UNIQUENESS THEOREMS FOR TOPOLOGICAL HIGHER-RANK GRAPH C^*-ALGEBRAS

JEAN RENAULT, AIDAN SIMS, DANA P. WILLIAMS, AND TRENT YEEND

Abstract. We consider the boundary-path groupoids of topological higher-rank graphs. We show that the all such groupoids are topologically amenable. We deduce that the C^*-algebras of topological higher-rank graphs are nuclear and prove versions of the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem. We then provide a necessary and sufficient condition for simplicity of a topological higher-rank graph C^*-algebra, and a condition under which it is also purely infinite.

1. Introduction

Groupoids are a powerful and widely applicable model for operator algebras. One area of operator-algebra theory in which they have been particularly prominent recently is the field of graph algebras and their analogues.

The inception of the field of graph C^*-algebras goes back to the work of Cuntz and Krieger [5], and the subsequent work of Enomoto and Watatani [6], on simple purely infinite C^*-algebras associated to finite binary matrices. However, the theory of graph C^*-algebras really took off after the work of Kumjian-Pask-Raeburn-Renault [12]. The analysis there was facilitated by realising the C^*-algebras of interest as groupoid C^*-algebras and employing Renault's structure theory [18].

Since then, the class of graph C^*-algebras has been generalised in various directions, including ultragraph C^*-algebras [22], higher-rank graph C^*-algebras [11] and topological graph C^*-algebras [10] to name a few. Though these generalisations have not all been developed using groupoid methods, in each case a natural groupoid model exists [8, 13, 15, 26].

In 2005, Yeend developed the notion of a topological higher-rank graph, simultaneously generalising Katsura's notion of a topological graph and Kumjian and Pask's notion of a higher-rank graph. Yeend associated to each topological higher-rank graph Λ a groupoid $\mathcal{G}_\Lambda$ and hence a C^*-algebra $C^*(\Lambda) := C^*(\mathcal{G}_\Lambda)$. Yeend's construction is sufficiently general to capture Katsura's algebras and the finitely aligned k-graph C^*-algebras of [16]. However, the question of amenability of $\mathcal{G}_\Lambda$ remained unresolved in general, so Yeend's key C^*-algebraic results held only under additional hypotheses. In addition, the injectivity hypothesis on Yeend's uniqueness theorems is phrased in terms of functions on $\mathcal{G}_\Lambda$ rather than in terms of the underlying topological k-graph Λ.

The first version of the current paper, posted by Renault, Sims and Yeend on the arXiv preprint server in 2009, aimed to resolve the question of amenability of Yeend's groupoid

Date: September 11, 2012.

1991 *Mathematics Subject Classification.* Primary 46L05.

Key words and phrases. Topological graph, higher rank graph, groupoid, amenable groupoid, amenability, graph algebra, Cuntz-Krieger algebra.

This research was supported by the Australian Research Council.

and to prove versions of the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem with an injectivity hypothesis involving only the algebra of continuous functions on the vertex set of the topological graph. Our approach to amenability was to show that the kernel of the canonical $\mathbb{Z}^k$-valued cocycle c on $\mathcal{G}_\Lambda$ was amenable (by providing a measure-theoretic direct-limit decomposition into equivalence relations) and then bootstrap up to amenability of $\mathcal{G}_\Lambda$ by composing invariant means on $c^{-1}(0)$ with the mean on $\mathbb{Z}^k$. Shortly after submission, Williams spotted an error in our bootstrapping argument. The paper was withdrawn and Williams became involved as we considered how to repair the gap. In the mean time, versions of the key results, namely the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem, were proved respectively in [4] and [24] using the machinery of product systems (though our Proposition 4.2 is required to identify the C^*-algebras described there with Yeend's C^*-algebra). As a result, this project lay dormant for some time.

Two recent developments brought the project out of mothballs. The first is Spielberg's clever argument [21, Proposition 9.3] which combines groupoid theory and coaction theory to show that if c is a cocycle from an étale Hausdorff groupoid $\mathcal{G}$ into a countable abelian group G and $c^{-1}(0)$ is amenable, then $\mathcal{G}$ is amenable; this fixed the gap in our original argument. The second is the recent characterisation of simplicity for the C^*-algebra of a second-countable locally compact Hausdorff étale amenable groupoid $\mathcal{G}$ in [3]: Yeend showed in [27] that $\mathcal{G}_\Lambda$ has all these properties except for amenability, and simplicity was not addressed in either [4] or [24].

In this revised article, we combine Spielberg's argument with our previous analysis to prove that $\mathcal{G}_\Lambda$ is amenable and prove a gauge-invariant uniqueness theorem. The proof of what is now Proposition 3.1 has been significantly simplified by [4, Proposition 5.16]. We then use the results of [3] to prove a version of the Cuntz-Krieger uniqueness theorem and to provide a necessary and sufficient condition for simplicity of $C^*(\Lambda)$. We conclude by providing a sufficient condition for $C^*(\Lambda)$ to be purely infinite.

Acknowledgement. We thank Toke Carlsen for his reading of a preliminary draft of the manuscript, and for his valuable comments and input.

2. Background

Our results require a fair amount of background. We do not give full detail, especially as regards the theory of groupoids. For more detail see, for example, [2, 18, 27].

We regard $\mathbb{N}^k$ as a semigroup with identity 0, or sometimes as a category with a single object and composition defined by the addition operation. For $m, n \in \mathbb{N}^k$, we say $m \leq n$ if $m_i \leq n_i$ for all $i \in \{1, \ldots, k\}$. We write $m \vee n$ for the coordinatewise maximum of m and n. We frequently work also with the set $(\mathbb{N} \cup \{\infty\})^k$; we extend the addition operation and the order $\leq$ from $\mathbb{N}^k$ to $(\mathbb{N} \cup \{\infty\})^k$ in the obvious way.

