AMENABILITY FOR FELL BUNDLES OVER GROUPOIDS

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Abstract. We establish conditions under which the universal and reduced norms coincide for a Fell bundle over a groupoid. Specifically, we prove that the full and reduced C^*-algebras of any Fell bundle over a measurewise amenable groupoid coincide, and also that for a groupoid G whose orbit space is T_0, the full and reduced algebras of a Fell bundle over G coincide if the full and reduced algebras of the restriction of the bundle to each isotropy group coincide.

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Introduction

If G is an amenable group, then the reduced crossed product and full crossed product for any action of G on a C^*-algebra coincide. This result was proved for discrete groups by Zeller-Meyer in [20] and in general by Takai in [18]. Since the C^*-algebra of a Fell bundle over a groupoid G is a very general sort of crossed product by G, it is reasonable to expect the universal norm and reduced norm to coincide on $\Gamma_c(G;\mathscr{B})$ when G is suitably amenable. Immediately the situation is complicated because amenability for groupoids is not as clear cut as it is for groups. There are three reasonable notions of amenability for a second countable locally compact Hausdorff groupoid: (topological) amenability, measurewise amenability and, for lack of a better term, "metric amenability" by which we simply mean that the reduced norm and universal norm on $C_c(G)$ coincide.

Amenability implies measurewise amenability which in turn implies metric amenability. While there are situations where the converses hold, it is unknown if they hold in general. Our main result here, Theorem 1, is that if G is measurewise amenable as defined in [1], then the reduced norm and universal norm on $\Gamma_c(G;\mathscr{B})$ coincide

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for any Fell bundle $\mathscr{B}$ over G. This result subsumes the usual result for group dynamical systems and the result for groupoid dynamical systems; for a discussion of this, see [16, Examples 10 and 11]. The result for groupoid systems is also asserted in [1, Proposition 6.1.10] where they cite [15, Theorem 3.6]. Since it is usually hard to determine if a groupoid found in the wild is amenable in any given one the three flavors mentioned above, we also prove in Theorem 4 that groupoids which act nicely on their unit spaces in the sense that that $G\backslash G^{(0)}$ is T_0 and whose stability groups are all amenable are themselves measurewise amenable. This result may be known to experts, but seems worth advertising. We also show that if G is a groupoid whose orbit space is T_0 and if $\mathscr{B}$ is a Fell bundle over G such that the full and reduced C^*-algebras of the restriction of $\mathscr{B}$ to each isotropy group in G coincide, then the full and reduced C^*-algebras of the whole bundle coincide. This is a formally stronger result than the combination of Theorem 4 and Theorem 1: there are many examples of Fell bundles over non-amenable groups whose full and reduced C^*-algebras coincide (see, for example, [3]).

We start with very short sections on Fell bundles and amenable groupoids to clarify our definitions and point to the relevant literature. In Section 3 we point out a simple strengthening of the disintegration theorem for Fell bundles (from [10]) which is needed here. For readability, the details are shifted to Appendix A. In Section 4 we prove our main theorem. In Section 6 we show that groupoids with T_0 orbit space and amenable stability groups are measurewise amenable. In Section 6 we prove that bundles over groupoids with T_0 orbit space whose restrictions to isotropy groups are metrically amenable are themselves metrically amenable.

Since we appeal to the disintegration theorem for Fell bundles, we require separability for our results. In particular, all the groupoids and spaces that appear will be assumed to be second countable, locally compact and Hausdorff. Except when it is clearly not the case, for example $B(\mathcal{H})$ and other multiplier algebras, all the algebras and Banach spaces that appear are separable. We also assume that our Fell bundles are always saturated. The underlying Banach bundles are only required to be upper semicontinuous.

1. Fell bundles

We will refer to [10, §1] for details of the definition of a Fell bundle $p : \mathscr{B} \to G$ over a groupoid as well as of the construction of the associated C^*-algebra $C^*(G, \mathscr{B})$. (The examples in [10, §2] would be very helpful supplementary reading.) Roughly speaking, a Fell bundle $p : \mathscr{B} \to G$ is an upper-semicontinuous Banach bundle endowed with a partial multiplication compatible with p such that the fibres $A(u)$ over units u are C^*-algebras and such that each fibre $B(x)$ is an $A(r(x))$–$A(s(x))$-imprimitivity bimodule with respect to the inner products and actions induced by the multiplication on $\mathscr{B}$. In particular, when x and y are composable, multiplication in $\mathscr{B}$ implements isomorphisms $B(x) \otimes_{A(s(x))} B(y) \cong B(xy)$. The space $\Gamma_c(G; \mathscr{B})$ of continuous sections of $\mathscr{B}$ then carries a natural convolution and involution. The C^*-algebra $C^*(G, \mathscr{B})$ is the completion of $\Gamma_c(G; \mathscr{B})$ with respect to the universal norm for representations which are continuous with respect to the inductive-limit topology on G.

Regarding our notation: as above, we use a roman letter, $B(x)$, for the fibre over x together with its Banach space structure, but we will use both $A(u)$ and $B(u)$ for the fibre over a unit u so as to distinguish its dual roles. The Fell bundle axioms imply that $A := \Gamma_0(G^{(0)};\mathscr{B})$ is a C^*-algebra which is called the C^*-algebra of $\mathscr{B}$ over $G^{(0)}$; in particular it is a $C_0(G^{(0)})$-algebra. So for $u \in G^{(0)}$ we write $A(u)$ for the fibre over u when we are thinking of it as a C^*-algebra, and we write $B(u)$ when we are thinking of it instead as an $A(u)-A(u)$-imprimitivity bimodule. We assume that our Fell bundles are separable, so in addition to G being second countable, we assume that the Banach space $\Gamma_0(G;\mathscr{B})$ is separable. By axiom, our Fell bundles are saturated in that $B(x)B(y) = B(xy)$, where $B(x)B(y)$ denotes $\overline{\operatorname{span}}\{\, b_x b_y : b_x \in B(x),\, b_y \in B(y)\,\}$. If F is a locally closed subset[1] of $G^{(0)}$, then we abuse notation slightly and write $\Gamma_c(F;\mathscr{B})$ in place of $\Gamma_c(F;\mathscr{B}|_F)$ (as we have already done for $A = \Gamma_0(G^{(0)};\mathscr{B})$ above). If we let $G(F) := G|_F = \{\, x \in G : s(x) \in F \text{ and } r(x) \in F\,\}$ be the restriction of G to F, then $G(F)$ is a locally compact groupoid with Haar system $\{\lambda^u\}_{u\in F}$. As above, we write $C^*(G(F),\mathscr{B})$ in place of $C^*(G(F),\mathscr{B}|_{G(F)})$.

