E, V)$ of directed graphs $E$ using a construction which once again involved appending sources to $E$, and show that the Cuntz-Krieger algebra $C^*(E/H)$ of the quotient graph is canonically isomorphic to a relative graph algebra associated to $s^{-1}(E^0 \setminus H)$.

For higher-rank graphs, the situation is more complicated. A higher-rank graph $\Lambda$ can be thought of as a graph in which the paths have a shape or *degree* in $\mathbb{N}^k$ rather than a length in $\mathbb{N}$. Associated to each higher-rank graph $\Lambda$ there is a $C^*$-algebra $C^*(\Lambda)$ generated by partial isometries associated to paths in $\Lambda$ and carrying a strongly continuous gauge action $\gamma$ of $\mathbb{T}^k$. The decompositions of a path in $\Lambda$ must be in bijective correspondence with the decompositions of its degree in $\mathbb{N}^k$; this is called the *factorisation property*. The factorisation property poses significant complications for an analysis of the gauge-invariant ideals of $C^*(\Lambda)$ using methods like those of [2]. The point is that it is not clear how to generalise the quotient graph construction from [2] to the higher-rank setting: because of the factorisation property, the addition of a source locally will have global effects on the higher-rank graph, so it is unclear how to reconcile multiple such operations.

In this paper we analyse the relative Cuntz-Krieger algebras associated to higher-rank graphs $\Lambda$ with a view to studying the gauge-invariant ideal structure of $C^*(\Lambda)$. Since the analysis of relative graph algebras in [6] involves appending sources to graphs, we would face the same difficulties in generalising it to the higher-rank setting as we would face in generalising the quotient graph construction of [2]. Instead, we study the relative Cuntz-Krieger algebras of finitely aligned higher-rank graphs by regarding them as universal objects generated by families of partial isometries. Our main objective is to establish versions of the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem for relative Cuntz-Krieger algebras associated to higher-rank graphs, and we achieve these aims in Theorem 6.1 and Theorem 6.3. The motivation for this is that the rôles played by $C^*(E/H)$ and the usual uniqueness theorems for graph algebras in [2] will be filled by a relative Cuntz-Krieger algebra associated to $s^{-1}(\Lambda^0 \setminus H)$ and the uniqueness theorems Theorem 6.1 and Theorem 6.3 in an analysis of the gauge-invariant ideal structure of $C^*(\Lambda)$ for a finitely aligned $k$-graph $\Lambda$.

In Section 2, we given the definition of a $k$-graph and establish the notation we will need in later sections. In Section 3, we associate a relative Cuntz-Krieger algebra $C^*(\Lambda; \mathcal{E})$ to each pair $\Lambda, \mathcal{E}$ where $\Lambda$ is a finitely aligned $k$-graph, and $\mathcal{E}$ is a collection of *finite exhaustive* subsets of $\Lambda$. We establish the existence of the *core* subalgebra $C^*(\Lambda; \mathcal{E})^\gamma$ which is the fixed-point algebra for the gauge action, and adapt the methods of [9, Section 3] to show that $C^*(\Lambda; \mathcal{E})^\gamma$ is AF. In Section 4, we say what it means for a collection $\mathcal{E}$ of finite exhaustive sets to be *satiated*, and for such $\mathcal{E}$ we use the description of $C^*(\Lambda; \mathcal{E})^\gamma$ obtained in Section 3 to establish elementary conditions on a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family under which it determines an injective homomorphism of $C^*(\Lambda; \mathcal{E})^\gamma$. In Section 5, we show how to produce from an arbitrary collection $\mathcal{E}$ of finite exhaustive sets an enveloping collection $\overline{\mathcal{E}}$ such that $\overline{\mathcal{E}}$ is satiated and $C^*(\Lambda; \mathcal{E}) = C^*(\Lambda; \overline{\mathcal{E}})$. In Section 6, we prove versions of the gauge-invariant and Cuntz-Krieger uniqueness theorems for

$C^*(\Lambda; \mathcal{E})$ when $\mathcal{E}$ is satiated; the results of Section 5 show how to apply these theorems to $C^*(\Lambda; \mathcal{E})$ when $\mathcal{E}$ is not satiated.

## 2. Higher-rank graphs

We regard $\mathbb{N}^k$ as an additive semigroup with identity 0. For $m, n \in \mathbb{N}^k$, we write $m \vee n$ for their coordinate-wise maximum and $m \wedge n$ for their coordinate-wise minimum.

**Definition 2.1.** Let $k \in \mathbb{N} \setminus \{0\}$. A *graph of rank k*, or *k-graph*, is a pair $(\Lambda, d)$ where $\Lambda$ is a countable category and $d$ is a functor from $\Lambda$ to $\mathbb{N}^k$ which satisfies the *factorisation property*: For all $\lambda \in \mathrm{Mor}(\Lambda)$ and all $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m+n$, there exist unique morphisms $\mu$ and $\nu$ in $\mathrm{Mor}(\Lambda)$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

Since we are regarding $k$-graphs as generalised graphs, we refer to elements of $\mathrm{Mor}(\Lambda)$ as *paths* and to elements of $\mathrm{Obj}(\Lambda)$ as *vertices* and we write $r$ and $s$ for the codomain and domain maps.

The factorisation property allows us to identify $\mathrm{Obj}(\Lambda)$ with $\{\lambda \in \mathrm{Mor}(\Lambda) : d(\lambda) = 0\}$. So we write $\lambda \in \Lambda$ in place of $\lambda \in \mathrm{Mor}(\Lambda)$, and when $d(\lambda) = 0$, we regard $\lambda$ as a vertex of $\Lambda$.

Given $\lambda \in \Lambda$ and $E \subset \Lambda$, we define $\lambda E := \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}$ and $E\lambda := \{\mu\lambda : \mu \in E, s(\mu) = r(\lambda)\}$. In particular if $d(v) = 0$, then $v$ is a vertex of $\Lambda$ and $vE = \{\lambda \in E : r(\lambda) = v\}$; similarly, $Ev = \{\lambda \in \Lambda : s(\lambda) = v\}$. We write

$$\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\}.$$

The factorisation property ensures that if $l \leq m \leq n \in \mathbb{N}^k$ and if $d(\lambda) = n$, then there exist unique paths denoted $\lambda(0, l)$, $\lambda(l, m)$ and $\lambda(m, n)$ such that $d(\lambda(0, l)) = l$, $d(\lambda(l, m)) = m-l$, and $d(\lambda(m, n)) = n-m$ and such that $\lambda = \lambda(0, l)\lambda(l, m)\lambda(m, n)$.

Given $k \in \mathbb{N} \setminus \{0\}$, and $k$-graphs $(\Lambda_1, d_1)$ and $(\Lambda_2, d_2)$, we call a covariant functor $x : \Lambda_1 \to \Lambda_2$ a *graph morphism* if it satisfies $d_2 \circ x = d_1$.

**Definition 2.2.** Let $(\Lambda, d)$ be a $k$-graph. Given $\mu, \nu \in \Lambda$, we say that $\lambda$ is a *minimal common extension* of $\mu$ and $\nu$ if $d(\lambda) = d(\mu) \vee d(\nu)$, $\lambda(0, d(\mu)) = \mu$, and $\lambda(0, d(\nu)) = \nu$. We denote the collection of all minimal common extensions of $\mu$ and $\nu$ by $\mathrm{MCE}(\mu, \nu)$. We write $\Lambda^{\min}(\mu, \nu)$ for the collection

$$\Lambda^{\min}(\mu, \nu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \mu\alpha = \nu\beta \in \mathrm{MCE}(\mu, \nu)\}.$$

If $E \subset \Lambda$ and $\mu \in \Lambda$, then we write $\mathrm{Ext}(\mu; E)$ for the set

$$\mathrm{Ext}(\mu; E) := \{\alpha \in s(\mu)\Lambda : (\alpha, \beta) \in \Lambda^{\min}(\mu, \nu) \text{ for some } \nu \in E\}$$

of extensions of $\mu$ with respect to $E$. We say that $\Lambda$ is finitely aligned if $\mathrm{MCE}(\mu, \nu)$ is finite (possibly empty) for all $\mu, \nu \in \Lambda$.

Let $v \in \Lambda^0$ and $E \subset v\Lambda$. We say that $E$ is *exhaustive* if $\mathrm{Ext}(\lambda; E)$ is nonempty for all $\lambda \in v\Lambda$.

**Lemma 2.3.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, let $v \in \Lambda^0$, let $E \subset v\Lambda$ be finite and exhausitve, and let $\mu \in v\Lambda$. Then $\mathrm{Ext}(\mu; E)$ is a finite exhaustive subset of $s(\mu)\Lambda$. Moreover $\mu \in E\Lambda$ if and only if $s(\mu) \in \mathrm{Ext}(\mu; E)$.*

*Proof.* Let $E' := \mathrm{Ext}(\mu; E)$. Since $E$ is finite and $\Lambda$ is finitely aligned we know that $E'$ is finite, and $E' \subset s(\mu)\Lambda$ by definition, so we need only check that $E'$ is exhaustive. Let $\sigma \in s(\mu)\Lambda$. Since $E$ is exhaustive, there exists $\lambda \in E$ with $\Lambda^{\min}(\lambda, \mu\sigma) \neq \emptyset$, say $(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu\sigma)$. So $\lambda\alpha = \mu\sigma\beta$. Setting $\tau := (\mu\sigma\beta)(d(\mu), d(\lambda) \vee d(\mu))$, we have $\tau \in \operatorname{Ext}(\mu;\{\lambda\}) \subset E'$ by the factorisation property, and $\mu\sigma\beta = \mu\tau\tau'$ for some $\tau'$. But then the factorisation property gives $\sigma\beta = \tau\tau'$, so $(\sigma\beta)(0, d(\sigma) \vee d(\tau)) \in \operatorname{MCE}(\sigma,\tau)$. Since $\sigma \in s(\mu)\Lambda$ was arbitrary, it follows that $E'$ is exhaustive. The last statement of the lemma follows from the factorisation property. □

## 3. Relative Cuntz-Krieger algebras

**Notation 3.1.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph. We define

$$\operatorname{FE}(\Lambda) := \textstyle\bigcup_{v\in\Lambda^0}\{E \subset v\Lambda \setminus \{v\} : \ E \text{ is finite and exhaustive}\}.$$

For $E \in \operatorname{FE}(\Lambda)$ we write $r(E)$ for the vertex $v \in \Lambda^0$ such that $E \subset v\Lambda$.