2.1. **Groupoids.** For details of what follows, see [18]. A groupoid $\mathcal{G}$ is a small category with inverses. We denote the domain and codomain maps by s and r, the unit space by $\mathcal{G}^{(0)}$, and the collection of composable pairs by $\mathcal{G}^{(2)}$. A topological groupoid is a groupoid endowed with a topology under which both the inversion map and the composition map are continuous. In this paper, we consider only locally compact Hausdorff groupoids. A groupoid is étale if the range map is a local homeomorphism; when the topology is

Hausdorff, it follows that $\mathcal{G}^{(0)}$ is both open and closed in $\mathcal{G}$. For each unit u of an étale groupoid $\mathcal{G}$ the sets $\mathcal{G}^u := r^{-1}(u)$ and $\mathcal{G}_u := s^{-1}(u)$ are discrete.

The *isotropy* at a unit $u \in \mathcal{G}^{(0)}$ is the group $\mathcal{G}^u_u : r^{-1}(u) \cap s^{-1}(u)$. The phrase "points with trivial isotropy" is frequently used in the literature to refer to the units u of a groupoid $\mathcal{G}$ such that the isotropy at u is the trivial group. We say that a groupoid $\mathcal{G}$ is *principal* if every unit u has trivial isotropy; algebraically, these are just equivalence relations, but topologically they can be quite different.

The *orbit* of a unit u of a groupoid $\mathcal{G}$ is the set

$$[u] = \{r(x) : s(x) = u\};$$

that is, $[u] = r(\mathcal{G}_u) = s(\mathcal{G}^u)$. A subset U of $\mathcal{G}^{(0)}$ is *invariant* if $[u] \subseteq U$ whenever $u \in U$.

2.2. **Groupoid C^*-algebras.** We will now summarise the constructions of the full- and reduced groupoid C^*-algebras of an étale locally compact Hausdorff groupoid $\mathcal{G}$. These constructions can be carried through for any groupoid admitting a Haar system, but the formulae are simpler in our situation. For full details see [18, Section II.1]; or for a detailed treatment of étale groupoids, see [14] or [7, Section 3].

Consider the space $C_c(\mathcal{G})$ of compactly supported complex-valued functions on $\mathcal{G}$. For $x \in \mathcal{G}$ and $f \in C_c(\mathcal{G})$, the set $\{y : r(y) = r(x), f(y) \neq 0\}$ is both compact and discrete and hence finite. Thus we may sensibly define an operation $* : C_c(\mathcal{G}) \times C_c(\mathcal{G}) \to C_c(\mathcal{G})$ by

$$(f * g)(x) = \sum_{r(y)=r(x)} f(y)g(y^{-1}x).$$

The space $C_c(\mathcal{G})$ becomes a topological *-algebra with the involution $f^*(x) = \overline{f(x^{-1})}$ and the convolution product $*$ defined above.

A representation of $C_c(\mathcal{G})$ is a nondegenerate *-homomorphism $\pi : C_c(\mathcal{G}) \to B(\mathcal{H})$ which is continuous from the inductive limit topology on $C_c(\mathcal{G})$ to the strong-operator topology on $\mathcal{B}(\mathcal{H})$. Renault's disintegration theorem [19, Proposition 4.2] together with [18, Propositions II.1.7 and II.1.11] implies that there is a pre-C^*-norm on $C_c(\mathcal{G})$ determined by

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ is a representation of } C_c(\mathcal{G})\}.$$

The full groupoid C^*-algebra $C^*(\mathcal{G})$ is the C^*-completion of $C_c(\mathcal{G})$ with respect to this norm.

To define the reduced groupoid C^*-algebra, fix $u \in \mathcal{G}^{(0)}$ and let $\ell^2(\mathcal{G}_u)$ be the Hilbert space with orthonormal basis $\{\xi_x : x \in \mathcal{G}_u\}$. There is a representation $\operatorname{Ind}\epsilon_u : C_c(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_u))$ such that for $f \in C_c(\mathcal{G})$ and $x \in \mathcal{G}_u$, we have

$$\operatorname{Ind}\epsilon_u(f)\xi_x = \sum_{y \in \mathcal{G}_u} f(yx^{-1})\xi_y.$$

The reduced groupoid C^*-algebra $C^*_r(\mathcal{G})$ is then the completion of $C_c(\mathcal{G})$ in the C^*-norm

$$\|f\|_r = \sup_{u \in \mathcal{G}^{(0)}} \|\operatorname{Ind}\epsilon_u(f)\|.$$

There are at least two notions of amenability for groupoids: (topological) amenability [2, Definition 2.2.8], and measurewise amenability [2, Definition 3.3.1]. Topological amenability of $\mathcal{G}$ implies measurewise amenability of $\mathcal{G}$ [2, page 83] which in turn implies that $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$ coincide [2, Proposition 6.18]. If $\mathcal{G}$ is both étale and second-countable, then it has countable orbits and a continuous Haar system (consisting of counting measures), and in this case [2, Theorem 3.3.7] implies that measurewise and topological amenability are equivalent.

2.3. **Topological higher-rank graphs.** For the details of this and the next section, see [25, 27]. A k-graph is a small category Λ equipped with a functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the factorisation property: if $d(\lambda) = m + n$ then there exist unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$. We call d the *degree map* on Λ, and denote $d^{-1}(n)$ by Λ^n.[1] An argument using the factorisation property shows that Λ^0 is equal to the collection of identity morphisms of Λ. So the domain and codomain functions determine maps $s, r : \Lambda \to \Lambda^0$ which we call the source and range maps.

We regard a k-graph Λ as a kind of generalised directed graph: we think of Λ^0 as a collection of vertices; we think of each $\lambda \in \Lambda$ as a path from $s(\lambda)$ to $r(\lambda)$; and the degree map d plays the role of a generalised length function.