Recall the definition of the reduced norm on $\Gamma_c(G;\mathscr{B})$ from [16]. If π is a representation of $A = \Gamma_0(G^{(0)};\mathscr{B})$, then using [19, Example F.25] and the discussion preceding [11, Definition 7.9], we can assume that there is a Borel Hilbert bumdle $G^{(0)} * \mathscr{H}$, a finite Radon measure μ on $G^{(0)}$ and representations π_u of A on $\mathcal{H}(u)$, factoring through A_u, such that

$$\pi = \int_{G^{(0)}}^{\oplus} \pi_u \, d\mu(u).$$

For $u \in G^{(0)}$, we frequently regard the π_u as representations of $A(u)$. Even if π is nondegenerate, we can only assume that μ-almost all of the π_u are nondegenerate. Indeed, we could have $\pi_u = 0$ for a null set of u. The formula $(\operatorname{Ind}\pi)(f)(g\otimes h) = (f*g)\otimes h$ for $f, g \in \Gamma_c(G;\mathscr{B})$ and $h \in L^2(G^{(0)} * \mathscr{H}, \mu)$ determines a representation $\operatorname{Ind}\pi$ of $\Gamma_c(G;\mathscr{B})$ on the completion of $\Gamma_c(G;\mathscr{B}) \odot L^2(G^{(0)} * \mathscr{H}, \mu)$ with respect to the inner product

$$\begin{aligned}
(f\otimes h \mid g \otimes k) &= \bigl(\pi(g^* * f)h \mid k\bigr) \\
&= \int_{G^{(0)}} \int_G \bigl(\pi_u\bigl(g(x^{-1})^* f(x^{-1})\bigr)h(u) \mid k(u)\bigr)\, d\lambda^u(x)\, d\mu(u) \\
&= \int_{G^{(0)}} \int_G \bigl(\pi_u\bigl(g(x)^* f(x)\bigr)h(u) \mid k(u)\bigr)\, d\lambda_u(x)\, d\mu(u). \qquad (1)
\end{aligned}$$

The reduced norm on $\Gamma_c(G;\mathscr{B})$ is given by

$$\|f\|_r := \sup\{\, \|(\operatorname{Ind}\pi)(f)\| : \pi \text{ is a representation of } A\,\}.$$

Since $\ker(\operatorname{Ind}\pi)$ depends only on $\ker\pi$ (by, for example, [12, Corollary 2.73]) this definition of $\|\cdot\|_r$ agrees with other definitions in the literature—for example Exel's in [3] and Moutuou and Tu's in [8]. So $C^*_r(G,\mathscr{B})$ is the quotient of $C^*(G,\mathscr{B})$ by the kernel $I_{C^*_r(G,\mathscr{B})}$ of $\operatorname{Ind}\pi$ for any faithful representation π of the C^*-algebra $A = \Gamma_0(G^{(0)};\mathscr{B})$ of $\mathscr{B}$ over $G^{(0)}$.

[1] Recall that a subset of a locally compact Hausdorff space is locally compact if and only if it is locally closed and is locally closed if and only if it is the intersection of a closed set and an open set [19, Lemmas 1.25 and 1.26].

2. Amenable groupoids

Let G be a second-countable locally compact groupoid with Haar system λ. Renault [14, p. 92] originally defined G to be topologically amenable, or just amenable, if there is a net $\{f_i\} \subset C_c(G)$ such that

(a) the functions $u \mapsto f_i * f_i^*(u)$ are uniformly bounded on $C_0(G^{(0)})$, and
(b) $f_i * f_i^* \to 1$ uniformly on compact subsets of G.

Later, in the extensive treatment by Anatharaman-Delaroche and Renault, an *a priori* different definition was given: [1, Definition 2.2.8]; however, [1, Proposition 2.2.13(iv)] and its proof show that the two notions of amenability are equivalent. It is not hard to see, using standard criteria such as [19, Proposition A.17], that a group is amenable as a groupoid if and only if it is amenable as a group.

Let μ be a quasi-invariant measure on $G^{(0)}$, and let $\nu := \mu \circ \lambda$ be the induced measure on G (that is, $\nu(\cdot) = \int_{G^{(0)}} \lambda^u(\cdot)\, d\mu(u)$). In [1, Definition 3.2.8], μ is called amenable if there exists a suitably invariant mean on $L^\infty(G, \nu)$. The pair (G, λ) is measurewise amenable if every quasi-invariant measure μ is amenable [1, Definition 3.2.8]. Since $L^\infty(G, \nu)$ depends only on the equivalence class of ν, if μ' is equivalent to μ and μ is amenable, then so is μ'. Since [1] considers only σ-finite measures, to demonstrate that (G, λ) is measurewise amenable, it suffices to show that every *finite* quasi-invariant measure μ is amenable.

It follows from [1, Theorem 2.2.17] and [1, Theoerem 3.2.16] that amenability and measurewise amenability, respectively, are preserved under groupoid equivalence. Theorem 17 of [17] implies that metric amenability is preserved as well. In particular, none of the three flavors of amenability of G depend on the choice of Haar system λ.

In this note, we will use the characterization of amenability of (G, λ, μ) given in [1, Proposition 3.2.14(v)]. If G is amenable then it is measurewise amenable by [1, Proposition 3.3.5]. If G is measurewise amenable then it is metrically amenable by [1, Proposition 6.1.8].

