SIMPLICITY OF C^*-ALGEBRAS ASSOCIATED TO HIGHER-RANK GRAPHS

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Abstract. We prove that if Λ is a row-finite k-graph with no sources, then the associated C^*-algebra is simple if and only if Λ is cofinal and satisfies Kumjian and Pask's aperiodicity condition, known as Condition (A). We prove that the aperiodicity condition is equivalent to a suitably modified version of Robertson and Steger's original nonperiodicity condition (H3) which in particular involves only finite paths. We also characterise both cofinality and aperiodicity of Λ in terms of ideals in $C^*(\Lambda)$.

1. Introduction

Consider a directed graph E which is row-finite and has no sinks in the sense that the set of outgoing edges from each vertex is both finite and nonempty. As in [5], we say E satisfies Condition (L) if every cycle in E has an exit, and is cofinal if every vertex connects to every infinite path. There is an elegant relationship between these conditions and the ideal-structure of the graph algebra $C^*(E)$ (see [7] for an overview). In particular:

(1) every ideal of $C^*(E)$ contains at least one of the canonical generators of $C^*(E)$ if and only if E satisfies Condition (L) [5, Theorem 3.7]; and
(2) $C^*(E)$ is simple if and only if E satisfies Condition (L) and is cofinal [1, Proposition 5.1].

The k-graphs developed by Kumjian and Pask in [4] are a generalisation of directed graphs designed to model the higher-rank Cuntz-Krieger algebras of [10]. In [4], Kumjian and Pask identified a generalisation of Condition (L) for higher-rank graphs which they called the *aperiodicity condition* or Condition (A), and showed that this condition guarantees that every ideal of the C^*-algebra contains at least one of the canonical generators. They also identified a cofinality condition on higher-rank graphs which together with the aperiodicity condition implies that the associated C^*-algebra is simple. These two results generalise the "if" directions of statements (1) and (2) of the previous paragraph. However, the generalisations to k-graphs of the "only if" directions of (1) and (2) above have not been established. Moreover, the aperiodicity condition is phrased in terms of infinite paths, and is difficult to verify in practise.

If we remove the hypotheses that a directed graph E is row-finite and has no sources, Condition (L) and cofinality as in [5] still yield the same consequences for $C^*(E)$ [2], [3]. For k-graphs, however, different conditions ([8, Condition (B)]

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and [9, Condition (C)]) have arisen as each hypothesis has been removed. In the situation considered by Kumjian and Pask, Conditions (B) and (C) are equivalent and imply aperiodicity condition, but whether the reverse implication holds is an open question.

A number of authors (see for example [4], [8], [7]) have pointed to the shortcomings of the aperiodicity condition outlined above as significant open problems. In this paper we resolve many of them for row-finite k-graphs with no sources. In summary: Theorem 3.1 shows that a row-finite k-graph with no sources satisfies the aperiodicity condition and is cofinal if and only if its C^*-algebra is simple. More specifically, Proposition 3.4 establishes statement (2) above for row-finite k-graphs with sources, and Proposition 3.5 establishes (1) with Condition (L) replaced by the aperiodicity condition. Additionally, Proposition 3.6 characterises the k-graphs for which every ideal of $C^*(\Lambda)$ is gauge-invariant (c.f. [6, Section 6]). In Lemma 3.2, we identify a generalisation of Robertson and Steger's nonperiodicity condition (H3) to row-finite k-graphs with no sources, and show that this condition, the aperiodicity condition, and Condition (B) are all equivalent. Significantly, our generalisation of (H3) involves only finite paths.

The k-graph versions of cofinality, aperiodicity, the Cuntz-Krieger uniqueness theorem, and the simplicity theorem for k-graphs used in this paper are all due to Kumjian and Pask in [4]. Nonetheless, we have generally referenced [8] throughout. This should not be interpreted as a dismissal of the ground-breaking work of Kumjian and Pask, which is undoubtedly the original and definitive article. Our choice was made only because [8] is the earliest single paper containing both the uniqueness theorems *and* a description of the gauge-invariant ideal structure for k-graph algebras, and hence allowed us to restrict our referencing to a single paper.