Given sets $X, Y \subseteq \Lambda$ we write XY for the set $\{\mu\nu : \mu \in X, \nu \in Y, s(\mu) = r(\nu)\}$. In particular, for $V \subseteq \Lambda^0$ and $X \subseteq \Lambda$, $VX = \{\lambda \in X : r(\lambda) \in V\}$ and $XV = \{\lambda \in X : s(\lambda) \in V\}$. By the usual abuse of notation, for a singleton set $\{\lambda\} \subseteq \Lambda$, we write $\Lambda\lambda$ and $\lambda\Lambda$ in place of $\Lambda\{\lambda\}$ and $\{\lambda\}\Lambda$.

The factorisation property implies that if $\lambda \in \Lambda$ and $m \le n \le d(\lambda)$, then there are unique $\lambda(0, m) \in \Lambda^m$, $\lambda(m, n) \in \Lambda^{n-m}$ and $\lambda(n, d(\lambda)) \in \Lambda^{d(\lambda)-n}$ such that $\lambda = \lambda(0, m)\lambda(m, n)\lambda(n, d(\lambda))$. We think of $\lambda(m, n)$ as the segment of λ from position m to position n along λ.

Given μ and ν in Λ, we say that λ is a common extension of μ and ν if we can factorise $\lambda = \mu\mu'$ and $\lambda = \nu\nu'$ for some $\mu', \nu' \in \Lambda$. We say that λ is a minimal common extension of μ and ν if it is a common extension such that $d(\lambda) = d(\mu) \vee d(\nu)$. We denote by $\mathrm{MCE}(\mu, \nu)$ the set of all minimal common extensions of μ and ν. If $r(\mu) \ne r(\nu)$, then $\mathrm{MCE}(\mu, \nu) = \emptyset$. Given subsets $X, Y \subseteq \Lambda$, we define

$$\mathrm{MCE}(X, Y) := \bigcup_{\mu \in X, \nu \in Y} \mathrm{MCE}(\mu, \nu).$$

A topological k-graph is a k-graph Λ endowed with a second-countable locally compact Hausdorff topology such that each Λ^n is open, the range map is continuous, composition is continuous and open, and the source map is a local homeomorphism.

We say that the topological k-graph Λ is compactly aligned if, for every pair of compact subsets $X, Y \subseteq \Lambda$, the set $\mathrm{MCE}(X, Y)$ is also compact. Given $v \in \Lambda^0$ we say that a subset E of Λ is compact exhaustive for v if E is compact, $r(E)$ is a neighbourhood of v, and for all $\lambda \in r(E)\Lambda$ there exists $\mu \in E$ such that $\mathrm{MCE}(\lambda, \mu) \ne \emptyset$.

An important class of examples of higher-rank graphs, which we will use to make sense of the notion of an infinite path in a k-graph, are the (discrete) higher-rank graphs $\Omega_{k,m}$. These are defined as follows. For fixed $k \ge 1$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, the k-graph $\Omega_{k,m}$ has morphisms $\{(p, q) : p, q \in \mathbb{N}^k, p \le q \le m\}$. The range and source maps are $r(p, q) = (p, p)$ and $s(p, q) = (q, q)$, composition is determined by $(p, q)(q, r) = (p, r)$, and the degree map is given by $d(p, q) = q - p$. We usually abbreviate a vertex (p, p) of $\Omega_{k,m}$ as p.

A k-graph morphism x from a k-graph Λ to a k-graph Γ is a functor $x : \Lambda \to \Gamma$ which intertwines the degree maps. Given a k-graph Λ, each $\lambda \in \Lambda$ determines, and is determined

[1] When $k = 1$ so that $n \in \mathbb{N}$, there is a slight clash of notation here with the usual notation for the product space $\prod_{i=1}^{n} \Lambda$; but the meaning is usually clear from context.

by, the unique k-graph morphism $x_\lambda : \Omega_{k,d(\lambda)} \to \Lambda$ such that $x_\lambda(m,n) = \lambda(m,n)$ for all $m \leq n \leq d(\lambda)$. By analogy, for arbitrary $m \in (\mathbb{N} \cup \{\infty\})^k$, we regard a k-graph morphism $x : \Omega_{k,m} \to \Lambda$ as a (possibly infinite) path in Λ, and define $r(x) := x(0)$, and $d(x) = m$. If $m_i < \infty$ for all i, then we also write $s(x)$ for $x(m)$, but if $m_i = \infty$ for some i, then x has no source.

For $m \in (\mathbb{N} \cup \{\infty\})^k$, a boundary path of degree m in a topological k-graph Λ is a k-graph morphism $x : \Omega_{k,m} \to \Lambda$ such that for every $n \in \mathbb{N}^k$ with $n \leq m$ and each compact exhaustive set E for $x(n,n)$, there is an element λ of E such that $d(x) \geq n + d(\lambda)$ and $x(n, n+d(\lambda)) = \lambda$. Fix a boundary path x of degree m. For each $n \in \mathbb{N}^k$ with $n \leq m$, there is a unique boundary path $\sigma^n(x)$ of degree $m-n$ defined by $\sigma^n(x)(p,q) = x(n+p, n+q)$ for all $(p,q) \in \Omega_{k,m-n}$. Given $\mu \in \Lambda$ with $s(\mu) = r(x)$, there is a unique boundary path μx of degree $d(\mu)+m$ such that $(\mu x)(0, d(\mu)) = \mu$ and such that $(\mu x)(p+d(\mu), q+d(\mu)) = x(p,q)$ for all $(p,q) \in \Omega_{k,m}$. We have $\sigma^{d(\mu)}(\mu x) = x = x(0,n)\sigma^n(x)$.