**Definition 3.2.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and let $\mathcal{E}$ be a subset of $\operatorname{FE}(\Lambda)$. A *relative Cuntz-Krieger $(\Lambda;\mathcal{E})$-family* is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in a $C^*$-algebra satisfying

(TCK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(TCK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(TCK3) $t_\lambda^* t_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$ for all $\lambda,\mu \in \Lambda$; and
(CK) $\prod_{\lambda\in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$ for all $E \in \mathcal{E}$.

*Remark* 3.3. Relation (CK) is well-defined because (TCK3) ensures that the projections $\{t_\lambda t_\lambda^* : \lambda \in \Lambda\}$ pairwise commute. Note also that (TCK3) together with the $C^*$-identity show that $t_v \neq 0$ for all $v \in \Lambda^0$ if and only if $t_\lambda \neq 0$ for all $\lambda \in \Lambda$.

For each finitely aligned $k$-graph $\Lambda$, and each subset $\mathcal{E}$ of $\operatorname{FE}(\Lambda)$ there exists a $C^*$-algebra $C^*(\Lambda;\mathcal{E})$ generated by a relative Cuntz-Krieger $(\Lambda;\mathcal{E})$-family $\{s_\mathcal{E}(\lambda) : \lambda \in \Lambda\}$ which is universal in the sense that if $\{t_\lambda : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda;\mathcal{E})$-family in a $C^*$-algebra $B$, then there exists a unique homomorphism $\pi_t^\mathcal{E} : C^*(\Lambda;\mathcal{E}) \to B$ such that $\pi_t^\mathcal{E}(s_\mathcal{E}(\lambda)) = t_\lambda$ for all $\lambda \in \Lambda$.

For $z = (z_1,\dots,z_k) \in \mathbb{T}^k$ and $n = (n_1,\dots,n_k) \in \mathbb{N}^k$, we write $z^n$ for the product $\prod_{i=1}^k z_i^{n_i} \in \mathbb{T}$. The universal property of $C^*(\Lambda;\mathcal{E})$ guarantees that there exists a strongly continuous *gauge action* $\gamma$ of $\mathbb{T}^k$ on $C^*(\Lambda;\mathcal{E})$ such that $\gamma_z(s_\mathcal{E}(\lambda)) = z^{d(\lambda)}s_\mathcal{E}(\lambda)$ for all $\lambda \in \Lambda$. Averaging over this gauge action gives a faithful conditional expectation $\Phi_\mathcal{E}^\gamma$ from $C^*(\Lambda;\mathcal{E})$ to the fixed point algebra

$$C^*(\Lambda;\mathcal{E})^\gamma = \overline{\operatorname{span}}\{s_\mathcal{E}(\lambda)s_\mathcal{E}(\mu)^* : d(\lambda) = d(\mu)\}.$$

We refer to $C^*(\Lambda;\mathcal{E})^\gamma$ as the *core* of $C^*(\Lambda;\mathcal{E})$. The remainder of this section is devoted to showing that $C^*(\Lambda;\mathcal{E})^\gamma$ is AF. This material is adapted directly from [9, Section 3].

Recall from [9] that for a finitely aligned $k$-graph $(\Lambda, d)$ and a finite subset $E \subset \Lambda$, the set $\Pi E$ is the smallest subset of $\Lambda$ such that $E \subset \Pi E$ and such that

$$\lambda,\mu,\sigma \in G \text{ with } d(\lambda) = d(\mu) \text{ and } s(\lambda) = s(\mu) \text{ implies } \lambda\operatorname{Ext}(\mu;\{\sigma\}) \subset G. \tag{3.1}$$

We write $\Pi E \times_{d,s} \Pi E$ for the set $\{(\lambda,\mu) \in \Pi E \times \Pi E : d(\lambda) = d(\mu), s(\lambda) = s(\mu)\}$.

Lemma 3.2 of [9] shows that $\Pi E$ is finite, that if $\lambda,\mu \in \Pi E \times_{d,s} \Pi E$, then for $\nu \in s(\lambda)\Lambda$, we have $\lambda\nu \in \Pi E$ if and only if $\mu\nu \in \Pi E$, and that if $\lambda,\mu \in \Pi E$ and $(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)$, then $\lambda\alpha \in \Pi E$.

**Definition 3.4.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph, let $E \subset \Lambda$ be finite, let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ and let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. We define

$$M^t_{\Pi E} := \operatorname{span}\{t_\lambda t_\mu^* : (\lambda, \mu) \in \Pi E \times_{d,s} \Pi E\}.$$

For $\lambda, \mu \in \Pi E \times_{d,s} \Pi E$, we define

$$\Theta(t)^{\Pi E}_{\lambda,\mu} := t_\lambda \Big( \prod_{\lambda\nu \in \Pi E \setminus \{\lambda\}} (t_{s(\lambda)} - t_\nu t_\nu^*) \Big) t_\mu^*. \tag{3.2}$$

It is straightforward to check that Lemmas 3.2 and 3.11 and Proposition 3.9 of [9] apply to any family of partial isometries satisfying (TCK1)–(TCK3); for details see [10, Chapter 3]. Hence each $M^t_{\Pi E}$ is a finite-dimensional $C^*$-algebra [9, Lemma 3.2]. Moreover, for all $(\lambda, \mu), (\sigma, \tau) \in \Pi E \times_{d,s} \Pi E$, we have

$$\big(\Theta(t)^{\Pi E}_{\lambda,\mu}\big)^* = \Theta(t)^{\Pi E}_{\mu,\lambda} \quad \text{and} \quad \Theta(t)^{\Pi E}_{\lambda,\mu} \Theta(t)^{\Pi E}_{\sigma,\tau} = \delta_{\mu,\sigma} \Theta(t)^{\Pi E}_{\lambda,\tau} \tag{3.3}$$

by [9, Proposition 3.9], and for $\lambda, \mu \in \Pi E \times_{d,s} \Pi E$, we have

$$t_\lambda t_\mu^* = \textstyle\sum_{\lambda\nu \in \Pi E} \Theta(t)^{\Pi E}_{\lambda\nu,\mu\nu} \tag{3.4}$$

by [9, Lemma 3.11].

We can now show that the core is AF, and give a condition under which a representation of $C^*(\Lambda; \mathcal{E})$ is faithful on the core.

**Proposition 3.5.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and let $\mathcal{E}$ be a subset of $\mathrm{FE}(\Lambda)$. Then $C^*(\Lambda; \mathcal{E})^\gamma$ is an AF algebra. If $\{t_\lambda : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family, then $\pi^{\mathcal{E}}_t$ is injective on $C^*(\Lambda; \mathcal{E})^\gamma$ if and only if $\Theta(t)^{\Pi E}_{\lambda,\mu}$ is nonzero whenever $\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu}$ is nonzero.*

*Proof.* We have $C^*(\Lambda; \mathcal{E})^\gamma = \bigcup_{E \subset \Lambda \text{ finite}} M^{s_{\mathcal{E}}}_{\Pi E}$, so $C^*(\Lambda; \mathcal{E})^\gamma$ is an AF algebra. If $E$ is a finite subset of $\Lambda$, then (3.3) shows that $\{\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu} : (\lambda, \mu) \in \Pi E \times_{d,s} \Pi E, \Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu} \neq 0\}$ is a collection of nonzero matrix units, and (3.4) shows that these matrix units span $M^{s_{\mathcal{E}}}_{\Pi E}$. So if $\Theta(t)^{\Pi E}_{\lambda,\mu}$ is nonzero whenever $\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu}$ is nonzero, then $\pi^{\mathcal{E}}_t$ is injective on each $M^{s_{\mathcal{E}}}_{\Pi E}$, and hence on $C^*(\Lambda; \mathcal{E})^\gamma$ by [1, Lemma 1.3]. □

## 4. Nonzero matrix units and the $\mathcal{E}$-compatible boundary path representation

In this section, we identify the *satiated* subsets of $\mathrm{FE}(\Lambda)$, and when $\mathcal{E}$ is satiated, we characterise the $\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu}$ which are nonzero in $C^*(\Lambda; \mathcal{E})$.

**Definition 4.1.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph. We say that a subset $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ is *satiated* if it satisfies

- (S1) If $G \in \mathcal{E}$ and $E \in \mathrm{FE}(\Lambda)$ with $G \subset E$ then $E \in \mathcal{E}$.
- (S2) If $G \in \mathcal{E}$ with $r(G) = v$ and if $\mu \in v\Lambda \setminus G\Lambda$, then $\mathrm{Ext}(\mu; G) \in \mathcal{E}$.
- (S3) If $G \in \mathcal{E}$ and $0 < n_\lambda \le d(\lambda)$ for each $\lambda \in G$, then $\{\lambda(0, n_\lambda) : \lambda \in G\} \in \mathcal{E}$.
- (S4) If $G \in \mathcal{E}$, $G' \subset G$, and $G'_\lambda \in \mathcal{E}$ with $r(G'_\lambda) = s(\lambda)$ for each $\lambda \in G'$, then $\big((G \setminus G') \cup \big(\bigcup_{\lambda \in G'} \lambda G'_\lambda\big)\big) \in \mathcal{E}$.

The remainder of this section is devoted to proving the following theorem.

**Theorem 4.2.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and suppose that $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ is satiated. Let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. The homomorphism $\pi^{\mathcal{E}}_t$ is injective on $C^*(\Lambda; \mathcal{E})^\gamma$ if and only if*

(1) $t_v \neq 0$ *for all* $v \in \Lambda^0$, *and*
(2) $\prod_{\lambda \in F}(t_{r(F)} - t_\lambda t_\lambda^*) \neq 0$ *for all* $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$.