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2. Preliminaries

In this section we gather the notation and conventions we need regarding k-graphs. For a more detailed and rigorous treatment of k-graphs, see [4], [8].

We regard $\mathbb{N}^k$ as a monoid under addition, and denote its generators by $e_1, \dots, e_k$. We write n_i for the i^{th} coordinate of $n \in \mathbb{N}^k$. For $m, n \in \mathbb{N}^k$, we say $m \le n$ if $m_i \le n_i$ for each i. We write $m \vee n$ for the coordinate-wise maximum of m and n.

Higher-rank graphs. Fix $k > 0$. We think of a k-graph as a collection Λ of paths endowed with a *degree* function $d : \Lambda \to \mathbb{N}^k$ such that concatenation of paths has the following *factorisation property*: if $d(\lambda) = m + n$, there are unique paths $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$. For $n \in \mathbb{N}^k$, we write Λ^n for $d^{-1}(n)$. One can make this notion rigorous using category-theory; see [4, Definition 1.1] for the formal definition.

The vertices of Λ are the paths of degree 0. For a given path λ, the factorisation property ensures that there are unique vertices, called the range and source of λ and denoted $r(\lambda)$ and $s(\lambda)$, such that $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$. For $v \in \Lambda^0$ and $E \subseteq \Lambda$, we write vE for $E \cap r^{-1}(v)$ and Ev for $E \cap s^{-1}(v)$.

If $\lambda \in \Lambda$ with $d(\lambda) = l$, and $0 \le m \le n \le l$, there exist unique paths $\lambda(0, m) \in \Lambda^m$, $\lambda(m, n) \in \Lambda^{n-m}$ and $\lambda(n, l) \in \Lambda^{l-n}$ such that $\lambda = \lambda(0, m)\lambda(m, n)\lambda(n, l)$.

We say that a k-graph Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and we say Λ has *no sources* if $v\Lambda^n$ is always nonempty.

Infinite paths, the shift map, and cofinality, and aperiodicity. We denote by Ω_k the k-graph with paths $\{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$, $r(m,n) = (m,m)$, $s(m,n) = (n,n)$, $(m,n)(n,p) = (m,p)$, and $d(m,n) = n - m$. For brevity, we generally write n for the vertex (n,n) of Ω_k.

Given $k \in \mathbb{N}\backslash\{0\}$ and k-graphs Λ and Γ, a *graph morphism* from Λ to Γ is a function $x : \Lambda \to \Gamma$ which respects both connectivity and degree. Given a k-graph Λ, an *infinite path* in Λ is a graph morphism $x : \Omega_k \to \Lambda$. We write Λ^∞ for the collection of all infinite paths in Λ. We denote $x(0)$ by $r(x)$, and call it the *range* of x, and for $v \in \Lambda^0$, we write $v\Lambda^\infty$ for the set $\{x \in \Lambda^\infty : r(x) = v\}$. For $p \in \mathbb{N}^k$, we write $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ for the shift map determined by $\sigma^p(x)(n) = x(n+p)$.

We say Λ is *cofinal* if, for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n)$ is nonempty.

As in [4], we say that Λ satisfies the *aperiodicity condition* (or is *aperiodic* if, for every vertex $v \in \Lambda^0$ there is an infinite path $x \in v\Lambda^\infty$ such that $\sigma^m(x) \neq \sigma^n(x)$ for all $m \neq n \in \mathbb{N}^k$. The aperiodicity condition is also referred to as Condition (A) in [4] and in other k-graph literature. As in [8], we say that Λ satisfies *Condition (B)* if for every vertex $v \in \Lambda^0$, there is an infinite path $x \in v\Lambda^\infty$ such that $\mu x \neq \nu x$ for all $\mu \neq \nu \in \Lambda v$.

C^*-algebras of k-graphs. Let Λ be a row-finite k-graph with no sources. The associated C^*-algebra $C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ which satisfy the Cuntz-Krieger relations:

(1) $\{s_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
(2) $s_\mu s_\nu = s_{\mu\nu}$ when $s(\mu) = r(\nu)$;
(3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(4) $s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

By *universal* we mean that if $\{t_\lambda : \lambda \in \Lambda\}$ is any collection of partial isometries in a C^*-algebra A which satisfy the Cuntz-Krieger relations (1)–(4) above, then there is a homomorphism $\pi_t : C^*(\Lambda) \to A$ satisfying $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$. Proposition 2.11 of [4] implies that the generators $\{s_\lambda : \lambda \in \Lambda\}$ of $C^*(\Lambda)$ are all nonzero.