We denote the collection of all boundary paths in Λ by $\partial\Lambda$. For a subset U of Λ, we denote by $Z(U)$ the collection

$$\{x \in \partial\Lambda : x(0,n) \in U \text{ for some } n \in \mathbb{N}^k \text{ with } n \leq d(x)\}.$$

The collection of sets

$$\{Z(U) \cap Z(F)^c : U \subseteq \Lambda^m \text{ is relatively compact and open, } F \subseteq \overline{U}\Lambda \text{ is compact}\} \tag{2.1}$$

form a basis for a locally compact Hausdorff topology on $\partial\Lambda$.

2.4. **The boundary-path groupoid of a topological higher-rank graph.** Given a compactly aligned topological k-graph Λ, we define a set $\mathcal{G}_\Lambda$ by

$$\mathcal{G}_\Lambda := \{(x, m-n, y) : m,n \in \mathbb{N}^k, x,y \in \partial\Lambda, m \leq d(x), n \leq d(y) \text{ and } \sigma^m(x) = \sigma^n(y)\}.$$

Define $\mathcal{G}_\Lambda^{(0)} := \{(x,0,x) : x \in \partial\Lambda\}$, and identify it with $\partial\Lambda$ via $(x,0,x) \mapsto x$. For $(x,p,y) \in \mathcal{G}_\Lambda$, define $r(x,p,y) = x$ and $s(x,p,y) = y$. With structure maps

$$(x,p,y)^{-1} = (y,-p,x), \qquad \text{and} \qquad (x,p,y)(y,q,z) = (x,p+q,z),$$

the set $\mathcal{G}_\Lambda$ becomes a groupoid with unit space $\partial\Lambda$, and $c(x,p,y) := p$ defines a continuous 1-cocycle $c : \mathcal{G}_\Lambda \to \mathbb{Z}^k$.

For $U, V \subseteq \Lambda$, define $U *_s V := \{(\mu,\nu) \in U \times V : s(\mu) = s(\nu)\}$. For $F \subseteq \Lambda *_s \Lambda$ and $p \in \mathbb{Z}^k$, define

$$Z(F,p) := \{(\mu x, p, \nu x) : (\mu,\nu) \in F, d(\mu) - d(\nu) = p, s(\mu) = s(\nu), x \in s(\mu)\partial\Lambda\}.$$

The following follows from Yeend's results, but it is worthwhile to state it explicitly.

Lemma 2.1. *Let Λ be a compactly aligned topological k-graph. Then*

$$\begin{aligned}\{Z(U *_s V, p-q) \cap Z(F, p-q)^c : p,q \in \mathbb{N}^k,\ U \subseteq \Lambda^p \text{ and } V \subseteq \Lambda^q,\\ U,V \text{ are relatively compact and open,}\\ \text{and } F \text{ is a compact subset of } \textstyle\bigcup_{\alpha\in\Lambda} \overline{U}\alpha \times \overline{V}\alpha\}\end{aligned}$$

is a basis for a locally compact Hausdorff topology on $\mathcal{G}_\Lambda$ under which $\mathcal{G}_\Lambda$ becomes a locally compact Hausdorff étale groupoid.

Proof. Proposition 3.6 and Theorem 3.16 of [27] imply that the sets of the form described in the lemma are a basis for a second-countable, locally compact, Hausdorff topology on the path groupoid G_Λ of Λ, and that G_Λ is an étale groupoid under this topology. Propositions 4.4 and 4.7 of [27] show that $\partial\Lambda$ is a closed invariant subset of $G_\Lambda^{(0)}$. Since $\mathcal{G}_\Lambda$ is by definition the restriction of G_Λ to $\partial\Lambda$, the result follows. □

Notation 2.2. Given $U \subseteq \Lambda^p$ and $V \subseteq \Lambda^q$ and a compact subset $F \subseteq \bigcup_{\alpha\in\Lambda} \overline{U}\alpha \times \overline{V}\alpha$, it is unambiguous to abbreviate the basic open set $Z(U *_s V, p-q) \cap Z(F, p-q)^c$ as $Z(U *_s V) \cap Z(F)^c$, and we will frequently do so.

The topological higher-rank graph C^*-algebra $C^*(\Lambda)$ is defined to be the full groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$.

3. Injectivity of representations on functions on the unit space

The uniqueness theorems in [27] start with a representation of $C^*(\mathcal{G}_\Lambda)$ which restricts to an injection of $C_0(\mathcal{G}_\Lambda^{(0)}) = C_0(\partial\Lambda)$. For graph C^*-algebras, topological graph C^*-algebras and higher-rank graph C^*-algebras, the usual hypothesis is that the given representation be injective on the embedded copy of $C_0(\Lambda^0)$. We show that the two hypotheses are equivalent by showing that injectivity on $C_0(\Lambda^0)$ implies injectivity on $C_0(\partial\Lambda)$. That is, the usual hypothesis also suffices for topological higher-rank graphs.

The definition of the topology on $\partial\Lambda$ ensures that the range map $r : x \mapsto x(0)$ is continuous from $\partial\Lambda$ to Λ^0. Proposition 4.3 of [27] implies that r is surjective. It therefore induces an injection

$$r^* : C_0(\Lambda^0) \hookrightarrow C_0(\partial\Lambda) \quad \text{such that } r^*(f) = f \circ r \quad \text{for all } f \in C_0(\Lambda^0).$$

Proposition 3.1. *Let Λ be a compactly aligned topological k-graph. Let π be a representation of $C^*(\mathcal{G}_\Lambda)$. If $\pi|_{r^*(C_c(\Lambda^0))}$ is injective, then $\pi|_{C_0(\mathcal{G}_\Lambda^{(0)})}$ is injective.*

Proof. The ideal $\ker(\pi)\cap C_0(\partial\Lambda)$ consists of all functions supported on some open invariant subset U of $\partial\Lambda$. So $X := \partial\Lambda \setminus U$ is a closed invariant set and π factors through a representation of $\mathcal{G}_\Lambda|_X$. That $\pi\circ r$ is injective on $C_c(\Lambda^0)$ implies that $X \cap Z(V) \neq \emptyset$ for every open $V \subseteq \Lambda^0$. Fix $v \in \Lambda^0$ and a fundamental sequence of compact neighbourhoods $(K_n)_{n=1}^\infty$ of $v \in \Lambda^0$. We have just seen that $X \cap K_n \neq \emptyset$ for all n, so fix a sequence $(x_n)_{n=1}^\infty$ with each $x_n \in X \cap K_n$. By compactness we may pass to a convergent subsequence with limit x, and then continuity of the range map ensures that $r(x) = v$. Hence $K \cap v\partial\Lambda \neq \emptyset$.