To prove Theorem 4.2, we construct a representation of $C^*(\Lambda; \mathcal{E})$ which satisfies conditions (1) and (2). As usual, we obtain this representation by definining an appropriate boundary-path space.

Recall from [8] that for $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, the $k$-graph $\Omega_{k,m}$ has vertices $\{n \in \mathbb{N}^k : n \leq m\}$, morphisms $\{(n_1, n_2) : n_1, n_2 \in \mathbb{N}^k, n_1 \leq n_2 \leq m\}$, degree map $d((n_1, n_2)) = n_2 - n_1$ and range and source maps $r((n_1, n_2)) = n_1$, $s((n_1, n_2)) = n_2$.

**Definition 4.3.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ be satiated. We say that a graph morphism $x : \Omega_{k,m} \to \Lambda$ is an *$\mathcal{E}$-compatible boundary path* of $\Lambda$ if for every $n \in \mathbb{N}^k$ such that $n \leq m$, and every $E \in \mathcal{E}$ such that $r(E) = x(n)$, there exists $\lambda \in E$ such that $x(n, n + d(\lambda)) = \lambda$. We denote the collection of all $\mathcal{E}$-compatible boundary paths of $\Lambda$ by $\partial(\Lambda; \mathcal{E})$. We write $d(x)$ for $m$ and $r(x)$ for $x(0)$.

If $x \in \partial(\Lambda; \mathcal{E})$ and $\lambda \in \Lambda r(x)$ then there is a unique graph morphism $\lambda x : \Omega_{k,d(\lambda)+d(x)} \to \Lambda$ such that $(\lambda x)(0, d(\lambda)) = \lambda$ and $(\lambda x)(d(\lambda), n + d(\lambda)) = x(0, n)$ for all $n \leq d(x)$. Likewise, if $n \in \mathbb{N}^k$ with $n \leq d(x)$, there is a unique graph morphism $x|_n^{d(x)} : \Omega_{k,d(x)-n} \to \Lambda$ such that $x|_n^{d(x)}(0, m) = x(n, n+m)$ whenever $n + m \leq d(x)$. These two constructions are inverse to each other in the sense that

$$(\lambda x)|_{d(\lambda)}^{d(\lambda x)} = x = \big(x(0, n)\big)\big(x|_n^{d(x)}\big) \quad \text{for all } \lambda \in \Lambda r(x) \text{ and all } n \leq d(x). \tag{4.1}$$

**Lemma 4.4.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ be satiated. Let $x \in \partial(\Lambda; \mathcal{E})$. If $n \in \mathbb{N}^k$ with $n \leq d(x)$, then $x|_n^{d(x)} \in \partial(\Lambda; \mathcal{E})$, and if $\lambda \in \Lambda r(x)$, then $\lambda x \in \partial(\Lambda; \mathcal{E})$.*

*Proof.* For the first statement, just note that each vertex on $x|_n^{d(x)}$ is also a vertex on $x$. For the second statement, suppose $n \in \mathbb{N}^k$ with $n \leq d(\lambda x)$, and suppose $E \in \mathcal{E}$ with $r(E) = (\lambda x)(n)$. Let $\lambda' = (\lambda x)(n, n \vee d(\lambda))$, and let $x' = x|_{(n \vee d(\lambda)) - d(\lambda)}^{d(x)}$, so that $(\lambda x)|_n^{d(\lambda x)} = \lambda' x'$, and $x' \in \partial(\Lambda; \mathcal{E})$ by the first statement of the lemma. We must show that there exists $\mu \in E$ such that $(\lambda' x')(0, d(\mu)) = \mu$. If there exists $\mu \in E$ with $d(\mu) \leq d(\lambda')$ and $\lambda'(0, d(\mu)) = \mu$, we are done, so we may assume that $\lambda' \notin E\Lambda$. By (S2), we have $\mathrm{Ext}(\lambda', E) \in \mathcal{E}$, and $r(\mathrm{Ext}(\lambda', E)) = s(\lambda') = r(x')$ by definition. Since $x' \in \partial(\Lambda; \mathcal{E})$, it follows that there exists $\alpha \in \mathrm{Ext}(\lambda'; E)$ such that $x'(0, d(\alpha)) = \alpha$; equivalently, there exists $\mu \in E$ and $(\alpha, \beta) \in \Lambda^{\min}(\lambda', \mu)$ such that $\alpha = x'(0, d(\alpha))$. But now $\lambda' \alpha = \mu \beta$, and in particular, $(\lambda' x')(0, d(\mu)) = \big(\lambda' x'(0, d(\alpha))\big)(0, d(\mu)) = (\mu\beta)(0, d(\mu)) = \mu$. □

**Definition 4.5.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ be satiated. Define partial isometries $\{S_\mathcal{E}(\lambda) : \lambda \in \Lambda\} \subset \mathcal{B}(\ell^2(\partial(\Lambda; \mathcal{E})))$ by

$$S_\mathcal{E}(\lambda) e_x := \begin{cases} e_{\lambda x} & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 4.6.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ be satiated. The collection $\{S_\mathcal{E}(\lambda) : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family which we call the* $\mathcal{E}$-compatible boundary-path representation *of* $\Lambda$.

*Proof.* First notice that Lemma 4.4 ensures that for $\lambda \in \Lambda$ and $x \in \partial(\Lambda; \mathcal{E})$, we have

$$S_{\mathcal{E}}(\lambda)^* e_x = \begin{cases} e_{x|^{d(x)}_{d(\lambda)}} & \text{if } x(0, d(\lambda)) = \lambda \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

For $v \in \Lambda^0$, we have that $S_{\mathcal{E}}(v)$ is the projection onto $\overline{\operatorname{span}}\{e_x : x \in v\partial(\Lambda; \mathcal{E})\}$ and hence $\{S_{\mathcal{E}}(v) : v \in \Lambda^0\}$ are mutually orthogonal projections, establishing (TCK1). Since composition in the category $\Lambda$ is associative, (TCK2) is straightforward to check. To see (TCK3) one uses (4.2) to apply both $S_{\mathcal{E}}(\lambda)^* S_{\mathcal{E}}(\mu)$ and $\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_{\mathcal{E}}(\alpha) S_{\mathcal{E}}(\beta)^*$ to an arbitrary basis element $e_x$; calculations like those of [7, Example 7.4] show that the two agree. Finally, for condition (CK) let $x \in \partial(\Lambda; \mathcal{E})$ and $E \in \mathcal{E}$ with $r(E) = r(x) = x(0)$. Then $x(0, n) \in E$ for some $n \leq d(x)$ by definition of $\partial(\Lambda; \mathcal{E})$, and we have $\big(S_{\mathcal{E}}(x(0)) - S_{\mathcal{E}}(x(0,n)) S_{\mathcal{E}}(x(0,n))^*\big) e_x = 0$ by (4.1) and (4.2). Since $\big(S_{\mathcal{E}}(x(0)) - S_{\mathcal{E}}(x(0,n)) S_{\mathcal{E}}(x(0,n))^*\big)$ is a term in $\prod_{\lambda\in E} \big(S_{\mathcal{E}}(r(E)) - s_{\mathcal{E}}(\lambda) s_{\mathcal{E}}(\lambda)^*\big)$, it follows that the kernel of the latter contains $e_x$. Since $x \in \partial(\Lambda; \mathcal{E})$ and $E \in \mathcal{E}$ were arbitrary, this establishes (CK). □

**Lemma 4.7.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, let $\mathcal{E}$ be a satiated subset of* $\operatorname{FE}(\Lambda)$*, and let $v \in \Lambda^0$. Then*

(1) *$v\partial(\Lambda; \mathcal{E})$ is nonempty.*
(2) *If $F \in v\operatorname{FE}(\Lambda) \setminus \mathcal{E}$, then $v\partial(\Lambda; \mathcal{E}) \setminus F\partial(\Lambda; \mathcal{E})$ is nonempty.*

To prove Lemma 4.7, we first need the following technical lemma.

**Lemma 4.8.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and suppose that $\mathcal{E} \subset \operatorname{FE}(\Lambda)$ is satiated. Suppose that $E \in \mathcal{E}$ and that $F \subset r(E)\Lambda \setminus \Lambda^0$ is finite and satisfies*

$$\mu \in E \implies \begin{cases} \mu \in F\Lambda & \text{or} \\ \operatorname{Ext}(\mu; F) \in \mathcal{E}. \end{cases}$$

*Then $F \in \mathcal{E}$.*

*Proof.* Define $G := E \setminus F\Lambda$, and for each $\mu \in G$, let $G_\mu := \operatorname{Ext}(\mu; F)$. Then each $G_\mu \in \mathcal{E}$ by hypothesis, so (S4) gives

$$E' := \big((E \setminus G) \cup \big(\textstyle\bigcup_{\mu\in G} \mu G_\mu\big)\big) \in \mathcal{E}.$$

For $\lambda \in E \setminus G$ we have $\lambda(0, n) \in F$ for some n; in this case, let $n_\lambda := n$. Since $E \in \mathcal{E}$ we have $E \cap \Lambda^0 = \emptyset$ and hence $n_\lambda > 0$. For $\mu \in G$ and $\lambda \in \mu G_\mu$, we have $\lambda = \mu\beta$ for some $\beta \in \operatorname{Ext}(\mu; F)$, so there exists $\sigma \in F$ and $\alpha \in \Lambda$ such that $(\alpha, \beta) \in \Lambda^{\min}(\sigma, \mu)$. Hence

$$\lambda(0, d(\sigma)) = (\mu\beta)(0, d(\sigma)) = (\sigma\alpha)(0, d(\sigma)) = \sigma \in F;$$

in this case, set $n_\lambda := d(\sigma)$. Since $F \cap \Lambda^0 = 0$ by hypothesis, we have that $n_\lambda > 0$. Now $E'' := \{\lambda(0, n_\lambda) : \lambda \in E'\} \subset F$. But $E' \in \mathcal{E}$, and hence (S3) ensures that $E'' \in \mathcal{E}$. Since $E'' \subset F$ and $F$ is finite, it now follows from (S1) that $F \in \mathcal{E}$. □

To prove Lemma 4.7, we also need the following a result due to Farthing, Muhly and Yeend.