3. Results

Definition 1. Let Λ be a row-finite k-graph with no sources, and let $v \in \Lambda^0$. We say that Λ has *local periodicity at v* if there exist $m \neq n \in \mathbb{N}^k$ such that $\sigma^m(x) = \sigma^n(x)$ for all $x \in v\Lambda^\infty$. We say that Λ has *no local periodicity* if for each $v \in \Lambda^0$ and each $m \neq n \in \mathbb{N}^k$ there exists $x \in v\Lambda^\infty$ such that $\sigma^m(x) \neq \sigma^n(x)$.

Note that the hypothesis that Λ has no local periodicity is only a slight weakening of the aperiodicity condition: if Λ satisfies the aperiodicity condition, then for fixed $v \in \Lambda^0$ and distinct $m, n \in \mathbb{N}^k$, the path x with range v such that $\sigma^p(x) \neq \sigma^q(x)$ whenever $p \neq q$ certainly satisfies $\sigma^m(x) \neq \sigma^n(x)$. Indeed, we shall show in Lemma 3.2 that they are equivalent.

Theorem 3.1. *Let Λ be a row-finite k-graph with no sources. Then $C^*(\Lambda)$ is simple if and only if both of the following conditions hold:*

(i) *Λ is cofinal; and*

(ii) Λ *has no local periodicity.*

We first show that the no local periodicity hypothesis, the aperiodicity condition, and [8, Condition (B)] are all equivalent to a version of [10, (H3)] which involves only finite paths of Λ.

Lemma 3.2. *Let Λ be a row-finite k-graph with no sources. Then the following are equivalent.*

(i) Λ *satisfies the aperiodicity condition.*
(ii) Λ *satisfies Condition (B).*
(iii) Λ *has no local periodicity.*
(iv) *For each vertex $v \in \Lambda^0$ and each pair $m \neq n \in \mathbb{N}^k$ there is a path $\lambda = \lambda_{v,m,n} \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and*

$$\lambda\big(m, m + d(\lambda) - (m \vee n)\big) \neq \lambda\big(n, n + d(\lambda) - (m \vee n)\big).$$

Condition (iv) of Lemma 3.2 is a generalisation of the nonperiodicity condition (H3) of [10]. It looks complicated, but as a formulation of aperiodicity in terms of finite paths, it is an important outcome of the paper. Consequently we give a pictorial explanation of the condition in Appendix A.

Proof of equivalence of (i), (iii) *and* (iv) *in Lemma 3.2.* (i) $\Longrightarrow$ (iii) was established in the remark immediately following Definition 1.

(iii) $\Longrightarrow$ (iv). Suppose Λ has no local periodicity, and fix $v \in \Lambda^0$ and $m \neq n \in \mathbb{N}^k$. Then there exists $x \in v\Lambda^\infty$ such that $\sigma^m(x) \neq \sigma^n(x)$. Hence $\sigma^m(x)(0,p) \neq \sigma^n(x)(0,p)$ for large enough $p \in \mathbb{N}^k$. Let $\lambda_{v,m,n} := x(0, (m \vee n) + p)$. Then

$$\lambda_{v,m,n}(m, m+p) = \sigma^m(x)(0,p) \neq \sigma^n(x)(0,p) = \lambda_{v,m,n}(n, n+p).$$