Proposition 5.16 of [4] implies that the only closed invariant set of $\partial\Lambda$ which intersects each $v\partial\Lambda$ is $\partial\Lambda$ itself. So $K = \partial\Lambda$, and hence $\ker(\pi)\cap C_0(\partial\Lambda) = \{0\}$. □

4. Amenability and the gauge-invariant uniqueness theorem

In this section we prove a variant of an Huef and Raeburn's gauge-invariant uniqueness theorem [9] for topological higher-rank graph C^*-algebras. A key ingredient is amenability of $\mathcal{G}_\Lambda$ which guarantees that $C^*(\mathcal{G}_\Lambda)$ and $C^*_r(\mathcal{G}_\Lambda)$ coincide; it follows from [18, Proposition 4.8] that the conditional expectation of $C^*(\mathcal{G}_\Lambda)$ onto $C_0(\mathcal{G}_\Lambda^{(0)})$ is faithful.

Recall that given a topological higher-rank graph Λ, we denote by c the canonical 1-cocycle $c : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ given by $c(x, m, y) = m$.

Theorem 4.1 (The gauge-invariant uniqueness theorem)**.** *Let Λ be a compactly aligned topological k-graph, and let $r^* : C_0(\Lambda^0) \to C_0(\partial\Lambda)$ be the homomorphism $f \mapsto f \circ r$.*

Suppose that $\pi : C^*(\Lambda) \to B$ is a homomorphism such that $\pi \circ r^*$ is injective on $C_c(\Lambda^0)$. Suppose that there is a strongly continuous action $\beta : \mathbb{T}^k \to \mathrm{Aut}(B)$ such that for each $n \in \mathbb{N}^k$ and $f \in C_c(\mathcal{G}_\Lambda)$ with $\mathrm{supp}(f) \subseteq c^{-1}(n)$, we have $\beta_z(\pi(f)) = z^n \pi(f)$. Then π is injective.

To prove the theorem, we first show that Yeend's boundary-path groupoid is amenable in the sense of [2], and then follow the standard argument of [11]. We begin by showing that the kernel of c is amenable. For us, amenability of $\mathcal{G}$ is important only as a hypothesis which ensures that $C^*(\mathcal{G}_\Lambda)$ and $C^*_r(\mathcal{G}_\Lambda)$ coincide, so we will not dwell on the rather technical definition. We thank Toke Carlsen for pointing out an error in an earlier version of the proof of this result.

Proposition 4.2. *Let Λ be a compactly aligned topological k-graph. Then the kernel $H := c^{-1}(0)$ of c is amenable, principal and satisfies $H^{(0)} = \mathcal{G}_\Lambda^{(0)}$.*

Proof. For $m \in \mathbb{N}^k$, let R_m denote the subgroupoid of H defined by

$$\begin{aligned} R_m &:= \{(x, 0, x) : x \in \partial\Lambda\} \cup \{(x, 0, y) : d(x) = d(y) \geq m, \sigma^m(x) = \sigma^m(y)\} \\ &= \{(x, 0, x) : x \in \partial\Lambda\} \cup \{(\alpha z, 0, \beta z) : z \in \partial\Lambda, \alpha, \beta \in \Lambda^m r(z)\}. \end{aligned}$$

Each R_m is an equivalence relation, and each R_m is also an F_σ set (that is, a countable union of closed sets) in $\partial\Lambda \times \partial\Lambda$ because $\mathcal{G}_\Lambda$ is locally compact. We claim that each R_m is proper as a Borel groupoid [2, Definition 2.1.2]. By [2, Examples 2.1.4(2)], this is equivalent to the quotient space being a standard Borel space. The Mackey-Glimm-Ramsay dichotomy [17, Theorem 2.1] implies that this in turn is equivalent here to the assertion that the orbits are locally closed.

Fix $m \in \mathbb{N}^k$. To see that the orbits in R_m are indeed locally closed, first observe that the orbit $[x]$ of x in R_m is equal to $\{x\}$ if $d(x) \not\geq m$, and is equal to $\{\alpha\sigma^m(x) : \alpha \in \Lambda^m, s(\alpha) = x(m)\}$ otherwise. In the first case, $[x] = \{x\}$ is in fact closed because the topology on $\mathcal{G}^0_\Lambda$ is Hausdorff. In the second case, we claim that

$$[x] = \Big(R_m^{(0)} \cap Z(\Lambda^m *_s \Lambda^m)\Big) \cap \overline{\{(\alpha\sigma^m(x), 0, \beta\sigma^m(x)) : d(\alpha) = d(\beta) = m\}}. \tag{4.1}$$

To see this, observe that the right-hand side clearly contains $[x]$, so we need only show the reverse inclusion. Fix

$$(w, p, z) \in \Big(R_m^{(0)} \cap Z(\Lambda^m *_s \Lambda^m)\Big) \cap \overline{\{(\alpha\sigma^m(x), 0, \beta\sigma^m(x)) : d(\alpha) = d(\beta) = m\}}.$$