3. The disintegation theorem revisited

Our main tool here is the disintegration theorem from [10]. Fix a nondegenerate representation L of $C^*(G, \mathscr{B})$. Then [10, Theorem 4.13] implies that there are a quasi-invariant measure μ on $G^{(0)}$, a Borel Hilbert bundle $G^{(0)} * \mathscr{H}$, and a Borel $*$-functor $b \mapsto \big(r(b), \pi(b), s(b)\big)$ (see [10, Definition 4.5]) from $\mathscr{B}$ into $\operatorname{End}(G^{(0)} * \mathscr{H})$ such that L is equivalent to the integrated form of the associated strict representation $(\mu, G^{(0)} * \mathscr{H}, \pi)$ of $\mathscr{B}$. For $h, k \in L^2(G^{(0)} * \mathscr{H}, \mu)$ and $f \in \Gamma_c(G; \mathscr{B})$, we then have

$$\big(L(f)h \mid k\big) = \int_G \Big(\pi(f(x))h(s(x)) \bigm| k(r(x)\Big)\Delta(x)^{-\frac{1}{2}}\, d\nu(x).$$

Regrettably, the authors of [10] neglected to point out that the Borel $*$-functor associated to L constructed in [10, Theorem 4.13] is *nondegenerate* in the sense that for all $x \in G$,

$$\overline{\pi(B(x))\mathcal{H}(s(x))} = \overline{\operatorname{span}}\{\, \pi(b)v : b \in B(x) \text{ and } v \in \mathcal{H}(s(x)) \,\} = \mathcal{H}(r(x)). \tag{2}$$

We outline why this is true in Appendix A, and at the same time, we tidy up some details of the proof of the disintegration theorem itself.

4. Fell bundles over amenable groupoids

Our first main theorem says that every Fell bundle over a measurewise amenable groupoid is metrically amenable.

Theorem 1. *Let G be a second-countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Suppose that $p:\mathscr{B}\to G$ is a separable Fell bundle over G. If G is measurewise amenable, then the reduced norm on $\Gamma_c(G;\mathscr{B})$ is equal to the universal norm, so $C^*_r(G,\mathscr{B})=C^*(G,\mathscr{B})$.*

Our proof follows the lines of Renault's proof of the corresponding result for groupoid C^*-algebras, suitably modified for the bundle context. Before getting into the proof, we need to do a little set-up. We will continue with the following notation for the remainder of the section.

Fix $a\in I_{C^*_r(G,\mathscr{B})}$ and let L be a nondegenerate representation of $C^*(G,\mathscr{B})$. As in Section 3, we may assume that L is the integrated form of a strict representation $\bigl(\mu,G^{(0)}*\mathscr{H},\pi\bigr)$ of $\mathscr{B}$ which is nondegenerate in the sense that (2) holds for all x. Fix a unit vector h in $L^2(G^{(0)}*\mathscr{H},\mu)$, and let ω_h be the associated vector state. To prove Theorem 1, it suffices to see that $\omega_h(a)=0$.

Let $G*\mathscr{H}_r$ be the pullback of $G^{(0)}*\mathscr{H}$ over the range map. We may describe it as follows. Let $(h_j)_{j=1}^\infty$ be a special orthorgonal fundamental sequence for $G^{(0)}*\mathscr{H}$ as in [19, Proposition F.6]. For each j, let $\tilde h_j(x)=h_j(r(x))\in\mathcal{H}(r(x))$. Then $G*\mathscr{H}_r$ is isomorphic to the Borel Hilbert bundle built from $\coprod_{x\in G}\mathcal{H}(r(x))$ with fundamental sequence $(\tilde h_j)_{j=1}^\infty$.

Let $\nu=\mu\circ\lambda$ be the measure on G induced by μ, and recall that ν^{-1} denotes the measure $\nu^{-1}(f)=\int_G f(x^{-1})d\nu(x)$. Since μ is quasi-invariant, ν and ν^{-1} are equivalent measures. By passing to an equivalent measure, we may assume that the Radon-Nikodym derivative $\Delta=d\nu/d\nu^{-1}$ is multiplicative from G to $(0,\infty)$ — there is a nice proof of this in [9, Theorem 3.15].[2]

Lemma 2. *Define $U:\Gamma_c(G;\mathscr{B})\odot L^2(G^{(0)}*\mathscr{H},\mu)\to L^2(G*\mathscr{H}_r,\nu^{-1})$ by $U(f\otimes h)(x)=\pi\bigl(f(x)\bigr)h\bigl(s(x)\bigr)$. Then U is isometric and extends to a unitary, also denoted by U, from $\mathcal{H}_{\operatorname{Ind}\pi_\mu}$ onto $L^2(G*\mathscr{H}_r,\nu^{-1})$. Furthermore, U intertwines the regular representation $\operatorname{Ind}\pi_\mu$ with the representation M_π of $C^*(G,\mathscr{B})$ on $L^2(G*\mathscr{H},\nu^{-1})$ given on $f\in\Gamma_c(G;\mathscr{B})$ by*

$$\bigl(M_\pi(f)\xi\mid\eta\bigr)=\int_G\int_G\Bigl(\pi(f(xy))\xi(y^{-1})\bigm|\eta(x)\Bigr)\,d\lambda^{s(x)}(y)\,d\nu^{-1}(x). \tag{3}$$

Proof. That π is a Borel $*$-functor, f is a continuous section and

$$\begin{aligned}\bigl(U(f\otimes h)(x)\mid\tilde h_j(x)\bigr)&=\bigl(\pi(f(x))h(s(x))\mid h_j(r(x)\bigr)\\&=\sum_{k=1}^\infty\bigl(h(s(x))\mid h_k(s(x))\bigr)\bigl(\pi(f(x))h_k(s(x))\mid h_j(r(x))\bigr),\end{aligned}$$

imply that $x\mapsto\bigl(U(f\otimes h)(x)\mid\tilde h_j(x)\bigr)$ is Borel. Thus $U(g\otimes h)\in B(G*\mathscr{H}_r)$. The representation π_μ comes from a Borel $*$-functor defined on all of $\mathscr{B}$, so the

[2]The proof in [9] unfortunatly remains unpublished, but it is based on Hahn's [4, Corollary 3.14].

formula (1) for the inner product on $\mathcal{H}_{\operatorname{Ind}\pi_\mu}$ becomes

$$\begin{aligned}\big(f\otimes h \mid g\otimes k\big) &= \int_{G^{(0)}}\int_G \big(\pi\big(g(x)^*f(x)\big)h(u) \mid k(u)\big)\,d\lambda_u(x)\,d\mu(u)\\ &= \int_G \big(\pi(f(x))h(s(x)) \mid \pi(g(x))k(s(x))\big)\,d\nu^{-1}(x)\\ &= \int_G \big(U(f\otimes h)(x) \mid U(g\otimes k)(x)\big)\,d\nu^{-1}(x).\end{aligned}$$

Hence $U(f\otimes h)\in L^2(G*\mathscr{H},\nu^{-1})$ and U is an isometry. Since π is nondegenerate, an argument like that of [19, Lemma F.17], shows that the range of U is dense. Hence, U is a unitary as claimed.