(iv) $\Longrightarrow$ (i). Suppose (iv) holds, and fix $v \in \Lambda^0$. Let $(m^i, n^i)_{i=1}^\infty$ be a listing of $\{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k : m \neq n\}$. Let $\lambda_1 := \lambda_{v,m^1,n^1}$ as in (iv), and inductively let $\lambda_i := \lambda_{s(\lambda_{i-1}),m^i,n^i}$. Let $\eta_i := \lambda_1\lambda_2 \dots \lambda_i$ for each $i \geq 1$. Since $\bigvee_{i \in \mathbb{N}} d(\eta_i) = \infty^k$, there is a unique infinite path $x \in \Lambda^\infty$ such that $x(0, d(\eta_i)) = \eta_i$ for all i (see [8, page 107]). Fix $m \neq n \in \mathbb{N}^k$. Then $(m,n) = (m^i, n^i)$ for some i, and then

$$\begin{aligned}\sigma^{d(\eta_{i-1})+m}&(x)(0, d(\lambda_i) - (m \vee n)) \\ &= \sigma^{d(\eta_{i-1})}(x)(m^i, m^i + d(\lambda_i) - (m^i \vee n^i)) \\ &= \lambda_i(m^i, m^i + d(\lambda_i) - (m^i \vee n^i))\end{aligned}$$

and likewise

$$\sigma^{d(\eta_{i-1})+n}(x)(0, d(\lambda_i) - (m \vee n)) = \lambda_i(n^i, n^i + d(\lambda_i) - (m^i \vee n^i))$$

Hence $\sigma^m(x) \neq \sigma^n(x)$ by definition of λ_i. Since m,n were arbitrary, x is aperiodic, and since v was arbitrary, it follows that Λ satisfies the aperiodicity condition. $\square$

To prove that Condition (B) is equivalent to the other conditions in Lemma 3.2, we use a technical lemma which we will use again in the proof of Proposition 3.5.

Lemma 3.3. *Let Λ be a row-finite k-graph with no sources. Suppose that Λ has local periodicity at v, and fix $m \neq n \in \mathbb{N}^k$ such that $\sigma^m(x) = \sigma^n(x)$ for all $x \in v\Lambda^\infty$. Fix $\mu \in v\Lambda^m$ and $\alpha \in s(\mu)\Lambda^{(m \vee n)-m}$ and let $\nu = (\mu\alpha)(0,n)$. Then $\mu\alpha y = \nu\alpha y$ for all $y \in s(\alpha)\Lambda^\infty$.*

Proof. Fix $y \in s(\alpha)\Lambda^\infty$, and let $x := \mu\alpha y$. Since $x \in v\Lambda^\infty$, $\sigma^m(x) = \sigma^n(x)$. We have $\sigma^m(x) = \alpha y$ by definition. Since $\nu = (\mu\alpha)(0, n)$, $x(0, n) = \nu$, so $x = \nu\sigma^n(x)$. As $\sigma^n(x) = \sigma^m(x)$, it follows that $\sigma^n(x) = \alpha y$, so $\mu\alpha y = x = \nu\sigma^n(x) = \nu\alpha y$. □

Proof of equivalence of (i) *and* (ii) *in Lemma 3.2.* Remark 4.4 of [8] shows that the aperiodicity condition implies Condition (B). Since (i), (iii) and (iv) are equivalent, it now suffices to show that Condition (B) implies that Λ has no local periodicity. We argue by contrapositive. Suppose Λ has local periodicity at v. Let m, n, μ, ν and α be as in Lemma 3.3. Since $d(\mu\alpha) = m + d(\alpha) \neq n + d(\alpha) = d(\nu\alpha)$, we have $\mu\alpha \neq \nu\alpha$. Since Lemma 3.3 implies that $\mu\alpha y = \nu\alpha y$ for all $y \in s(\alpha)\Lambda^\infty$, it follows that Λ does not satisfy Condition (B). □

Our next steps are to describe the significance of cofinality and of the aperiodicity condition in terms of ideals in $C^*(\Lambda)$. The proof of Proposition 3.4 is not new (see for example [1, Proposition 5.1] and [4, Proposition 4.8]) but the result has not to our knowledge been stated explicitly before now.

For $z \in \mathbb{T}^k$ and $n \in \mathbb{Z}^k$, we use the multi-index notation z^n for the product $z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k} \in \mathbb{T}$. Recall from [8, Section 4] that the universal property of $C^*(\Lambda)$ supplies automorphisms $\{\gamma_z : z \in \mathbb{T}^k\}$ of $C^*(\Lambda)$ which satisfy $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ for all $\lambda \in \Lambda$ and $z \in \mathbb{T}^k$. The map $z \mapsto \gamma_z$ is a strongly continuous action of $\mathbb{T}^k$ on $C^*(\Lambda)$. The fixed point algebra $C^*(\Lambda)^\gamma$ is called the *core* of $C^*(\Lambda)$ and is equal to $\overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}$.