Then $w = z \in \partial\Lambda$, $p = 0$, and $d(w) \geq m$. Fix a sequence of pairs $(\alpha_j, \beta_j) \in \Lambda^m x(m) \times \Lambda^m x(m)$ such that $(\alpha_j\sigma^m(x), 0, \beta_j\sigma^m(x)) \to (w, 0, w)$. In particular, $\alpha_j\sigma^m(x), \beta_j\sigma^m(x) \to w$ in $\partial\Lambda$. Then [27, Proposition 3.12](i) ensures that the $\alpha_j\sigma^m(x)(0, p \wedge d(x))$ converge to $w(0, m+p)$ for all $p \leq d(w) - m$, which implies that $d(w) \leq d(x)$ and then that in fact $(\alpha_j\sigma^m(x))(0, q) \to w(0, q)$ for all $q \leq d(w)$. It therefore suffices to show that $d(w) \geq d(x)$. Suppose for contradiction that $i \leq k$ satisfies $d(w)_i < d(x)_i$. Then $m_i \leq d(w)_i < d(x)_i$, which gives

$$(\alpha_j\sigma^m(x))(d(w), d(w) + e_i) = x(d(w), d(w) + e_i) \text{ for all } j \in \mathbb{N}.$$

In particular, the set

$$J_{d(w),i} = \{j \in \mathbb{N} : d(\alpha_j\sigma^m(x))_i \geq d(w) + e_i\}$$

is equal to $\mathbb{N}$ and hence infinite, but the sequence $(\alpha_j\sigma^m(x))(d(w), d(w)+e_i)_{j\in J_{d(w),i}}$ is the constant sequence, and in particular is not wandering, contradicting [27, Proposition 3.12](ii). This proves (4.1).

Since $\mathcal{G}_\Lambda$ is étale, the unit space $R_m^{(0)} = \mathcal{G}_\Lambda^{(0)}$ is open. Each $Z(\Lambda^n *_s \Lambda^n)$ is open in $\mathcal{G}_\Lambda$ by definition of the topology on $\mathcal{G}_\Lambda$, so

$$R_m^{(0)} \cap \bigcup_{n\geq m} Z(\Lambda^n *_s \Lambda^n)$$

is open in the relative topology on R_m. Hence $[x]$ is the intersection of an open set and a closed set in R_m and hence is locally closed. So R_n is a proper Borel groupoid; in particular it is measurewise amenable.

The groupoid $H = \bigcup_{n\in\mathbb{N}^k} R_n$ is therefore a direct limit (in the sense of [2, Section 5.3**f**]) of measurewise amenable groupoids, and hence is itself measurewise amenable by [2, Proposition 5.3.37]. Since $\mathcal{G}_\Lambda$ is étale by [27, Theorem 3.16 and Definition 4.8], H is also étale. Since it is second-countable, it follows that orbits are countable in H, so [2, Theorem 3.3.7] implies that H is topologically amenable.

The factorisation property in Λ implies that if $d(\alpha) = d(\beta)$, then for any $x \in \partial\Lambda$, we have $\alpha x = \beta x$ if and only if $\alpha = \beta$. So H is principal. □

In the earlier withdrawn version of this article, the first-, second- and fourth-named authors gave an incorrect proof that if $\mathcal{G}$ is a second-countable, locally compact, Hausdorff, étale, amenable groupoid and admits a continuous cocycle c into an amenable group such that the kernel of c is an amenable groupoid, then $\mathcal{G}$ itself is amenable. Our proof was flawed because it required strong surjectivity of c. The canonical $\mathbb{Z}^k$-valued cocycle on the groupoid of a topological k-graph is usually not strongly surjective unless the range-map in Λ is both proper and surjective on each Λ^n, in which case Yeend's original results [27] apply. Fortunately, this gap in our argument is filled in by a recent result of Spielberg [21].

Corollary 4.3. *Let Λ be a compactly aligned topological k-graph, and let $\mathcal{G}_\Lambda$ be the associated groupoid as in* [27]*. Then $\mathcal{G}_\Lambda$ is (topologically) amenable and $C^*(\Lambda)$ is nuclear.*

Proof. Proposition 9.3 of [21] says that if c is a continuous cocycle on a Hausdorff étale groupoid $\mathcal{G}$ taking values in a discrete abelian group G and the fixed-point groupoid for c is amenable, then $\mathcal{G}$ is amenable. To prove this result, Spielberg shows that $C^*(\mathcal{G})$ is nuclear and then applies [2, Corollary 6.2.14(ii)]. So the result follows from Spielberg's argument combined with Proposition 4.2. □

Proof of Theorem 4.1. Let $H := c^{-1}(0)$. Averaging over the gauge-action $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda))$ determines a faithful conditional expectation $\Phi^\gamma : C^*(\mathcal{G}_\Lambda) \to C^*(H)$. Averaging over the action $\beta : \mathbb{T}^k \to \mathrm{Aut}(B)$ determines a conditional expectation $\Phi^\beta : B \to \pi(C^*(H))$ such that $\pi \circ \Phi^\gamma = \Phi^\beta \circ \pi$, so by a standard argument it suffices to show that $\pi|_{C^*(H)}$ is injective.

By hypothesis π is injective on $r^*(C_c(\Lambda^0))$, and it follows from Proposition 3.1 that π is injective on $C_0(\mathcal{G}_\Lambda^{(0)}) = C_0(H^{(0)})$. Since Proposition 4.2 implies that H is both principal and amenable, it follows from [18, II, Proposition 4.6] that $\pi|_{C^*(H)}$ is injective. □

5. The Cuntz-Krieger uniqueness theorem and simplicity

In this section we use groupoid machinery to recover Yamashita's version of the Cuntz-Krieger uniqueness theorem [24]. (Yamashita's proof uses the technology of product systems and Cuntz-Pimsner algebras.) We also use the results of [3] to characterise simplicity of $C^*(\Lambda)$ in terms of the structure of Λ, and to establish a condition under which $C^*(\Lambda)$ is purely infinite.