For the last assertion, recall that $\operatorname{Ind}\pi_\mu(f)$ acts by convolution. Thus

$$\begin{aligned}\big(M_\pi(f)\xi \mid \eta\big) &= \int_G \big(M_\pi(f)\xi(x) \mid \eta(x)\big)\,d\nu^{-1}(x)\\ &= \int_G\int_G \big(\pi(f(y))\xi(y^{-1}x) \mid \eta(x)\big)\,d\lambda^{r(x)}(y)\,d\nu^{-1}(x)\\ &= \int_G\int_G \big(\pi(f(xy))\xi(y^{-1}) \mid \eta(x)\big)\,d\lambda^{s(x)}(y)\,d\nu^{-1}(x).\end{aligned}$$ □

To prove Theorem 1, we invoke measurewise amenability in the form of [1, Proposition 3.2.14(v)]. So we fix a sequence $(f_n)_{n=1}^\infty$ of Borel functions on G such that

(a) $u\mapsto \int_G |f_n(x)|^2\,d\lambda^u(x)$ is bounded on $G^{(0)}$,
(b) $f_n^* * f_n(u)\le 1$ for all $u\in G^{(0)}$ and
(c) $f_n^* * f_n\to 1$ in the weak-$*$ topology on $L^\infty(G,\nu)$.

To keep notation compact, we denote $f_n^* * f_n$ by e_n, so that for $y\in G$,

$$e_n(y) = \int_G \overline{f_n(x^{-1})}f_n(x^{-1}y)\,d\lambda^{r(y)}(x),$$

Proof of Theorem 1. Recall the notation fixed at the beginning of the section: in particular, L is the integrated form of a nondegenerate strict representation of $C^*(G,\mathscr{B})$ on a Hilbert bundle $G^{(0)}*G$, h is a unit vector in $L^2(G^{(0)}*\mathscr{H},\mu)$ and ω_h is the associated vector state. We claim that $|\omega_h(g)|\le \|(\operatorname{Ind}\pi_\mu)(g)\|$ for all $g\in\Gamma_c(G;\mathscr{B})$.

Fix $g\in\Gamma_c(G;\mathscr{B})$. Then

$$\omega_h(g) = \big(L(g)h \mid h\big) = \int_G \big(\pi(g(y))h(s(y)) \mid h(r(y))\big)\Delta(y)^{-\frac12}\,d\nu(y).$$

Define a sequence $(\alpha_n)_{n=1}^\infty$ of complex numbers by

$$\begin{aligned}\alpha_n &:= \int_G e_n(y)\big(\pi(g(y))h(s(y)) \mid h(r(y))\big)\Delta(y)^{-\frac12}\,d\nu(y)\\ &= \int_{G^{(0)}}\int_G\int_G \overline{f_n(x^{-1})}f_n(x^{-1}y)\big(\pi(g(y))h(s(y)) \mid h(r(y))\big)\Delta(y)^{-\frac12}\\ &\qquad\qquad d\lambda^u(x)\,d\lambda^u(y)\,d\mu(u).\end{aligned}\tag{4}$$

(It is tempting to write $\omega_h(e_n g)$ for α_n, but the e_n are assumed only to be Borel, so the pointwise products $e_n g$ may not belong to $\Gamma_c(G;\mathscr{B})$.) By assumption on the e_n, the α_n converge to $\omega_h(g)$. So it suffices to show that

$$|\alpha_n|\le \|(\operatorname{Ind}\pi_\mu)(g)\| \quad \text{for all } n\in\mathbb{N}.\tag{5}$$

Fix $n \in \mathbb{N}$. Define $h_n : G \to \mathscr{H}$ by

$$h_n(x) = \Delta(x)^{\frac{1}{2}} f_n(x^{-1}) h\big(r(x)\big).$$

Then for each j, the function

$$x \mapsto \big(h_n(x) \mid \tilde{h}_j(x)\big) = \Delta(x)^{\frac{1}{2}} f_n(x^{-1}) \big(h(r(x)) \mid h_j(r(x))\big)$$

is Borel, so $h_n \in L^2(G * \mathscr{H}, \nu^{-1})$. Starting from (4), we apply Fubini's theorem, substitute xy for x, and then use first that $\nu = \Delta\nu^{-1}$ and then that Δ is multiplicative to calculate:

$$\begin{aligned}
\alpha_n &= \int_G \int_G \overline{f_n(x^{-1})} f_n(y) \big(\pi(g(xy)) h(s(y)) \mid h(r(x))\big) \Delta(xy)^{-\frac{1}{2}} \, d\lambda^{s(x)}(y) \, d\nu(x) \\
&= \int_G \int_G \overline{f_n(x^{-1})} f_n(y) \big(\pi(g(xy)) h(s(y)) \mid h(r(x))\big) \\
&\qquad\qquad\qquad \Delta(xy)^{-\frac{1}{2}} \Delta(x) \, d\lambda^{s(x)}(y) \, d\nu^{-1}(x) \\
&= \int_G \int_G \big(\pi(g(xy)) h_n(y^{-1}) \mid h_n(x)\big) \, d\lambda^{s(x)}(y) \, d\nu^{-1}(x) \\
&= \big(M_\pi(g) h_n \mid h_n\big).
\end{aligned}$$

We have

$$\|h_n\|^2 = \int_G \|h_n(x)\|^2 \, d\nu^{-1}(x) = \int_G |f_n(x^{-1})|^2 \|h(r(x))\|^2 \Delta(x) \, d\nu^{-1}(x),$$

and since $\nu = \Delta\nu^{-1}$ and $e_n(u) \leq 1$ for all u, it follows that

$$\|h_n\|^2 = \int_{G^{(0)}} e_n(u) \|h(u)\|^2 \, d\mu(u) \leq \|h\|^2 = 1.$$

Hence the Cauchy-Schwarz inequality gives (5). Thus $|\omega_h(g)| \leq \|(\operatorname{Ind} \pi_\mu)(g)\|$ for all $g \in \Gamma_c(G; \mathscr{B})$ as claimed. Since $\Gamma_c(G; \mathscr{B})$ is dense in $C^*(G, \mathscr{B})$, it follows that

$$|\omega_h(a)| \leq \|(\operatorname{Ind} \pi_\mu)(a)\| \quad \text{for all } a \in C^*(G, \mathscr{B}).$$