Proposition 3.4. *Let Λ be a row-finite k-graph with no sources. The following are equivalent:*

(i) *Λ is cofinal.*
(ii) *If I is an ideal of $C^*(\Lambda)$ and $s_v \in I$ for some $v \in \Lambda^0$, then $I = C^*(\Lambda)$.*
(iii) *If I is an ideal of $C^*(\Lambda)$ and $I \cap C^*(\Lambda)^\gamma \neq \{0\}$, then $I = C^*(\Lambda)$.*

To prove the proposition, we need to recall some terminology from [8, Section 5]. We say that $H \subset \Lambda^0$ is *hereditary* if $r(\lambda) \in H$ implies $s(\lambda) \in H$ for all $\lambda \in \Lambda$, and that H is *saturated* if we have $v \in H$ whenever there exists $n \in \mathbb{N}^k$ such that $s(\lambda) \in H$ for all $\lambda \in v\Lambda^n$.

Proof. We first show that (i) and (ii) are equivalent, and then that (ii) and (iii) are equivalent.

(i) $\Longrightarrow$ (ii). Suppose that Λ is cofinal. An argument formally identical to the second paragraph of [1, Proposition 5.1] shows that the only nonempty saturated hereditary subset of Λ^0 is Λ^0 itself. Theorem 5.2 of [8] then implies that the only ideal of $C^*(\Lambda)$ which contains a vertex projection is $C^*(\Lambda)$ itself.

(ii) $\Longrightarrow$ (i). We argue by contrapositive. Suppose that Λ is not cofinal. We must construct an ideal I of $C^*(\Lambda)$ such that $I \neq C^*(\Lambda)$, but $s_v \in I$ for some $v \in \Lambda^0$.

Since Λ is not cofinal, there exists a vertex $v_0 \in \Lambda^0$ and an infinite path $x \in \Lambda^\infty$ such that $v_0\Lambda x(n) = \emptyset$ for all $n \in \mathbb{N}^k$. Let

$$H_x := \{w \in \Lambda^0 : w\Lambda x(n) = \emptyset \text{ for all } n \in \mathbb{N}^k\}.$$

Then $\emptyset \subsetneq H_x \subsetneq \Lambda^0$ because $v_0 \in H_x$ but $x(0) \notin H_x$. We claim that H_x is saturated and hereditary.

To see that H_x is hereditary, fix $u \in H_x$ and $v \in \Lambda^0$ with $u\Lambda v \neq \emptyset$, say $\lambda \in u\Lambda v$. If we suppose for contradiction that $v \notin H_x$, then there exists $\alpha \in v\Lambda x(n)$ for some $n \in \mathbb{N}^k$ and it follows that $\lambda\alpha \in u\Lambda x(n)$ contradicting $u \in H_x$.

To see that H_x is saturated, fix $v \in \Lambda^0$ and $m \in \mathbb{N}^k$ such that $s(\lambda) \in H_x$ for all $\lambda \in v\Lambda^m$. Suppose for contradiction that $v \notin H_x$. Then there exists $\alpha \in v\Lambda x(n)$ for some $n \in \mathbb{N}^k$. Let $\alpha' = x(n, n+m)$. We have $s(\alpha') = x(n+m)$ and $r(\alpha') = s(\alpha)$. By choice, $d(\alpha\alpha') \geq m$ so we may rewrite $\alpha\alpha' = \mu\nu$ where $\mu \in v\Lambda^m$. By choice of m, we have $s(\mu) \in H_x$, and since H_x is hereditary, it follows that $s(\nu) \in H_x$. But $s(\nu) = s(\alpha') = x(n+m)$, contradicting the definition of H_x.

Since $\emptyset \subsetneq H_x \subsetneq \Lambda^0$, Theorem 5.2 of [8] implies that I_{H_x} is a nontrivial gauge-invariant ideal of $C^*(\Lambda)$.