Recall from [27] that given a topological higher-rank graph Λ, a boundary path $x \in \partial\Lambda$ is said to be *aperiodic* if $\sigma^m(x) \neq \sigma^n(x)$ for all distinct $m, n \in \mathbb{N}^k$ with $m, n \leq d(x)$.

Theorem 5.1 (The Cuntz-Krieger uniqueness theorem)**.** *Let Λ be a compactly aligned topological k-graph. Let $r^* : C_0(\Lambda^0) \to C_0(\partial\Lambda)$ be the homomorphism $f \mapsto f \circ r$. The following are equivalent.*

(1) *For every open set $V \subseteq \Lambda^0$ there exists an aperiodic element $x \in Z(V)$.*
(2) *Every homomorphism $\pi : C^*(\Lambda) \to B$ such that $\pi \circ r^*$ is injective on $C_c(\Lambda^0)$ is an isomorphism.*

Remark 5.2. Condition (1) in Theorem 5.1 is precisely Yeend's aperiodicity condition (A) (see [27, Theorem 5.2]). Wright shows in Theorem 3.1 of [23] that Λ satisfies condition (A) if and only if

(5.1) for every pair U, V of open subsets of Λ such that $s(U) = s(V)$ and $s|_U, s|_V$ are homeomorphisms, there exists $\tau \in s(U)\Lambda$ such that $\mathrm{MCE}(U\tau, V\tau) = \emptyset$.

Since it does not involve elements of $\partial\Lambda$, which are hard to identify in practise, this condition is easier to check in practice than condition (1) of Theorem 5.1 (see [23, Section 4]). We give an independent, although somewhat round-about, proof of Wright's result in Lemma 5.6 below.

The relationship between Yeend's aperiodicity condition and Yamashita's Condition (B) [24, Definition 4.9] is not transparent. However, since [24, Theorem 4.14] says that Condition (B) implies (2) of Theorem 5.1, we deduce that Condition (1) is at least formally weaker than Condition (B).

As in [3], we say that a topological groupoid $\mathcal{G}$ is *topologically principal* if the set $\{u \in \mathcal{G}^{(0)} : \mathcal{G}_u^u = \{u\}\}$ of units with trivial isotropy is dense in $\mathcal{G}^{(0)}$, and we say that $\mathcal{G}$ is *minimal* if the only nonempty open invariant subset of $\mathcal{G}^{(0)}$ is $\mathcal{G}^{(0)}$ itself.

Proof of Theorem 5.1. Lemma 2.1 says that $\mathcal{G}_\Lambda$ is second-countable, locally compact, Hausdorff and étale. Corollary 4.3 implies that it is amenable.

Theorem 5.2 of [27] says that Λ satisfies (1) if and only if $\mathcal{G}_\Lambda$ is topologically principal. Combined with the preceding paragraph, [3, Proposition 5.5] implies that $\mathcal{G}_\Lambda$ is topologically principal if and only if every nontrivial ideal of $C^*(\mathcal{G}_\Lambda)$ has nontrivial intersection with $C_0(\mathcal{G}_\Lambda^{(0)})$. Finally, Proposition 3.1 implies that every nontrivial ideal of $C^*(\mathcal{G}_\Lambda)$ has nontrivial intersection with $C_0(\mathcal{G}_\Lambda^{(0)})$ if and only if $C^*(\Lambda)$ satisfies (2). □

We now employ the full strength of the characterisation [3, Theorem 5.1] of simplicity for C^*-algebras of second-countable locally-compact Hausdorff étale amenable groupoids to characterise simplicity of topological higher-rank graph C^*-algebras.

Theorem 5.3. *Let Λ be a compactly aligned topological k-graph. Then $C^*(\Lambda)$ is simple if and only if both of the following conditions are satisfied:*

(1) Λ *satisfies condition (1) of Theorem 5.1; and*
(2) *For every* $x \in \partial\Lambda$ *and open* $U \subseteq \Lambda^0$ *there exists* $n \in \mathbb{N}^k$ *such that* $n \leq d(x)$ *and* $U\Lambda x(n) \neq \emptyset$.

Lemma 5.4. *Let* Λ *be a compactly aligned topological* k*-graph. Suppose that* $V \subseteq \Lambda^m$ *is open,* $F \subseteq \overline{V}\Lambda$ *is compact and* $Z(V) \cap Z(F)^c \neq \emptyset$*. Then there exists* $p \geq m$ *and a nonempty open subset* W *of* Λ^p *such that* $Z(W) \subseteq Z(V) \cap Z(F)^c$*. In particular, if* U *is an open subset of* $\partial\Lambda$*, then there exist* $n \in \mathbb{N}^k$ *and an open subset* W *of* Λ^n *such that* $Z(W) \subseteq U$*.*

Proof. We follow the argument of Theorem 5.2 of [27]. Since $s|_{\Lambda^m}$ is a local homeomorphism, we may assume that it restricts to a homeomorphism on U. Let $E := \{\lambda(m, d(\lambda)) : \lambda \in F\}$. As in [27, Definition 3.10], the map $\lambda \mapsto \lambda(m, d(\lambda))$ is continuous on each $F \cap \Lambda^p$. Since F is compact and $d : \Lambda \to \mathbb{N}^k$ is continuous, $d(F)$ is finite, and it follows that E is compact. Fix $x \in Z(V) \cap Z(F)^c$m and let $\lambda := x(0, m)$. Since $x \notin Z(F)$, we have $\sigma^m(x) \notin Z(E)$. Since $\sigma^m(x) \in \partial\Lambda$ it follows that either $r(E)$ is not a neighbourhood of $x(m)$, or E is not exhaustive for $r(E)$. Suppose first that $r(E)$ is not a neighbourhood of $x(m)$. Since E is closed, it follows that there is an open neighbourhood S of $x(m)$ which does not intersect $r(E)$; and then $n := m$ and $W := US$ does the job. Now suppose that $r(E)$ is not exhaustive for $r(E)$. Then there exists $\lambda \in r(E)\Lambda$ such that $\lambda\Lambda \cap E\Lambda = \emptyset$. Since $E\Lambda$ closed there is then a neighbourhood S of λ in $\Lambda^{d(\lambda)}$ such that $S\Lambda \cap E\Lambda = \emptyset$. Now $n := m + d(\lambda)$ and $W := US$ does the job.