In particular, if $a \in I_{C^*_r(G,\mathscr{B})}$, then $(\operatorname{Ind} \pi_\mu)(a) = 0$, and hence $\omega_h(a) = 0$ as required. $\square$

Example 3. Recall from [2] that given a row-finite k-graph Λ with no sources, a Λ-system of C^*-correspondences consists of an assignment $v \mapsto A_v$ of C^*-algebras to vertices and an assignment $\lambda \mapsto X_\lambda$ of an $A_{r(\lambda)}$–$A_{s(\lambda)}$ correspondence to each path λ, together with isomorphisms $\chi_{\mu,\nu} : X_\mu \otimes_{A_{s(\mu)}} X_\nu \to X_{\mu\nu}$ for each composable pair $\mu, \nu \in \Lambda$, all subject to an appropriate associativity condition on the $\chi_{\mu,\nu}$ (see [2, Definitions 3.1.1 and 3.1.2] for details). Suppose that X is such a system, and suppose that each X_λ is nondegenerate as a left $A_{r(\lambda)}$-module, and full as a right Hilbert $A_{s(\lambda)}$-module, and that the left action of $A_{r(\lambda)}$ is by compact operators.

By [2, Theorem 4.3.1], the construction of Sections 4.1 and 4.2 of the same paper associates to X a saturated Fell bundle E_X over the k-graph groupoid G_Λ of [7]. Moreover, [2, Theorem 4.3.6] says that the C^*-algebra $C^*(A, X, \chi)$ of the Λ-system is isomorphic to the reduced C^*-algebra $C^*_r(G_\Lambda, E_X)$ of the Fell bundle.

Theorem 5.5 of [7] says that G_Λ is amenable, and hence also measurewise amenable. Hence our Theorem 1 implies that $C^*_r(G_\Lambda, E_X) = C^*(G_\Lambda, E_X)$; in particular $C^*(A, X, \chi) \cong C^*(G_\Lambda, E_X)$.

Since 1-graphs are precisely the path-categories E^* of countable directed graphs E, and since an E^*-system of correspondences can be constructed from any assignment of C^*-algebras A_v to vertices v, and $A_{r(e)}$–$A_{s(e)}$ C^*-correspondences X_e to edges e (see [2, Remark 3.1.5]), Example 3 provides a substantial library of examples of our result

5. Measurewise amenable groupoids

Our initial motivation for proving Theorem 1 was to show that if G has T_0 orbit space and amenable stability groups then the full and reduced C^*-algebras of any Fell bundle over G coincide: roughly, since $C^*(G,\mathscr{B})$ is a $C_0(G\backslash G^{(0)})$-algebra, representations will factor through restrictions to orbit groupoids $G([u])$, each of which is amenable because is is equivalent to the amenable stability group $G(u) := \{x \in G : r(x) = u = s(x)\}$ (see section 6 for details). However, the following argument shows that the result follows directly from Theorem 1. We thank Jean Renault for pointing us in the direction of [1, Proposition 5.3.4].

Theorem 4. *Suppose that G is a second countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Suppose that the orbit space $G\backslash G^{(0)}$ is T_0 and that each stability group $G(u)$ is amenable. Then G is measurewise amenable.*

Our proof requires some straightforward observations as well as some nontrivial results from [1].

Lemma 5. *Suppose that μ is a quasi-invariant finite measure on $G^{(0)}$ and that $F \subset G^{(0)}$ is a locally compact G-invariant subset such that $\mu(G^{(0)} \setminus F) = 0$. Then (G,λ,μ) is amenable if and only if $(G(F),\lambda|_F,\mu|_F)$ is amenable.*

Proof. Recall that (G,λ,μ) is amenable if there is an invariant mean on $L^\infty(G,\nu)$ where $\nu = \mu\circ\lambda$. Since $\mu|_F\circ\lambda|_F = \nu|_{G(F)}$, we have $L^\infty(G,\nu) \cong L^\infty(G(F),\mu|_F\circ\lambda|_F)$. In particular, an invariant mean on $L^\infty(G)$ gives an invariant mean on $L^\infty(G(F))$ and vice versa. □

Lemma 6. *Suppose that $G\backslash G^{(0)}$ is T_0. Then, as a Borel space, $G\backslash G^{(0)}$ is countably separated and each orbit $[u]$ is locally closed in G and hence locally compact.*

Proof. Since subsets of a locally compact Hausdorff space are locally compact if and only if they are locally closed (see [19, Lemma 1.26]), the lemma is an immediate consequence of the Mackey-Glimm-Ramsay dichotomy [13, Theorem 2.1]. □

Proof of Theorem 4. Suppose that μ is a finite quasi-invariant measure on $G^{(0)}$. It suffices to show that (G,λ,μ) is amenable. Let $p : G \to G\backslash G^{(0)}$ be the orbit map, and let $\underline{\mu}$ be the forward image $\underline{\mu}(f) = \mu(f \circ p)$ of μ under p. By Lemma 6, $G\backslash G^{(0)}$ is countably separated as a Borel space. Hence we can disintegrate μ — as, for example, in [19, Theorem I.5] — so that for each orbit $[u]$ there is a probability measure $\rho_{[u]}$ on $G^{(0)}$ supported on $[u]$ such that

$$\mu = \int_{G\backslash G^{(0)}} \rho_{[u]}\, d\underline{\mu}([u]).$$

It follows from [1, Proposition 5.3.4] that $\rho_{[u]}$ is quasi-invariant for almost all $[u]$ and that (G,λ,μ) is amenable if each $(G,\lambda,\rho_{[u]})$ is. Since $\rho_{[u]}(G^{(0)} \setminus [u]) = 0$, Lemma 5 implies that it is enough to see that each $(G([u]),\lambda|_{[u]},\mu|_{[u]})$ is amenable. Since $[u]$ is locally compact, $G([u])$ is a locally compact transitive groupoid equivalent to $G(u)$, which is assumed to be amenable. Hence [1, Theorem 2.2.13] implies that $G([u])$ is amenable, and therefore also measurewise amenable by [1, Proposition 3.3.5]. □

6. Fibrewise-amenable Fell bundles

In the preceding section, we showed that if $G\backslash G^{(0)}$ is T_0 and each stability group is amenable, then G is measurewise amenable. In particular, if $p : \mathscr{B} \to G$ is a bundle over such a groupoid, then its full and reduced algebras coincide. In this section, we show that it suffices that $G\backslash G^{(0)}$ is T_0 and that for each $u \in G^{(0)}$, the full and reduced algebras of the restriction of $\mathscr{B}$ to the isotropy group $G(u)$ coincide. To see that this is a strictly stronger theorem, and also that the hypothesis is genuinely checkable, we refer the reader to the results, for example, of [3].