**Lemma 4.9** (Farthing, Muhly and Yeend, 2003)**.** *Let $(\Lambda, d)$ be a $k$-graph. For $v \in \Lambda^0$, $E \subset v\Lambda$, $\lambda_1 \in v\Lambda$, and $\lambda_2 \in s(\lambda_1)\Lambda$, we have* $\operatorname{Ext}(\lambda_2; \operatorname{Ext}(\lambda_1; E)) = \operatorname{Ext}(\lambda_1\lambda_2; E)$.

*Proof.* The result is proved in [3], currently in draft form; a proof also appears in [10, Appendix A]. □

*Proof of Lemma 4.7.* The proofs of both statements of Lemma 4.7 proceed by constructing an $\mathcal{E}$-relative boundary path with the desired properties. The two constructions have a great deal in common, but the construction for statement (2) is somewhat more complicated. To avoid duplication, we present the full text of the proof of statement (2) below, but we typeset those parts of the proof which are germane only to statement (2) in slanted text, and enclose them in square brackets $\big[$*like this*$\big]$.

Define $P : (\mathbb{N} \setminus \{0\})^2 \to (\mathbb{N} \setminus \{0\})$ by

$$P(m, n) := \frac{(m + n - 1)(m + n - 2)}{2} + m.$$

Then $P$ is the position function corresponding to the diagonal listing of $(\mathbb{N} \setminus \{0\})^2$ in the sense that if $(m, n)$ is the $l^{\text{th}}$ term in the diagonal listing, then $P(m, n) = l$. For all $l \in \mathbb{N} \setminus \{0\}$, define $(i_l, j_l) := P^{-1}(l)$. Fix $v \in \Lambda^0$ $\big[$*and fix* $F \in v\,\mathrm{FE}(\Lambda) \setminus \mathcal{E}\big]$.

We claim that there exist a sequence $\{\lambda_l : l \geq 1\} \subset v\Lambda$ and listings $\{E_{l,j} : j \geq 1\}$ of $s(\lambda_l)\mathcal{E}$ for all $l \geq 1$ satisfying

(i) $\lambda_{l+1}(0, d(\lambda_l)) = \lambda_l$ for all $l \geq 1$,
(ii) $\lambda_{l+1}\big(d(\lambda_{i_l}), d(\lambda_{l+1})\big)$ belongs to $E_{i_l,j_l}\Lambda$ for all $l \geq 1$.
$\big[$*(iii)* $\mathrm{Ext}(\lambda_{l+1}; F)$ *belongs to* $\mathrm{FE}(\Lambda) \setminus \mathcal{E}$ *for all* $l \geq 0.\big]$

We prove the claim by induction on $l$. For a basis case, let $l = 0$ and define $\lambda_{l+1} = \lambda_1 := v$. For each $w \in \Lambda^0$, the collection of finite subsets of $w\Lambda$ is countable because $\Lambda$ is countable. In particular, $w\mathcal{E}$ is countable. Let $\{E_{1,j} : j \in \mathbb{N} \setminus \{0\}\}$ be any listing of $v\mathcal{E}$. Note that (i) and (ii) are trivial in this case because $l = 0$ $\big[$*and (iii) is satisfied because* $\mathrm{Ext}(v; F) = F\big]$.

Now suppose as an inductive hypothesis that $l \geq 1$, and that $\lambda_n$ and $\{E_{n,j} : j \geq 1\}$ exist and satisfy (i) and (ii) $\big[$*and (iii)*$\big]$ for $1 \leq n \leq l$.

Let $\lambda^l_{i_l} := \lambda_l\big(d(\lambda_{i_l}), d(\lambda_l)\big)$. Notice that $i_l < l$, so $E_{i_l,j_l} \in s(\lambda_{i_l})\mathcal{E}$ has already been defined by the inductive hypothesis. Suppose first that $\lambda^l_{i_l}$ belongs to $E_{i_l,j_l}\Lambda$. Define $\lambda_{l+1} := \lambda_l$, and $E_{i_{l+1},j} := E_{i_l,j}$ for all $j \geq 1$. We have that $\lambda_{l+1}$ satisfies (i) by definition, and satisfies (ii) because we supposed $\lambda^l_{i_l}$ to belong to $E_{i_l,j_l}\Lambda$. $\big[$*We have that* $\lambda_{l+1}$ *satisfies (iii) because* $\lambda_l$ *satisfies (iii) by the inductive hypothesis.*$\big]$

Now suppose that $\lambda^l_{i_l}$ does not belong to $E_{i_l,j_l}\Lambda$. Let $E := \mathrm{Ext}(\lambda^l_{i_l}; E_{i_l,j_l})$. Then $E \neq \emptyset$ because $E_{i_l,j_l} \in \mathrm{FE}(\Lambda)$. For $\alpha \in E$ we have that $\lambda^l_{i_l}\alpha = \mu\beta$ for some $\mu \in E$ and $(\alpha, \beta) \in \Lambda^{\min}(\lambda^l_{i_l}, \mu)$. It follows that for any $\nu_{l+1}$ in $E$ we have that $\lambda_{l+1} := \lambda_l\nu_{l+1}$ satisfies (ii). Such a choice of $\lambda_{l+1}$ trivially satisfies (i).

$\big[$*To complete the construction of* $\lambda_{l+1}$, *we need only show that there exists a choice of* $\nu_{l+1} \in E$ *such that* $\lambda_{l+1} := \lambda_l\nu_{l+1}$ *also satisfies (iii). Since* $\lambda_l$ *satisfies (iii), we have that* $F_l := \mathrm{Ext}(\lambda_l; F)$ *belongs to* $\mathrm{FE}(\Lambda) \setminus \mathcal{E}$. *By the contrapositive of Lemma 4.8, there exists* $\alpha \in E \setminus F_l\Lambda$ *such that* $\mathrm{Ext}(\alpha; F_l) \notin \mathcal{E}$. *But Lemma 2.3 ensures that* $\mathrm{Ext}(\alpha; F_l) \in \mathrm{FE}(\Lambda)$, *so* $\mathrm{Ext}(\alpha; F_l) \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$. *Let* $\nu_{l+1} := \alpha$, *and define* $\lambda_{l+1} := \lambda_l\nu_{l+1}$. *Then*

$$\mathrm{Ext}(\lambda_{l+1}; F) = \mathrm{Ext}(\lambda_l\nu_{l+1}; F) = \mathrm{Ext}(\nu_{l+1}; \mathrm{Ext}(\lambda_l; F)) \tag{4.3}$$

*by Lemma 4.9. But* $\operatorname{Ext}(\lambda_l; F) = F_l$ *by definition, so* (4.3) *gives* $\operatorname{Ext}(\lambda_{l+1}; F) = \operatorname{Ext}(\nu_{l+1}; F_l)$ *which belongs to* $\operatorname{FE}(\Lambda) \setminus \mathcal{E}$ *by choice of* $\nu_{l+1}$*. Hence* $\lambda_{l+1}$ *satisfies (iii) as required.*]

Let $m := \lim_{l\to\infty} d(\lambda_l) \in (\mathbb{N} \cup \{\infty\})^k$. Since $\{\lambda_l : l \geq 1\}$ satisfies (i), there exists a unique graph morphism $x : \Omega_{k,m} \to \Lambda$ such that $x(0, d(\lambda_l)) = \lambda_l$ for all $l \in \mathbb{N} \setminus \{0\}$.

We have that $r(x) = v$ by definition, so to see that $x \in v\partial(\Lambda; \mathcal{E})$, suppose that $M \in \mathbb{N}^k$ with $M \leq m$. Let $E \in x(M)\mathcal{E}$. We must show that there exists $N \geq M$ such that $x(M, N) \in E$. By definition of $x$ there exists $l \geq 1$ such that $M \leq d(\lambda_l)$. If $\lambda_l(M, d(\lambda_l))$ belongs to $E\Lambda$, then we are done, so suppose that $\lambda_l(M, d(\lambda_l)) \notin E\Lambda$. By (S3), it follows that $G := \operatorname{Ext}(\lambda_l(M, d(\lambda_l)); E) \in s(\lambda_l)\mathcal{E}$, and hence that $G = E_{i_l,j}$ for some $j \geq 1$. But then property (ii) ensures that $\lambda_{P(i_l,j)+1}(M, N) \in E$ for some $N$, and it follows that $x(M, N) \in E$ as required.

[*Finally we must show that* $x \notin F\Lambda$*. Suppose for contradiction that* $x \in F\Lambda$*. Then* $x(0, N) \in F$ *for some* $N$*, and it follows from the definition of* $x$ *that there exists* $l \geq 1$ *such that* $\lambda_l(0, N) = x(0, N) \in F$*. Hence* $s(\lambda_l)$ *belongs to* $\operatorname{Ext}(\lambda_l; F)$*. But for* $G \in \operatorname{FE}(\Lambda)$*, we have* $G \cap \Lambda^0 = \emptyset$ *by definition, so* $s(\lambda_l) \in \operatorname{Ext}(\lambda_l; F)$ *contradicts (iii). Hence* $x \notin F\Lambda$*.*] □

**Corollary 4.10.** *Let* $(\Lambda, d)$ *be a finitely aligned* $k$*-graph, and let* $\mathcal{E} \subset \operatorname{FE}(\Lambda)$ *be satiated. The vertex projections* $s_{\mathcal{E}}(v)$ *are all nonzero. Moreover, if* $E \subset v\Lambda \setminus \Lambda^0$ *is finite, then* $\prod_{\lambda\in E}(s_{\mathcal{E}}(v) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*) = 0$ *if and only if* $E \in \mathcal{E}$*.*

To prove Corollary 4.10, we make use of an equality established in [9]: let $(\Lambda, d)$ be a finitely aligned $k$-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a collection of partial isometries satisfying (TCK1)–(TCK3), let $v$ be an element of $\Lambda^0$, let $E$ be a finite subset of $v\Lambda$, and let $\mu$ be an element of $v\Lambda$. Then [9, Equation (3.4)] shows that

$$\Big(\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*)\Big)t_\mu t_\mu^* = t_\mu\Big(\prod_{\alpha\in\operatorname{Ext}(\mu;E)}(t_{s(\mu)} - t_\alpha t_\alpha^*)\Big)t_\mu^*, \tag{4.4}$$

with the convention that the empty product is equal to the unit of the multiplier algebra so that if $\operatorname{Ext}(\mu; E) = \emptyset$ then the right-hand side of (4.4) is equal to $t_\mu t_\mu^*$.