(ii) $\Longrightarrow$ (iii). Recall from [8, Section 4] that there is an isomorphism π from $C^*(\Lambda)^\gamma$ to $\varinjlim_{n\in\mathbb{N}^k} \bigoplus_{v\in\Lambda^0} M_{\Lambda^n v}(\mathbb{C})$ which takes $s_\lambda s_\lambda^*$ to the diagonal matrix unit $\theta_{\lambda,\lambda}$ in $M_{\Lambda^{d(\lambda)} s(\lambda)}(\mathbb{C})$. Let I be an ideal of $C^*(\Lambda)$ which intersects $C^*(\Lambda)^\gamma$ nontrivially. Then I must intersect $\pi^{-1}(\bigoplus_{v\in\Lambda^0} M_{\Lambda^n v}(\mathbb{C}))$ for some n, hence must intersect one of the summands $\pi^{-1}(M_{\Lambda^n v})$. As $\pi^{-1}(M_{\Lambda^n v})$ is simple, it follows that $I \cap \pi^{-1}(M_{\Lambda^n v}) = \pi^{-1}(M_{\Lambda^n v})$, so I contains $s_\lambda s_\lambda^*$ for some $\lambda \in \Lambda^n v$. Hence $s_v = s_\lambda^*(s_\lambda s_\lambda^*)s_\lambda \in I$, and (ii) implies that $I = C^*(\Lambda)$.

(iii) $\Longrightarrow$ (ii). Trivial. □

Let $\{\xi_x : x \in \Lambda^\infty\}$ denote the usual basis for $\ell^2(\Lambda^\infty)$. As in [8, Theorem 3.15], there is a family $\{S_\eta : \eta \in \Lambda\} \subset \mathcal{B}(\ell^2(\Lambda^\infty))$ satisfying the Cuntz-Krieger relations such that

$$S_\eta \xi_x = \begin{cases} \xi_{\eta x} & \text{if } r(x) = s(\eta) \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad S_\eta^* \xi_y = \begin{cases} \xi_{\sigma^{d(\eta)}(y)} & \text{if } y(0, d(\eta)) = \eta \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

The universal property of $C^*(\Lambda)$ gives a homomorphism $\pi_S : C^*(\Lambda) \to \mathcal{B}(\ell^2(\Lambda^\infty))$ satisfying $\pi_S(s_\eta) = S_\eta$ for all $\eta \in \Lambda$. We call π_S the infinite path representation.

Proposition 3.5. *Let Λ be a row-finite k-graph with no sources. The following are equivalent:*

(i) *Λ has no local periodicity.*
(ii) *Every nonzero ideal of $C^*(\Lambda)$ contains a vertex projection.*
(iii) *The infinite path representation π_S is faithful.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that for each $v \in \Lambda^0$ and each pair $m \neq n \in \mathbb{N}^k$ there exists $x \in v\Lambda^\infty$ such that $\sigma^m(x) \neq \sigma^n(x)$. Then Lemma 3.2 implies that Λ satisfies Condition (B). The Cuntz-Krieger uniqueness theorem [8, Theorem 4.3] therefore implies that every ideal of $C^*(\Lambda)$ contains a vertex projection.

(ii) $\Longrightarrow$ (iii). For $v \in \Lambda^0$, $\pi_S(s_v) = S_v$ is the projection onto $\overline{\text{span}}\{\xi_x : x \in v\Lambda^\infty\}$ and so is nonzero. So $\ker(\pi_S)$ contains no vertex projection and is trivial by (ii).

(iii) $\Longrightarrow$ (i). We argue by contrapositive. Suppose that Λ has local periodicity at $v \in \Lambda^0$. By Lemma 3.3 there exist $m \neq n \in \mathbb{N}^k$, $\mu \in v\Lambda^m$, $\nu \in v\Lambda^n s(\mu)$ and $\alpha \in s(\mu)\Lambda$ such that $\mu\alpha y = \nu\alpha y$ for all $y \in s(\alpha)\Lambda^\infty$.

We will show that $a := s_{\mu\alpha}s_{\mu\alpha}^* - s_{\nu\alpha}s_{\mu\alpha}^*$ belongs to $\ker(\pi_S) \setminus \{0\}$.