Theorem 7. *Let G be a second-countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, and let $p : \mathscr{B} \to G$ be a separable Fell bundle over G. Suppose that the orbit space $G\backslash G^{(0)}$ is T_0 and that for each unit u, the full and reduced cross-sectional algebras $C^*(G(u), \mathscr{B})$ and $C^*_r(G(u), \mathscr{B})$ coincide. Then the full and reduced norms on $\Gamma_c(G; \mathscr{B})$ are equal and hence $C^*_r(G, \mathscr{B}) = C^*(G, \mathscr{B})$.*

To prove the theorem, we first use the equivalence theorem of [16] to see that the full and reduced C^*-algebras of a Fell bundle over transitive groupoid coincide whenever the full and reduced algebras of its restriction to any isotropy group coincide.

Lemma 8. *Let G be a second-countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, and let $p : \mathscr{B} \to G$ be a separable Fell bundle over G. Suppose that G is transitive. Then the following are equivalent.*

(a) *For some unit u, the full and reduced cross-section algebras $C^*(G(u), \mathscr{B})$ and $C^*_r(G(u), \mathscr{B})$ coincide.*
(b) *For every unit u, the full and reduced cross-section algebras $C^*(G(u), \mathscr{B})$ and $C^*_r(G(u), \mathscr{B})$ coincide.*
(c) *The full and reduced norms on $\Gamma_c(G; \mathscr{B})$ are equal and hence $C^*_r(G, \mathscr{B}) = C^*(G, \mathscr{B})$.*

Proof. Fix $u \in G^{(0)}$. Then $G_u := s^{-1}(u)$ is a $(G, G(u))$-equivalence, and as in [5, Theorem 1], $\mathscr{E} := p^{-1}(G_u)$ implements an equivalence between $\mathscr{B}$ and $p^{-1}(G(u))$. Consequently, [16, Theorem 14] implies that the natural surjection of $C^*(G, \mathscr{B})$ onto $C^*_r(G, \mathscr{B})$ is an isomorphism if and only if the kernel I_r of the natural map of $C^*(G(u), \mathscr{B})$ onto $C^*_r(G(u), \mathscr{B})$ is trivial. Since $u \in G^{(0)}$ was arbitrary, the result follows. □

To finish off our proof of Theorem 7, we need the following special case of [6, Theorem 3.7]. As above, let $p : \mathscr{B} \to G$ be a separable Fell bundle over G with associated C^*-algebra $A = \Gamma_0(G^{(0)}; \mathscr{B})$. Recall from [6, Proposition 2.2] that G acts on $\operatorname{Prim} A$ which we identify with $\{ (u, P) : u \in G^{(0)} \text{ and } P \in \operatorname{Prim} A(u) \}$. Let U be an open G-invariant subset of $G^{(0)}$ with complement F. Then $\{ (u, P) \in \operatorname{Prim} A : u \in F \}$ is a closed invariant subset of $\operatorname{Prim}(A)$, and corresponds to the G-invariant ideal $\{ a \in A : a(u) = 0 \text{ for all } u \in F \}$ of A. By [6, Proposition 3.3], the corresponding bundle $\mathscr{B}_I$ is the one with fibres

$$B_I(x) = \begin{cases} B(x) & \text{if } x \in G(U) \\ \{0\} & \text{if } u \in G(F), \end{cases}$$

so we can identify it with the bundle $\mathscr{B}|_{G(U)}$ over $G(U)$. Moreover, $\mathscr{B}^I$ is the complementary bundle

$$B^I(x) = \begin{cases} \{0\} & \text{if } u \in G(U) \\ B(x) & \text{if } u \in G(F), \end{cases}$$

which we may identify with the bundle $\mathscr{B}|_{G(F)}$ over $G(F)$. Thus, as a special case of Theorem 3.7 of [6], we obtain the following result.

Lemma 9. *Let G be a second-countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, and let $p : \mathscr{B} \to G$ be a separable Fell bundle over G. Suppose that U is a G-invariant open subset of $G^{(0)}$ with complement F. There is a short exact sequence of C^*-algebras*

$$0 \longrightarrow C^*(G(U),\mathscr{B}) \xrightarrow{\iota} C^*(G,\mathscr{B}) \xrightarrow{q} C^*(G(F),\mathscr{B}) \longrightarrow 0,$$

where ι is induced by inclusion and q by restriction on sections.

As an application of Lemma 9, recall[3] that there is a nondegenerate map $M : C_0(G^{(0)}) \to M(C^*(G,\mathscr{B}))$ given on sections by

$$M(\phi)f(x) = \phi\big(r(x)\big)f(x).$$

Suppose that the orbit space $G\backslash G^{(0)}$ is Hausdorff. Then we may identify $C_0(G\backslash G^{(0)})$ with the subalgebra of $C_b(G^{(0)})$ consisting of functions which are constant on orbits and vanish at infinity on the orbit space. We extend M to $C_b(G^{(0)})$ and restrict to $C_0(G\backslash G^{(0)})$ to obtain a nondegenerate map of $C_0(G\backslash G^{(0)})$ into the center of $M(C^*(G,\mathscr{B}))$, making $C^*(G,\mathscr{B})$ into a $C_0(G\backslash G^{(0)})$-algebra. As usual, if $u \in G^{(0)}$, we let $[u]$ be the corresponding orbit in $G\backslash G^{(0)}$.