*Proof of Corollary 4.10.* Statement (1) of Lemma 4.7 shows that for $v \in \Lambda^0$, there exists $x \in v\partial(\Lambda; \mathcal{E})$, and then $S_{\mathcal{E}}(v)e_x = e_x \neq 0$. So $S_{\mathcal{E}}(v)$ is nonzero, and the universal property of $C^*(\Lambda; \mathcal{E})$ then shows that $s_{\mathcal{E}}(v)$ is nonzero.

For the second statement of the Corollary, the "if" direction is precisely (CK). For the reverse implication, suppose that $E \subset v\Lambda$ but $E \notin \mathcal{E}$. If $E \notin \operatorname{FE}(\Lambda)$ then there exists $\xi \in v\Lambda$ such that $\operatorname{Ext}(\xi; E) = \emptyset$. Equation 4.4 shows that

$$s_{\mathcal{E}}(\xi)s_{\mathcal{E}}(\xi)^* \prod_{\lambda\in E}(s_{\mathcal{E}}(v) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*) = s_{\mathcal{E}}(\xi)s_{\mathcal{E}}(\xi)^*,$$

and hence $\prod_{\lambda\in E}(s_{\mathcal{E}}(v) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*)$ is nonzero by Remark 3.3. On the other hand, if $E \in \operatorname{FE}(\Lambda) \setminus \mathcal{E}$, then statement (2) of Lemma 4.7 shows that there exists $x \in v\partial(\Lambda; \mathcal{E}) \setminus E\partial(\Lambda; \mathcal{E})$. We then have $\prod_{\lambda\in E}(S_{\mathcal{E}}(v) - S_{\mathcal{E}}(\lambda)S_{\mathcal{E}}(\lambda)^*)e_x = e_x \neq 0$, and then the universal property of $C^*(\Lambda; \mathcal{E})$ gives $\prod_{\lambda\in E}(s_{\mathcal{E}}(v) - s_{\mathcal{E}}(\lambda)s_{\mathcal{E}}(\lambda)^*) \neq 0$. □

*Proof of Theorem 4.2.* The "only if" implication follows from Corollary 4.10. Equation (3.2) and Corollary 4.10 show that for $E \subset \Lambda$ finite and $(\lambda, \mu) \in \Pi E \times_{d,s} \Pi E$, we have $\Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu} = 0$ if and only if $T^{\Pi E}(\lambda) := \{\nu \in \Lambda \setminus \Lambda^0 : \lambda\nu \in \Pi E\}$ belongs to $\mathcal{E}$. Hence for the "if" direction it suffices to establish that if $\Theta(t)^{\Pi E}_{\lambda,\mu} = 0$, then $T^{\Pi E}(\lambda)$ belongs to $\mathcal{E}$; indeed, by (3.2), it suffices to show that if $T^{\Pi E}(\lambda) \notin \mathcal{E}$, then $\prod_{\nu \in T^{\Pi E}(\lambda)}(t_{s(\lambda)} - t_\nu t_\nu^*) \neq 0$.

So suppose $T^{\Pi E}(\lambda) \notin \mathcal{E}$. If $T^{\Pi E}(\lambda) \notin \mathrm{FE}(\Lambda)$, then $\prod_{\nu \in T^{\Pi E}(\lambda)}(t_{s(\lambda)} - t_\nu t_\nu^*) \neq 0$ exactly as in the proof of Corollary 4.10. On the other hand, if $T^{\Pi E}(\lambda) \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$, then $\prod_{\nu \in T^{\Pi E}(\lambda)}(t_{s(\lambda)} - t_\nu t_\nu^*) \neq 0$ by assumption. □

## 5. Constructing satiations

In this section we show how to use Theorem 4.2 to characterise the homomorphisms of arbitrary relative Cuntz-Krieger algebras $C^*(\Lambda; \mathcal{E})$ which are injective on the core, and not just those for which $\mathcal{E}$ is satiated.

**Definition 5.1.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. We write $\overline{\mathcal{E}}$ for the smallest satiated subset of $\mathrm{FE}(\Lambda)$ which contains $\mathcal{E}$, and we call $\overline{\mathcal{E}}$ the *satiation* of $\mathcal{E}$.

The idea is to show that for any $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, we have $C^*(\Lambda; \mathcal{E}) = C^*(\Lambda; \overline{\mathcal{E}})$. To this end we define maps $\Sigma_1$–$\Sigma_4$ on subsets of $\mathrm{FE}(\Lambda)$, and show that iterated application of these maps produces $\overline{\mathcal{E}}$ from $\mathcal{E}$.

**Definition 5.2.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and for $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, define

$$\begin{aligned}
\Sigma_1(\mathcal{E}) &= \{F \subset \Lambda \setminus \Lambda^0 : F \text{ is finite, and there exists } E \in \mathcal{E} \text{ with } E \subset F\} \\
\Sigma_2(\mathcal{E}) &= \{\mathrm{Ext}(\mu; E) : E \in \mathcal{E}, \mu \in r(E)\Lambda \setminus E\Lambda\} \\
\Sigma_3(\mathcal{E}) &= \big\{\{\lambda(0, n_\lambda) : \lambda \in E\} : E \in \mathcal{E}, 0 < n_\lambda \le d(\lambda) \text{ for all } \lambda \in E\big\} \\
\Sigma_4(\mathcal{E}) &= \big\{(E \setminus F) \cup \big(\textstyle\bigcup_{\lambda \in F} \lambda F_\lambda\big) : E \in \mathcal{E}, F \subset E, \\
&\qquad\qquad\qquad\qquad F_\lambda \in s(\lambda)\mathcal{E} \text{ for all } \lambda \in F\big\}.
\end{aligned}$$

**Lemma 5.3.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Then $\mathcal{E} \subset \Sigma_i(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$ for $1 \le i \le 4$. Let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family and let $E \in \Sigma_i(\mathcal{E})$ for $1 \le i \le 4$. Then $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$.*

*Proof.* Let $E \in \mathcal{E}$. We trivially have $E \in \Sigma_1(\mathcal{E})$. To see that $E \in \Sigma_2(\mathcal{E})$, note that $r(E) \notin E\Lambda$ by definition, and $E = \mathrm{Ext}(r(E); E)$. To see that $E \in \Sigma_3(\mathcal{E})$, just take $n_\lambda := d(\lambda)$ for all $\lambda \in E$. Finally, to see that $E \in \Sigma_4(\mathcal{E})$, take $F = \emptyset \subset E$.

We will now establish that if $\{t_\lambda : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family and $E \in \Sigma_i(\mathcal{E})$, then $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$.

If $i = 1$, then $E = G \cup F$ for some $G \in \mathcal{E}$ and finite $F \subset r(G)\Lambda$, and $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) \le \prod_{\lambda \in G}(t_{r(G)} - t_\lambda t_\lambda^*) = 0$.

If $i = 2$, then $E = \mathrm{Ext}(\mu; G)$ for some $G \in \mathcal{E}$ and $\mu \in r(G)\Lambda \setminus G\Lambda$. So multiplying (4.4) by $t_\mu^*$ on the left and by $t_\mu$ on the right gives

$$\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = t_\mu^* \Big( \prod_{\sigma \in G}(t_{r(G)} - t_\sigma t_\sigma^*) \Big) t_\mu = 0.$$

If $i = 3$, then $E = \{\lambda(0, n_\lambda) : \lambda \in G\}$ for some $G \in \mathcal{E}$ and $0 < n_\lambda \le d(\lambda)$ for each $\lambda \in G$. Since $t_{r(E)} - t_{\lambda(0,n_\lambda)} t_{\lambda(0,n_\lambda)}^* \le t_{r(E)} - t_\lambda t_\lambda^*$ for all $\lambda \in E$, we then have

$$\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) \le \prod_{\mu \in G}(t_{r(G)} - t_\mu t_\mu^*) = 0.$$

If $i = 4$, then $E = G \setminus G' \cup \big(\bigcup_{\lambda \in G'} \lambda G'_\lambda\big)$ for some $G \in \mathcal{E}$, $G' \subset G$, and $G'_\lambda \in s(\lambda)\mathcal{E}$ for each $\lambda \in G'$. Lemma C.7 of [9] shows that for $\lambda \in G'$, we have $t_{r(G)} - t_\lambda t_\lambda^* = \prod_{\mu \in G'_\lambda}(t_{r(G)} - t_{\lambda\mu} t_{\lambda\mu}^*)$. Hence

$$\begin{aligned}\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) &= \Big(\prod_{\lambda \in G \setminus G'}(t_{r(E)} - t_\lambda t_\lambda^*)\Big) \prod_{\lambda \in G'} \Big(\prod_{\mu \in G'_\lambda}(t_{r(E)} - t_{\lambda\mu} t_{\lambda\mu}^*)\Big)\\ &= \prod_{\lambda \in G}(t_{r(G)} - t_\lambda t_\lambda^*).\end{aligned}$$