We begin by showing that a is nonzero. The gauge action γ of $\mathbb{T}^k$ on $C^*(\Lambda)$ satisfies $\gamma_z(s_{\mu\alpha}s_{\mu\alpha}^*) = s_{\mu\alpha}s_{\mu\alpha}^*$ and $\gamma_z(s_{\nu\alpha}s_{\mu\alpha}^*) = z^{n-m}s_{\nu\alpha}s_{\mu\alpha}^*$. Fix $\omega \in \mathbb{T}^k$ such that $\omega^{n-m} = -1$. Suppose for contradiction that $a = 0$. Then

$$0 = a + \gamma_\omega(a) = s_{\mu\alpha}s_{\mu\alpha}^* - s_{\nu\alpha}s_{\mu\alpha}^* + s_{\mu\alpha}s_{\mu\alpha}^* + s_{\nu\alpha}s_{\mu\alpha}^* = 2s_{\mu\alpha}s_{\mu\alpha}^*, \tag{3.2}$$

contradicting [8, Theorem 3.15] which shows that $s_\lambda \neq 0$ for all $\lambda \in \Lambda$.

To see that $\pi_S(a) = 0$, we fix $x \in \Lambda^\infty$ and show that $\pi_S(a)\xi_x = 0$. By (3.1),

$$S_{\mu\alpha}S_{\mu\alpha}^*\xi_x = \begin{cases} \xi_x & \text{if } x(0, m \vee n) = \mu\alpha \\ 0 & \text{otherwise.} \end{cases}$$

We consider two cases: either $x(0, m \vee n) \neq \mu\alpha$, or $x = \mu\alpha y$ for some $y \in \Lambda^\infty$. In the first case, we have $\pi_S(a)\xi_x = 0$ by (3.1). In the second case, our choice of μ, ν, α ensures that $x = \mu\alpha y = \nu\alpha y$. Equation (3.1) therefore implies that $S_{\nu\alpha}S_{\mu\alpha}^*\xi_x = S_{\nu\alpha}\xi_y = \xi_x$. Hence $\pi_S(a)\xi_x = 0$.

As x was arbitrary, $\pi_S(a)$ annihilates all basis elements of $\ell^2(\Lambda^\infty)$, so is equal to zero. Since $\pi_S(s_v) = S_v \neq 0$ for each $v \in \Lambda^0$, $\ker(\pi_S)$ is an ideal of $C^*(\Lambda)$ which contains no vertex projection. □

Proof of Theorem 3.1. If $C^*(\Lambda)$ is simple, then Proposition 3.4 implies that Λ is cofinal, and Proposition 3.5 implies that Λ has no local periodicity. Conversely, if Λ is cofinal and has no local periodicity and I is an ideal of $C^*(\Lambda)$, then Proposition 3.5 implies $s_v \in I$ for some $v \in \Lambda^0$, and then Proposition 3.4 implies that $I = C^*(\Lambda)$. □

Recall from [8, Section 5] that if $H \subset \Lambda^0$ is hereditary, then $\Lambda \setminus \Lambda H := \{\lambda \in \Lambda : s(\lambda) \notin H\}$ is itself a locally convex row-finite k-graph. It is easy to check that if H is also saturated, then $\Lambda \setminus \Lambda H$ also has no sources. As in [8], given a saturated hereditary $H \subset \Lambda^0$, we denote by I_H the ideal generated by $\{s_v : v \in H\}$; and given an ideal $I \subset C^*(\Lambda)$, we denote by H_I the collection $\{v \in \Lambda^0 : s_v \in I\}$.

Proposition 3.6. *Let Λ be a row-finite k-graph with no sources. Then the following are equivalent:*

(i) *Every ideal of $C^*(\Lambda)$ is gauge-invariant.*
(ii) *For every saturated hereditary $H \subset \Lambda^0$, $\Lambda \setminus \Lambda H$ has no local periodicity.*

Proof. (ii) $\implies$ (i). Lemma 3.2 shows that each $\Lambda \setminus \Lambda H$ satisfies Condition (B). Hence Theorem 5.3 of [8] implies that every ideal of $C^*(\Lambda)$ is gauge invariant.