Corollary 10. *Let G be a second-countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, and let $p : \mathscr{B} \to G$ be a separable Fell bundle over G. If $G\backslash G^{(0)}$ is Hausdorff, then $C^*(G,\mathscr{B})$ is a $C_0(G\backslash G^{(0)})$-algebra with fibres $C^*(G,\mathscr{B})([u]) \cong C^*(G([u]),\mathscr{B})$.*

Proof. Recall that $C^*(G,\mathscr{B})([u])$ is the quotient of $C^*(G,\mathscr{B})$ by the ideal $J_{[u]} = \overline{\operatorname{span}}\{\, \phi\cdot a : \phi \in C_0(G\backslash G^{(0)}),\ \phi([u]) = 0 \text{ and } a \in C^*(G,\mathscr{B}) \,\}$. Using Lemma 9, we can identify $J_{[u]}$ with $C^*(G(U),\mathscr{B})$, where $U = G^{(0)} \setminus [u]$, and $C^*(G,\mathscr{B})/J_{[u]}$ with $C^*(G([u]),\mathscr{B})$ as claimed. □

Proof of Theorem 7. Fix in irreducible representation π of $C^*(G;\mathscr{B})$ and an element $f \in \Gamma_c(G;\mathscr{B})$. It suffices to show that $\|\pi(f)\| \le \|f\|_{C_r^*(G;\mathscr{B})}$.

By [13, Theorem 2.1], the orbit space $G\backslash G^{(0)}$ is locally Hausdorff and every orbit $[u]$ is locally closed in $G^{(0)}$. Since $G\backslash G^{(0)}$ is second countable, [19, Lemma 6.3] implies that there is a countable ordinal γ and a nested open cover $\{\, U_n : 0 \le n \le \gamma \,\}$ of $G\backslash G^{(0)}$ such that $U_0 = \emptyset$, $U_\gamma = G\backslash G^{(0)}$ and $U_{n+1} \setminus U_n$ is Hausdorff (and dense) in $(G\backslash G^{(0)}) \setminus U_n$. For $n \le \gamma$, let $V_n := \{\, u \in G^{(0)} : [u] \in U_n \,\}$. Then each V_n is an open invariant subset of $G^{(0)}$. Using Lemma 9, we can identify $C^*(G(V_n),\mathscr{B})$ with an ideal in $C^*(G,\mathscr{B})$. In fact, $\{\, C^*(G(V_n),\mathscr{B}) \,\}_{n\le\gamma}$ is a composition series of ideals in $C^*(G,\mathscr{B})$. By [19, Lemma 8.13], there exists $0 < n \le \gamma$ such that π lives on the subquotient $C^*(G(V_n),\mathscr{B})/C^*(G(V_{n-1}),\mathscr{B})$; that is, π is the canonical lift $\bar\rho$ of an irreducible representation ρ of the ideal $C^*(G(V_n),\mathscr{B})$ such that

[3]A proof can be constructed along the lines of [10, Proposition 4.2].

$\ker\rho \supset C^*(G(V_{n-1}),\mathscr{B})$. Lemma 9 implies that $C^*(G(V_n),\mathscr{B})/C^*(G(V_{n-1}),\mathscr{B}) \cong C^*(G(V_n \setminus V_{n-1}),\mathscr{B})$. By construction, $G(V_n \setminus V_{n-1})$ has Hausdorff orbit space $U_n \setminus U_{n-1}$. Hence $C^*(G(V_n \setminus V_{n-1}),\mathscr{B})$ is a $C_0(U_n \setminus U_{n-1})$-algebra and ρ factors through a fibre $C^*(G(V_n \setminus V_{n-1}),\mathscr{B})([u]) \cong C^*(G([u]),\mathscr{B})$ for some $u \in V_n$ by [19, Proposition C.5]. Since, by assumption, $C^*(G[u]),\mathscr{B}) = C^*_r(G([u]),\mathscr{B})$, we have $\ker(\operatorname{Ind}\pi_{[u]}) \subset \ker\rho$ where $\pi_{[u]}$ factors through a faithful representation of the quotient $A_{V_n}([u])$ of the C^*-algebra $A(V_n)$ of $C^*(G(V_n),\mathscr{B})$ corresponding to the closed set $[u] \subset V_n$. (Note that $A(V_n)$ is the ideal of A corresponding to $V_n \subset G^{(0)}$.) The kernel of $\pi = \bar\rho$ is then contained in the kernel of the canonical lift of $\operatorname{Ind}\pi_{[u]}$ to $C^*(G,\mathscr{B})$. It is not hard to check that the canonical lift of $\operatorname{Ind}\pi_{[u]}$ to $C^*(G,\mathscr{B})$ is $\operatorname{Ind}\bar\pi_{[u]}$ where $\bar\pi_{[u]}$ is the canonical lift of $\pi_{[u]}$ to A. Hence $\|\pi(f)\| = \|\bar\rho(f)\| \le \|\operatorname{Ind}\bar\pi_{[u]}(f)\| \le \|f\|_{C^*_r(G,\mathscr{B})}$. □

Appendix A. Nondegenerate Borel $*$-functors

Let $p : \mathscr{B} \to G$ be a Fell bundle over a second-countable locally compact Hausdorff groupoid, and let $G^{(0)} * \mathscr{H}$ be a Borel Hilbert bundle. A *Borel $*$-functor* $\hat\pi$ from $\mathscr{B}$ to $\operatorname{End}(G^{(0)} * \mathscr{H})$ is a map

$$\hat\pi : b \mapsto \big(r(b), \pi(b), s(b)\big)$$

such that $\pi(b) \in B\big(\mathcal{H}(s(b)), \mathcal{H}(r(b))\big)$ for all b and such that π respects adjoints and the partial linear and multiplicative structure of $\mathscr{B}$ (see [10, Definition 4.5]). Following [10, §4], a strict representation of $\mathscr{B}$ is a triple $(\mu, G^{(0)} * \mathscr{H}, \hat\pi)$ consisting of a quasi-invariant measure μ on $G^{(0)}$, a Borel Hilbert bundle $G^{(0)} * \mathscr{H}$ and a Borel $*$-functor $\hat\pi$. It is common practice to use $\hat\pi$ and π interchangeably, and we will drop the caret henceforth. A strict representation determines a bounded representation via integration (see [10, Proposition 4.10]); indeed, a Borel $*$-functor defined on $p^{-1}(G|_F)$ for any μ-conull set $F \subset G^{(0)}$ is sufficient. Nevertheless, it is convenient to have π defined everywhere.