It remains only to show that $\Sigma_i(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$. For this, first notice that $E \in \Sigma_i(\mathcal{E})$ implies that $E \cap \Lambda^0 = \emptyset$ and that $E$ is finite by definition of $\Sigma_1$–$\Sigma_4$. Now let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family in which $t_v \neq 0$ for all $v \in \Lambda^0$; such a family exists by Corollary 4.10. Suppose that $v \in \Lambda^0$ and that $E$ is a finite subset of $v\Lambda \setminus \Lambda^0$ with $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = 0$, and suppose for contradiction that $E \notin \mathrm{FE}(\Lambda)$. Then there exists $\mu \in v\Lambda$ such that $\Lambda^{\min}(\mu; \lambda) = \emptyset$ for all $\lambda \in E$. Equation 4.4 gives $t_\mu t_\mu^* \prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = t_\mu t_\mu^*$, and hence $t_\mu t_\mu^* = 0$, contradicting $t_v \neq 0$ for all $v \in \Lambda^0$. Since we have already established that if $\{t_\lambda : \lambda \in \Lambda\}$ is a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family and $E \in \Sigma_i(\mathcal{E})$, then $\prod_{\lambda \in E}(t_{r(E)} - t_\lambda t_\lambda^*) = 0$, it follows that $\Sigma_i(\mathcal{E}) \subset \mathrm{FE}(\Lambda)$ as required. □

**Notation 5.4.** We write $\Sigma$ for the map $\Sigma_4 \circ \Sigma_3 \circ \Sigma_2 \circ \Sigma_1$. For $n \in \mathbb{N}$ and $\mathcal{E} \subset \mathrm{FE}(\Lambda)$, we write $\Sigma^n(\mathcal{E})$ for

$$\overbrace{\Sigma \circ \Sigma \circ \cdots \circ \Sigma}^{n \text{ terms}}(\mathcal{E}),$$

and write $\Sigma^\infty(\mathcal{E})$ for $\bigcup_{n=1}^\infty \Sigma^n(\mathcal{E})$.

**Proposition 5.5.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$. Then $\Sigma^\infty(\mathcal{E}) = \overline{\mathcal{E}}$.*

*Proof.* The definitions of the maps $\Sigma_1$–$\Sigma_4$ show that $\Sigma_i(\mathcal{E}) \subset \overline{\mathcal{E}}$ for all $i$, and hence that $\Sigma^\infty(\mathcal{E}) \subset \overline{\mathcal{E}}$. Hence, it suffices to show that $\Sigma^\infty(\mathcal{E})$ is satiated. If $G \in \Sigma^\infty(\mathcal{E})$ and $E$ is constructed from $G$ as in (S1), (S2) or (S3), then we have $G \in \Sigma^n(\mathcal{E})$ for some $n \in \mathbb{N}$, and then since we have $\mathcal{E} \subset \Sigma_i(\mathcal{E})$ for all $i$ by Lemma 5.3, it follows that $E \in \Sigma^{n+1}(\mathcal{E}) \subset \Sigma^\infty(\mathcal{E})$ as required. If $F \in \Sigma^\infty(\mathcal{E})$, $G \subset F$, and $G_\lambda \in s(\lambda)\Sigma^\infty(\mathcal{E})$ for all $\lambda \in G$, then there exist $n \in \mathbb{N}$ with $F \in \Sigma^n(\mathcal{E})$, and $n_\lambda \in \mathbb{N}$ such that $G_\lambda \in \Sigma^{n_\lambda}(\mathcal{E})$ for each $\lambda \in \Lambda$. Let $N := \max\{n, n_\lambda : \lambda \in G\}$. Again since Lemma 5.3 shows that $\mathcal{E} \subset \Sigma_i(\mathcal{E})$ for all $i$, we have that $F$ and each $G_\lambda$ belong to $\Sigma^N(\mathcal{E})$. The definition of $\Sigma_4$ together with another application of Lemma 5.3 shows that $E \in \Sigma^{N+1}(\mathcal{E}) \subset \Sigma^\infty(\mathcal{E})$, and the proof is complete. □

**Corollary 5.6.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and let $\mathcal{E}$ be any subset of $\mathrm{FE}(\Lambda)$. Then $C^*(\Lambda; \mathcal{E}) = C^*(\Lambda; \overline{\mathcal{E}})$.*

*Proof.* An induction on $n$ using the last statement of Lemma 5.3 shows that if $F \in \Sigma^n(\mathcal{E})$, then $\prod_{\mu \in F}(t_{r(F)} - t_\mu t_\mu^*) = 0$ for all $n \in \mathbb{N}$. Hence $\prod_{\mu \in F}(t_{r(F)} - t_\mu t_\mu^*) = 0$ for all $F \in \overline{\mathcal{E}}$ by Proposition 5.5. It follows that every relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family is a relative Cuntz-Krieger $(\Lambda; \overline{\mathcal{E}})$-family. On the other hand $\mathcal{E} \subset \overline{\mathcal{E}}$ by definition, so every Cuntz-Krieger $(\Lambda; \overline{\mathcal{E}})$-family is trivially a relative Cuntz-Krieger $(\Lambda; \mathcal{E})$-family. The universal properties of $C^*(\Lambda; \mathcal{E})$ and $C^*(\Lambda; \overline{\mathcal{E}})$ now show that the two algebras coincide. □

## 6. Uniqueness theorems

In this section we prove versions of the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem for $C^*(\Lambda;\mathcal{E})$.

**Theorem 6.1.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ be satiated. Let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda;\mathcal{E})$-family in a $C^*$-algebra $B$, and suppose that*

(1) *$t_v \neq 0$ for all $v \in \Lambda^0$;*
(2) *$\prod_{\lambda\in F}(t_{r(F)} - t_\lambda t_\lambda^*) \neq 0$ for all $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$; and*
(3) *there exists an action $\theta : \mathbb{T}^k \to \mathrm{Aut}(B)$ such that $\theta_z(t_\lambda) = z^{d(\lambda)}t_\lambda$ for all $z \in \mathbb{T}^k$ and $\lambda \in \Lambda$.*

*Then $\pi_t^{\mathcal{E}}$ is injective.*

*Proof.* Theorem 4.2 and Conditions (1) and (2) guarantee that $\pi_t^{\mathcal{E}}$ is injective on $C^*(\Lambda;\mathcal{E})^\gamma$. Assume without loss of generality that $B = C^*(\{t_\lambda : \lambda \in \Lambda\})$. Since the polynomials are continuous on $\mathbb{T}^k$, and since $B = \overline{\mathrm{span}}\{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda\}$ by (TCK3), we have that $\theta$ is strongly continuous. Since $\pi_t^{\mathcal{E}}$ is equivariant in $\theta$ and $\gamma$, averaging over $\theta$ gives a norm-decreasing linear map $\Phi_{\mathcal{E}}^\theta$ on $B$ which satisfies $\Phi_{\mathcal{E}}^\theta \circ \pi_t^{\mathcal{E}} = \pi_t^{\mathcal{E}} \circ \Phi_{\mathcal{E}}^\gamma$. The result now follows from an argument identical to that of [9, Proposition 4.1]. □

To state our Cuntz-Krieger uniqueness theorem, we first need to establish some notation.

**Definition 6.2.** Let $(\Lambda, d)$ be a $k$-graph, and let $x : \Omega_{k,d(x)} \to \Lambda$ and $y : \Omega_{k,d(y)} \to \Lambda$ be graph morphisms. We say that a graph morphism $z : \Omega_{k,d(z)} \to \Lambda$ is a *minimal common extension* of $x$ and $y$ if it satisfies

(1) $d(z)_j = \max\{d(x)_j, d(y)_j\}$ for $1 \leq j \leq k$; and
(2) $z|_{\Omega_{k,d(x)}} = x$ and $z|_{\Omega_{k,d(y)}} = y$.

We write $\mathrm{MCE}(x, y)$ for the collection of minimal common extensions of $x$ and $y$.

It turns out that to obtain a Cuntz-Krieger uniqueness theorem for relative Cuntz-Krieger algebras, the appropriate analogue of an aperiodic path is a path $x \in \partial(\Lambda;\mathcal{E})$ such that

$$\text{for distinct } \lambda, \mu \in \Lambda r(x), \text{ we have } \mathrm{MCE}(\lambda x, \mu x) = \emptyset. \tag{6.1}$$

**Theorem 6.3.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph and let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ be satiated. Suppose that $(\Lambda,\mathcal{E})$ satisfies*

$$\begin{aligned}&\text{For all } v \in \Lambda^0 \text{ there exists } x \in v\partial(\Lambda;\mathcal{E}) \text{ satisfying (6.1)},\\ &\quad\text{and for all } v \in \Lambda^0 \text{ and } F \in v\,\mathrm{FE}(\Lambda) \setminus \mathcal{E} \text{ there exists}\\ &\quad x \in v\partial(\Lambda;\mathcal{E}) \setminus F\partial(\Lambda;\mathcal{E}) \text{ satisfying (6.1)}.\end{aligned} \tag{C}$$

*Let $\{t_\lambda : \lambda \in \Lambda\}$ be a relative Cuntz-Krieger $(\Lambda;\mathcal{E})$-family such that $t_v \neq 0$ for all $v \in \Lambda^0$, and $\prod_{\lambda\in F}(t_{r(F)} - t_\lambda t_\lambda^*) \neq 0$ for all $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$. Then $\pi_t^{\mathcal{E}}$ is injective.*

The remainder of the section is devoted to proving Theorem 6.3. We first need some technical lemmas.

**Lemma 6.4.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and suppose that $x : \Omega_{k,d(x)} \to \Lambda$ is a graph morphism satisfying (6.1). Suppose that $\lambda \neq \mu$ with $s(\lambda) = s(\mu) = r(x)$. Then there exists $n^x_{\lambda,\mu} \in \mathbb{N}^k$ such that*

$$n^x_{\lambda,\mu} \leq d(x) \quad \text{and} \quad \Lambda^{\min}(\lambda x(0, n^x_{\lambda,\mu}), \mu x(0, n^x_{\lambda,\mu})) = \emptyset.$$

*Proof.* Suppose for contradiction that for all $n \in \mathbb{N}^k$ with $n \leq d(x)$, we have $\Lambda^{\min}(\lambda x(0,n), \mu x(0,n)) \neq \emptyset$.