(i) $\implies$ (ii). We argue by contrapositive. Suppose that there is a saturated hereditary subset H of Λ such that $\Lambda \setminus \Lambda H$ has local periodicity at v, say. Let $\{t_\lambda : \lambda \in \Lambda \setminus \Lambda H\}$ denote the universal generating Cuntz-Krieger family for $C^*(\Lambda \setminus \Lambda H)$. Theorem 5.2 of [8] shows that there is an isomorphism ϕ of $C^*(\Lambda)/I_H$ onto $C^*(\Lambda \setminus \Lambda H)$ satisfying $\phi(s_\lambda + I_H) = t_\lambda$ for all $\lambda \in \Lambda \setminus \Lambda H$. Let q_{I_H} denote the quotient map from $C^*(\Lambda)$ to $C^*(\Lambda)/I_H$.

The argument of Proposition 3.5 gives a nonzero element $a = t_{\mu\alpha}t_{\mu\alpha}^* - t_{\nu\alpha}t_{\mu\alpha}^*$ of $C^*(\Lambda \setminus \Lambda H)$ which satisfies $\pi_T(a) = 0$ where π_T is the infinite-path representation of $C^*(\Lambda \setminus \Lambda H)$. Let $b := s_{\mu\alpha}s_{\mu\alpha}^* - s_{\nu\alpha}s_{\mu\alpha}^* \in C^*(\Lambda)$. By definition of b, we have $\phi \circ q_{I_H}(b) = a \neq 0$, but $\pi_T \circ \phi \circ q_{I_H}(b) = 0$. Since ϕ is an isomorphism, the kernel of $\phi \circ q_{I_H}$ is precisely I_H. Theorem 5.2 of [8] implies that $H_{I_H} = H$. Since the kernel of π_T contains no of the vertex projections of $C^*(\Lambda \setminus \Lambda H)$, the ideal $J = \ker(\pi_T \circ \phi \circ q_{I_H})$ also satisfies $H_J = H$. Now $J \neq I_H$ because $b \in J \setminus I_H$. Theorem 5.2 of [8] implies that $I \mapsto H_I$ is a bijection between gauge-invariant ideals of $C^*(\Lambda)$ and saturated hereditary subsets of Λ^0 with inverse $H \mapsto I_H$. Since $J \neq I = I_{H_J}$, it follows that J is a nontrivial ideal of $C^*(\Lambda)$ which is not gauge-invariant. □

Appendix A. Finite paths and aperiodicity

Condition (iv) of Lemma 3.2 insists that for each $v \in \Lambda^0$ and each $m \neq n \in \mathbb{N}^k$ there exists a path $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

In this appendix we attempt to provide some intuition for what this condition says.

Fix a vertex v in a k-graph Λ, and a pair $m \neq n \in \mathbb{N}^k$. In Figure 1, a path λ in $v\Lambda^{(n\vee m)+l}$ is represented by the large rectangle. Regarded as a scale diagram,

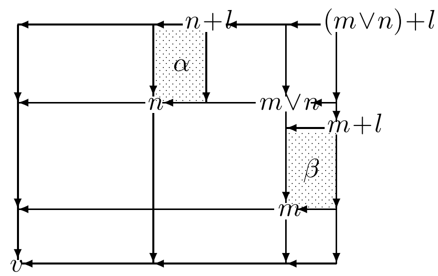

Figure 1. Condition (iv), Lemma 3.2 when m, n are not comparable.

Figure 1 illustrates the configuration $k = 2$, $m = (10, 2)$, $n = (5, 6)$. However, we can use this picture to represent the k-dimensional situation by using horizontal distance to represent the directions in which m is bigger than n and vertical distance to represent the directions in which n is bigger than m.

If Λ satisfies Condition (iv) of Lemma 3.2, then there exists a path λ as in the diagram whose degree $(m \vee n) + l$ is greater than both m and n and whose segment from m to $m + l$ is distinct from the segment from n to $n + l$. (In the picture, $l = (2, 3)$, but more generally it is the difference, represented by the top-right rectangle, between the least upper bound of m and n and the degree of λ.)

The factorisation property ensures that for each path from top right to bottom left in the picture there is a unique factorisation of λ into segments of the corresponding degrees. Condition (iv) of Lemma 3.2 insists that there exists λ as shown in Figure 1 for which the shaded segments α and β are distinct.

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