The purpose of this section is to point out that the disintegration theorem [10, Theorem 4.13] for Fell bundles can be strengthened to assert that π can be taken to be nondegenerate as defined in §3. At the same time, we correct an error in the construction of π in [10].

In the proof of [10, Theorem 4.13], starting from a pre-representation L of $\mathscr{B}$ on a dense subspace $\mathcal{H}_0$ of a Hilbert space H, the authors showed that for any orthonormal basis $\{\zeta_i : i \in \mathbb{N}\}$ for $\operatorname{span}\{L(f)\xi : f \in \Gamma_c(G;\mathscr{B}), \xi \in \mathcal{H}_0\}$, setting $\mathcal{H}'_{00} := \operatorname{span}\{\zeta_i : i \in \mathbb{N}\}$, there is a saturated Borel μ-conull set $F \subset G^{(0)}$ and a Borel Hilbert bundle $F * \mathscr{H}$ whose fibres $\mathcal{H}(u)$ are Hilbert-space completions of $\Gamma_c(G^u;\mathscr{B}) \odot \mathcal{H}'_{00}$ (see [10, Lemma 5.18] and [10, Lemma 5.20]). For $f \in \Gamma_c(G;\mathscr{B})$ and $h \in \mathcal{H}'_{00}$, the class of $f \otimes h$ in $\mathcal{H}(u)$ is denoted by $f \otimes_u h$. The space $\mathcal{H}(u)$ may be trivial for some u. For each $z \in G|_F$, $b \in B(z)$ and $f \in \Gamma_c(G;\mathscr{B})$, let $\check\pi(b)f$ denote a section satisfying

$$\check\pi(b)f(x) = \Delta(z)^{\frac{1}{2}} b f(z^{-1}x) \quad \text{for } x \in G^{r(b)}.$$

The Borel $*$-functor in the disintegration of L constructed in [10] is defined by

$$\pi(b)(f \otimes_{s(b)} \zeta_i) = \check\pi(b)f \otimes_{r(b)} \zeta_i.$$

Since F is saturated, G is the disjoint union of $G|_F$ and $G|_{G^{(0)}\setminus F}$. Since the latter is ν-null, we can extend π to all of G by defining it as we please on $p^{-1}(G|_{G^{(0)}\setminus F})$, and this will not affect the integrated representation. To ensure that π is still a genuine Borel $*$-functor, one sets $\mathcal{H}(u) := \{0\}$ for each $u \notin F$ and $\pi(b) := 0$ for $b \notin p^{-1}(G|_F)$. (In [10], the authors mistakenly let $(G^{(0)} \setminus F) * \mathscr{H}$ be a constant field and let $\pi(b)$ be the identity operator, but such a π is not a $*$-functor since as it doesn't preserve the partial linear structure.) We claim that π is nondegenerate in the sense that (2) holds for all $z \in G$. It holds trivially for $z \notin G|_F$, so fix $z \in G|_F$, and let $u := r(z)$. We start with two observations.

(A) If $f_i \to f$ in the inductive limit topology on $\Gamma_c(G^u; \mathscr{B})$ then $f_i \otimes_u \zeta_k \to f \otimes_u \zeta_k$ in $\mathcal{H}(u)$. To see this, observe that equation (5.19) of [10] is bounded by $K\|f\|_\infty \|g\|_\infty$ where K is constant depending only on $\operatorname{supp} f$ and $\operatorname{supp} g$.

(B) If $\{ e_i \}$ is an approximate identity in $A(u)$, and, for each i, $e_i g$ represents any section in $\Gamma_c(G; \mathscr{B})$ such that $(e_i g)(x) = e_i g(x)$ for $x \in G^u$, then $e_i g \to g$ in the inductive limit topology on $\Gamma_c(G^u; \mathscr{B})$. This follows from a compactness argument using that $A(u)$ acts nondegenerately on $B(x)$.

By (B), to establish (2) for z, it suffices to see that each $e_i g \otimes_{r(z)} \zeta_k$ belongs to $\overline{\pi(B(z)\mathcal{H}(s(z))}$. Fix $b_1, \dots b_n \in B(z)$ such that

$$\sum_j b_j b_j^* \sim e_i.$$

Then by (A), we have

$$\sum_j \pi(b_j)(\check{\pi}(b_j^*) g \otimes \zeta_k) \sim e_i g \otimes \zeta_k,$$

and this suffices.

Remark 11. Just as $*$-functors are automatically bounded (see [10, Remark 4.6]), there is a sense in which the Borel $*$-functor appearing in any strict representation $(\mu, G^{(0)} * \mathscr{H}, \pi)$ is essentially nondegenerate. We claim that

$$\overline{\pi\big(B(x)\big)\mathcal{H}\big((s(x)\big)} = \overline{\pi\big(A(r(x)\big)\mathcal{H}\big((r(x)\big)} \quad \text{for all } x \in G. \tag{6}$$

The right-hand side of (6) is the essential space of the representation $\pi_{r(x)}$ of $A\big(r(x)\big)$ determined by π, so (2) holds whenever $\pi_{r(x)}$ is nondegenerate. So if the representation π_μ of $A = \Gamma_0(G^{(0)}; \mathscr{B})$ determined by π is nondegenerate, then π_u is nondegenerate for μ-almost all u, so (2) holds on a ν-conull subset of G (where, as usual, $\nu = \mu \circ \lambda$).

To establish (6), we use that $\mathscr{B}$ is saturated: one the one hand,

$$\overline{\pi\big(B(x)\big)\mathcal{H}\big(s(x)\big)} = \overline{\pi\big(A(r(x))\big)\pi\big(B(x)\big)\mathcal{H}\big(s(x)\big)} \subset \overline{\pi\big(A(r(x))\big)\mathcal{H}\big(r(x)\big)},$$

while on the other hand,

$$\overline{\pi\big(B(x)\big)\mathcal{H}\big(s(x)\big)} \supset \overline{\pi\big(B(x)\big)\pi\big(B(x^*)\big)\mathcal{H}\big(r(x)\big)} = \overline{\pi\big(A\big(r(x)\big)\big)\mathcal{H}\big(r(x)\big)}.$$

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