For each $i \in \mathbb{N}$, define $n(i) \in \mathbb{N}^k$ by $n(i)_j := \min\{d(x)_j, i\}$. By assumption, there exists $(\alpha_i, \beta_i) \in \Lambda^{\min}(\lambda x(0,n(i)), \mu x(0,n(i)))$ for each $i \in \mathbb{N}$; since $\Lambda^{\min}(\lambda, \mu)$ is finite, there must exist a pair $(\eta_1, \zeta_1)$ belonging to $\Lambda^{\min}(\lambda, \mu)$ and an infinite subset $I_1 \subset \mathbb{N}$ such that for all $i \in I_1$,

$$(\lambda x(0,n(i))\alpha_i)(0, d(\lambda) \vee d(\mu)) = \lambda\eta_1 = \mu\zeta_1.$$

Set $i_1 := \min I_1$. For each $j \in I_1$ with $j > i_1$, we have

$$\begin{aligned}\big(\lambda x(0,n(j))\alpha_j\big)&\big(0, d(\lambda x(0,n(i_1))) \vee d(\mu x(0,n(i_1)))\big)\\ &= \big(\lambda x(0,n(j))\alpha_j\big)\big(0, (d(\lambda) \vee d(\mu)) + n(i_1)\big)\\ &\in \mathrm{MCE}(\lambda x(0,n(i_1)), \mu x(0,n(i_1))).\end{aligned}$$

Since $\mathrm{MCE}(\lambda x(0,n(i_1)), \mu x(0,n(i_1)))$ is finite, there exists a pair $(\eta_2, \zeta_2)$ belonging to $\Lambda^{\min}(\lambda x(0,n(i_1)), \mu x(0,n(i_1)))$ and an infinite subset $I_2 \subset I_1 \setminus \{i_1\}$ such that for each $i \in I_2$, we have

$$\big(\lambda x(0,n(j))\alpha_j\big)\big(0, (d(\lambda) \vee d(\mu)) + n(i)\big) = \lambda x(0,n(i_1))\eta_2 = \mu x(0,n(i_1))\zeta_2.$$

Since $I_2 \subset I_1$, a straightforward calculation using the fact that $\lambda x(0,n(i_1))\eta_2$ is an initial segment of $\lambda x(0,n(i))\alpha_i$ for any $i \in I_2$ shows that

$$(\lambda x(0,n(i_1))\eta_2)(0, d(\lambda) \vee d(\mu)) = \lambda\eta_1 = \mu\zeta_1.$$

Set $i_2 := \min I_2$. Iterating this procedure, we obtain a sequence

$$\{\sigma_l := \lambda x(0,n(i_l))\eta_{l+1} : l \in \mathbb{N}\}$$

such that $d(\sigma_l) = (d(\lambda) \vee d(\mu)) + n(i_l)$, and $\sigma_{l+1}(0, d(\sigma_l)) = \sigma_l$ for all $l$. There is a unique graph morphism $y : \Omega_{k,d(y)} \to \Lambda$ such that

$$d(y) = \lim_{l\to\infty} (d(\lambda) \vee d(\mu)) + n(i_l) = d(\lambda x) \vee d(\mu x),$$

and $y(0, d(\sigma_l)) = \sigma_l$ for all $l$. We then have

$$\begin{aligned}y(0, d(\lambda) + n(i_l)) &= \sigma_l(0, d(\lambda) + n(i_l))\\ &= \big(\lambda x(0,n(i_l))\eta_{l+1}\big)(0, d(\lambda) + n(i_l)) = \lambda x(0,n(i_l)).\end{aligned}$$

Since $n(i_l) \to d(x)$, it follows that $y|_{\Omega_{k,d(\lambda)+d(x)}} = \lambda x$. Similarly, $y|_{\Omega_{k,d(\mu)+d(x)}} = \mu x$. It follows that $y \in \mathrm{MCE}(\lambda x, \mu x)$, contradicting (6.1). □

**Lemma 6.5.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ be satiated, and suppose that $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$. Let $x \in r(F)\partial(\Lambda; \mathcal{E}) \setminus F\partial(\Lambda; \mathcal{E})$, and let $n \in \mathbb{N}^k$ with $n \leq d(x)$. Then $\mathrm{Ext}(x(0,n); F) \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$.*

*Proof.* By Lemma 2.3, we have $\mathrm{Ext}(x(0,n); F) \in \mathrm{FE}(\Lambda)$. Suppose for contradiction that $\mathrm{Ext}(x(0,n); F) \in \mathcal{E}$. Since $x \in \partial(\Lambda; \mathcal{E})$, there exists $m > n$ such that $m \leq d(x)$ and $x(n,m) \in \mathrm{Ext}(x(0,n); F)$. So there exists $\lambda \in F$ and $\alpha \in s(\lambda)\Lambda$ such that $(\alpha, x(n,m)) \in \Lambda^{\min}(\lambda, x(0,n))$. But then $x(0,m) = x(0,n)x(n,m) = \lambda\alpha$, contradicting the assumption that $x$ does not belong to $F\partial(\Lambda; \mathcal{E})$. □

**Corollary 6.6.** *Let $(\Lambda, d)$ be a finitely aligned $k$-graph, let $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ be satiated, and suppose that $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$. Let $x \in r(F)\partial(\Lambda; \mathcal{E}) \setminus F\partial(\Lambda; \mathcal{E})$, and let $n \in \mathbb{N}^k$*

*with* $n \le d(x)$*. Let* $\{t_\lambda : \lambda \in \Lambda\}$ *be a relative Cuntz-Krieger* $(\Lambda; \mathcal{E})$*-family such that* $t_v \neq 0$ *for all* $v \in \Lambda^0$*, and* $\prod_{\lambda \in F}(t_{r(F)} - t_\lambda t_\lambda^*) \neq 0$ *for all* $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$*. Then*

$$\prod_{\lambda \in F}(t_{r(F)} - t_\lambda t_\lambda^*) t_{x(0,n)} t_{x(0,n)}^* = t_{x(0,n)} \Big( \prod_{\beta \in \mathrm{Ext}(x(0,n);F)} (t_{x(n)} - t_\beta t_\beta^*) \Big) t_{x(0,n)}^*,$$

*and in particular is nonzero.*

*Proof.* The displayed equation is an instance of (4.4). Lemma 6.5 ensures that $\mathrm{Ext}(x(0,n); F) \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$, and then $\prod_{\beta \in \mathrm{Ext}(x(0,n);F)}(t_{x(n)} - t_\beta t_\beta^*) \neq 0$ by hypothesis. □

**Lemma 6.7.** *Let* $(\Lambda, d)$ *be a finitely aligned* $k$*-graph, let* $\mathcal{E} \subset \mathrm{FE}(\Lambda)$ *be satiated, and suppose that* $(\Lambda, \mathcal{E})$ *satisfies condition* (C)*. Let* $\{t_\lambda : \lambda \in \Lambda\}$ *be a relative Cuntz-Krieger* $(\Lambda; \mathcal{E})$*-family such that* $t_v \neq 0$ *for all* $v \in \Lambda^0$*, and* $\prod_{\lambda \in F}(t_{r(F)} - t_\lambda t_\lambda^*) \neq 0$ *for all* $F \in \mathrm{FE}(\Lambda) \setminus \mathcal{E}$*. Let* $\pi_t^{\mathcal{E}}$ *be the representation of* $C^*(\Lambda; \mathcal{E})$ *determined by* $\pi_t^{\mathcal{E}}(s_{\mathcal{E}}(\lambda)) = t_\lambda$*. Let* $a \in \mathrm{span}\{s_{\mathcal{E}}(\lambda) s_{\mathcal{E}}(\mu)^* : \lambda, \mu \in \Lambda\} \subset C^*(\Lambda; \mathcal{E})$*. Then* $\|\pi_t^{\mathcal{E}}(\Phi^\gamma(a))\| \le \|\pi_t^{\mathcal{E}}(a)\|$*.*

*Proof.* Express $a = \sum_{\lambda,\mu \in \Pi E} a_{\lambda,\mu} s_{\mathcal{E}}(\lambda) s_{\mathcal{E}}(\mu)^*$ for some finite $E \subset \Lambda$, and express $\Phi^\gamma(a) = \sum_{(\lambda,\mu) \in \Pi E \times_{d,s} \Pi E} b_{\lambda,\mu} \Theta(s_{\mathcal{E}})^{\Pi E}_{\lambda,\mu}$; so we have

$$\pi_t^{\mathcal{E}}(a) = \sum_{\lambda,\mu \in \Pi E} a_{\lambda,\mu} t_\lambda t_\mu^* \quad \text{and} \quad \pi_t^{\mathcal{E}}(\Phi^\gamma(a)) = \sum_{(\lambda,\mu) \in \Pi E \times_{d,s} \Pi E} b_{\lambda,\mu} \Theta(t)^{\Pi E}_{\lambda,\mu}.$$

Since the $\Theta(t)^{\Pi E}_{\lambda,\mu}$ are matrix units, there exists $n$ in $d(\Pi E)$ and $v \in s(\Pi E \cap \Lambda^n)$ such that

$$\|\pi_t^{\mathcal{E}}(\Phi^\gamma(a))\| = \Big\| \sum_{\lambda,\mu \in (\Pi E) v \cap \Lambda^n} b_{\lambda,\mu} \Theta(t)^{\Pi E}_{\lambda,\mu} \Big\|.$$

Write $T^{\Pi E}(n,v)$ for $T^{\Pi E}(\lambda)$ where $\lambda \in (\Pi E) v \cap \Lambda^n$, so

$$T^{\Pi E}(n,v) = \{\nu \in \Lambda \setminus \Lambda^0 : \lambda\nu \in \Pi E \text{ for any } \lambda \in (\Pi E) v \cap \Lambda^n\}.$$

Equation (3.1) ensures that $T^{\Pi E}_{n,v}$ is well-defined. If $T^{\Pi E}(n,v)$ belongs to $\mathcal{E}$, then we must have $\Phi^\gamma(a) = 0$ in which case the result is trivial. So suppose that $T^{\Pi E}(n,v) \notin \mathcal{E}$. We claim that there exists $x \in v\partial(\Lambda; \mathcal{E}) \setminus T^{\Pi E}(n,v)\partial(\Lambda; \mathcal{E})$ satisfying (6.1). To see this, note that if $T^{\Pi E}(n,v) \in \mathrm{FE}(\Lambda)$, then such an $x$ exists because $(\Lambda, \mathcal{E})$ satisfies condition (C), whereas if $T^{\Pi E}(n,v) \notin \mathrm{FE}(\Lambda)$, then there exists $\sigma \in v\Lambda$ with $\mathrm{Ext}(\sigma; T^{\Pi E}(n,v)) = \emptyset$, and condition (C) gives $x' \in s(\sigma)\Lambda$ satisfying (6.1); it is then easy to check that $x := \sigma x'$ also satisfies (6.1) and does not have an initial segment in $T^{\Pi E}(n,v)$.

For all $\lambda \in \Pi E$ with $d(\lambda) \le n$, $\mu \in (\Pi E)s(\lambda)$ with $\mu \neq \lambda$, and $\nu \in s(\lambda)\Lambda$ such that $\lambda\nu \in (\Pi E)v \cap \Lambda^n$, the factorisation property ensures that $\lambda\nu \neq \mu\nu$. Hence Lemma 6.4 shows that there exists $n^x_{\lambda\nu,\mu\nu} \in \mathbb{N}^k$ with $n^x_{\lambda\nu,\mu\nu} \le d(x)$ such that $\Lambda^{\min}\big(\lambda\nu x(0, n^x_{\lambda\nu,\mu\nu}), \mu\nu x(0, n^x_{\lambda\nu,\mu\nu})\big) = \emptyset$. Define

$$N := \bigvee \{n^x_{\lambda\nu,\mu\nu} : \lambda, \mu \in \Pi E, d(\lambda) \neq d(\mu), \lambda\nu \in (\Pi E)v \cap \Lambda^n\}.$$

Since each $n^x_{\lambda\nu,\mu\nu} \le d(x)$, we have $N \le d(x)$, and for each $\lambda, \mu, \nu$ as above, we have that $\Lambda^{\min}(\lambda\nu x(0,N), \mu\nu x(0,N)) = \emptyset$.

Define projections $P_1$ and $P_2$ by

$$P_1 := \sum_{\lambda \in (\Pi E)v \cap \Lambda^n} \Theta(t)^{\Pi E}_{\lambda,\lambda} \quad \text{and} \quad P_2 := \sum_{\lambda \in (\Pi E)v \cap \Lambda^n} t_{\lambda x(0,N)} t^*_{\lambda x(0,N)}.$$

We have $P_1\pi_t^{\mathcal{E}}(\Phi^\gamma(a)) = \sum_{\lambda,\mu\in(\Pi E)v\cap\Lambda^n} b_{\lambda,\mu}\Theta(t)^{\Pi E}_{\lambda,\mu}$ and hence $\|P_1\pi_t^{\mathcal{E}}(\Phi^\gamma(a))\| = \|\pi_t^{\mathcal{E}}(\Phi^\gamma(a))\|$. For $\lambda,\sigma\in(\Pi E)v\cap\Lambda^n$, we have $t^*_{\sigma x(0,N)}t_\lambda = \delta_{\sigma,\lambda}t^*_{x(0,N)}$ by (TCK3), and so for $\lambda,\mu\in(\Pi E)v\cap\Lambda^n$, we have

$$\begin{aligned}P_2\Theta(t)^{\Pi E}_{\lambda,\mu}P_2 &= P_2t_\lambda\Big(\prod_{\nu\in T^{\Pi E}(n,v)}(t_v - t_\nu t_\nu^*)\Big)t_\mu^*P_2\\ &= t_{\lambda x(0,N)}t^*_{x(0,N)}\Big(\prod_{\nu\in T^{\Pi E}(n,v)}(t_v - t_\nu t_\nu^*)\Big)t_{x(0,N)}t^*_{\mu x(0,N)}\\ &= t_{\lambda x(0,N)}\Big(\prod_{\beta\in\mathrm{Ext}(x(0,N);T^{\Pi E}(n,v))} t_{x(N)} - t_\beta t_\beta^*\Big)t^*_{\mu x(0,N)}\end{aligned}$$

by Corollary 6.6; Corollary 6.6 also shows that this last expression is nonzero. For $\lambda\in(\Pi E)v\cap\Lambda^n$, we have $t_{\lambda x(0,N)}\in\Lambda^{n+N}$, and it follows that for $\lambda,\mu\in(\Pi E)v\cap\Lambda^n$, we have $t^*_{\lambda x(0,N)}t_{\mu x(0,N)} = \delta_{\lambda,\mu}t_{x(n)}$. Hence

$$\big\{P_2\Theta(t)^{\Pi E}_{\lambda,\mu}P_2 : \lambda,\mu\in(\Pi E)v\cap\Lambda^n\big\}$$

is a collection of nonzero matrix units, and compression by $P_2$ therefore implements an isomorphism of $M^t_{\Pi E}(n,v)$. It follows that

$$\|P_2(P_1\pi_t^{\mathcal{E}}(\Phi^\gamma(a)))P_2\| = \|P_1\pi_t^{\mathcal{E}}(\Phi^\gamma(a))\| = \|\pi_t^{\mathcal{E}}(\Phi^\gamma(a))\|.$$

On the other hand, we have $\|P_2(P_1\pi_t^{\mathcal{E}}(a))P_2)\| \le \|\pi_t^{\mathcal{E}}(a)\|$ because $P_1$ and $P_2$ are projections. Thus, the proof of Lemma 6.7 will be complete if we can establish that $P_2(P_1\pi_t^{\mathcal{E}}(a))P_2) = P_2(P_1\pi_t^{\mathcal{E}}(\Phi^\gamma(a)))P_2)$. To do this, it suffices to show that if $\lambda,\mu\in\Pi E$ with $d(\lambda)\ne d(\mu)$ and $s(\lambda)=s(\mu)$, we have $P_2(P_1t_\lambda t_\mu^*)P_2 = 0$. To see this, fix $\lambda,\mu\in\Pi E$ with $d(\lambda)\ne d(\mu)$ and $s(\lambda)=s(\mu)$, and calculate

$$\begin{aligned}P_2P_1t_\lambda t_\mu^*P_2 &= P_2\Big(\sum_{\lambda\nu\in(\Pi E)v\cap\Lambda^n} t_{\lambda\nu}\Big(\prod_{\substack{\lambda\nu\sigma'\in\Pi E\\ d(\sigma')>0}}(t_{s(\nu)} - t_{\sigma'}t^*_{\sigma'})\Big)t^*_{\mu\nu}\Big)P_2\\ &= \sum_{\lambda\nu\in(\Pi E)v\cap\Lambda^n}\Big(\Big(\prod_{\substack{\lambda\nu\sigma'\in\Pi E\\ d(\sigma')>0}}(t_{\lambda\nu}t^*_{\lambda\nu} - t_{\lambda\nu\sigma'}t^*_{\lambda\nu\sigma'})\Big)P_2t_{\lambda\nu}t^*_{\mu\nu}P_2\Big),\end{aligned}$$

because (TCK3) ensures that the projections $\{t_\lambda t_\lambda^* : \lambda\in\Lambda\}$ pairwise commute. So it suffices to show that $P_2t_{\lambda\nu}t^*_{\mu\nu}P_2 = 0$ for all $\nu$ such that $\lambda\nu\in(\Pi E)v\cap\Lambda^n$. Fix such a $\nu$. We have that $\sigma,\tau\in\Lambda^n$ implies $t_\sigma^*t_\tau = \delta_{\sigma,\tau}t_{s(\sigma)}$ by (TCK3). It follows that for $\sigma\in(\Pi E)v\cap\Lambda^n$, we have

$$t^*_{\sigma x(0,N)}t_{\lambda\nu} = t^*_{x(0,N)}t^*_\sigma t_{\lambda\nu} = \delta_{\sigma,\lambda\nu}t^*_{x(0,N)}.$$

Consequently, $P_2t_{\lambda\nu} = t_{\lambda\nu x(0,N)}t^*_{x(0,N)}$. Hence

$$\begin{aligned}P_2t_{\lambda\nu}t^*_{\mu\nu}P_2 &= t_{\lambda\nu x(0,N)}t^*_{x(0,N)}t^*_{\mu\nu}P_2\\ &= t_{\lambda\nu x(0,N)}\sum_{\tau\in(\Pi E)v\cap\Lambda^n} t^*_{\mu\nu x(0,N)}t_{\tau x(0,N)}t^*_{\tau x(0,N)}.\end{aligned}$$

Since $d(\mu)\ne d(\lambda)$, we have $d(\mu\nu)\ne d(\tau\nu)$, and hence $\mu\nu\ne\tau\nu$ for each $\tau\in(\Pi E)v\cap\Lambda^n$. It follows that $\Lambda^{\min}\big(\mu\nu x(0,N),\tau\nu x(0,N)\big)=\emptyset$ for all $\tau\in(\Pi E)v\cap\Lambda^n$ by our choices of $x$ and $N$. Hence the final line of the above calculation is equal to zero by (TCK3), proving the Lemma. $\square$

*Proof of Theorem 6.3.* Lemma 6.7 shows that the formula

$$t_\lambda t_\mu^* \mapsto \delta_{d(\lambda),d(\mu)} t_\lambda t_\mu^*$$

extends to a norm-decreasing linear map $\Phi^t$ on $\pi_t^{\mathcal{E}}(C^*(\Lambda;\mathcal{E}))$. Replacing $\Phi^\theta$ with $\Phi^t$ in the proof Theorem 6.1 now establishes the result. □

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