The final statement follows because the $Z(V) \cap Z(F)^c$ are a base for the topology on $\partial\Lambda$. □

Lemma 5.5. *Let* Λ *be a compactly aligned topological* k*-graph. The following are equivalent:*

(1) Λ *satisfies condition (2) of Theorem 5.3.*
(2) $\mathcal{G}_\Lambda^{(0)}$ *contains no nontrivial open invariant subsets.*

Proof. First suppose that Λ satisfies condition (2) of Theorem 5.3. Fix $x \in \partial\Lambda$. It suffices to show that $\overline{[x]} = \partial\lambda$. To see this, fix $y \in \mathcal{G}_\Lambda^{(0)} = \partial\Lambda$. Each neighbourhood of y contains a basic open neighbourhood $Z(U) \cap Z(F)^c$ of y where $U \subseteq \Lambda^m$ is relatively compact and $F \subseteq \overline{U}\Lambda$ is compact. Lemma 5.4 yields $p \in \mathbb{N}^k$ with $p \geq m$ and an open subset W of Λ^p such that $Z(W) \subseteq Z(U) \cap Z(F)^c$. Proposition 4.3 of [27] implies that each $v\partial\Lambda \neq \emptyset$, and so $W\Lambda \cap F\Lambda = \emptyset$. Since $s(W)$ is open, condition (2) of Theorem 5.3 gives us $n \leq d(x)$ such that $W\Lambda x(n) \neq \emptyset$, say $\alpha \in W\Lambda x(n)$. Then $\alpha\sigma^n(x) \in\in [x] \cap Z(W) \subseteq [x] \cap Z(U) \cap Z(F)^c$. Hence $y \in \overline{[x]}$.

Now suppose that $\mathcal{G}_\Lambda^{(0)}$ has no nontrivial open invariant subsets. Fix an open $U \subseteq \Lambda^0$ and an element $x \in \partial\Lambda$. Since $\overline{[x]}$ is a nonempty closed invariant set, it is all of $\partial\Lambda$. Since $Z(U)$ is open it follows that $[x] \cap Z(U)$ is nonempty. By definition of $\mathcal{G}$, we have $[x] = \{\lambda\sigma^n(x) : n \in \mathbb{N}^k, \lambda \in \Lambda x(n)\}$, so Λ satisfies condition (2) of Theorem 5.3. □

Proof of Theorem 5.3. As in the proof of Theorem 5.1, $\mathcal{G}_\Lambda$ is second-countable, locally compact, Hausdorff, étale and amenable. Hence Theorem 5.1 of [3] implies that $C^*(\Lambda)$ is simple if and only if $\mathcal{G}_\Lambda$ is topologically principal and minimal. Theorem 5.1 of [27] implies that $\mathcal{G}_\Lambda$ is topologically principal if and only if Λ satisfies condition (1) of Theorem 5.1. So the result follows from Lemma 5.5. □

Lemma 3.3 of [3] implies that a second-countable locally compact Hausdorff groupoid $\mathcal{G}$ is topologically principal if and only if it satisfies the apparently weaker condition (genuinely weaker in the absence of the assumption that $\mathcal{G}$ is second countable) that the interior of the isotropy subgroupoid $\bigcup_{u\in\mathcal{G}^{(0)}} \mathcal{G}_u^u$ of $\mathcal{G}$ is precisely $\mathcal{G}^{(0)}$. Since this condition should be easier to check, we describe what it says for a topological k-graph: it is a topological analogue of the condition called "no local periodicity" in [20]. The third condition below is Wright's finite-paths aperiodicity condition [23, Theorem 3.1(C)]; as mentioned above, our argument below recovers the equivalence (1) $\iff$ (3) of [23, Theorem 3.1] via results of [3] and [27].

Throughout the rest of the section we make frequent use of the notational convenience of Notation 2.2; that is, we write $Z(U *_s V) \cap Z(F)^c$ in place of $Z(U *_s V, p-q) \cap Z(F, p-q)^c$ when the former is unambiguous.

Lemma 5.6. *Let Λ be a compactly aligned topological k-graph. The following are equivalent:*

(1) *Λ satisfies condition (1) of Theorem 5.1.*
(2) *For every open set $V \subseteq \Lambda^0$ and every pair m, n of distinct elements of $\mathbb{N}^k$ there exists $x \in Z(V)$ such that either $d(x) \not\geq m \vee n$ or $\sigma^m(x) \neq \sigma^n(x)$.*
(3) *Λ satisfies* (5.1).

Proof. We first prove (1) $\iff$ (2). We have seen that Λ satisfies condition (1) of Theorem 5.1 if and only if $\mathcal{G}_\Lambda$ is topologically principal. Lemmas 3.1 and 3.3 of [3] show that $\mathcal{G}_\Lambda$ is topologically principal if and only if

(5.2) every open subset of $\mathcal{G}_\Lambda \setminus \mathcal{G}_\Lambda^{(0)}$ contains an element (x, m, y) such that $x \neq y$.

So it suffices to show that (5.2) is equivalent to (2).

First suppose that Λ satisfies (2). Fix an open set $O \subseteq \mathcal{G}_\Lambda \setminus \mathcal{G}_\Lambda^{(0)}$.

By definition of the topology on $\mathcal{G}_\Lambda$, the set O contains a nonempty subset of the form $Z(U *_s V) \cap Z(F)^c$ where $U \subseteq \Lambda^p$ and $V \subseteq \Lambda^q$ are relatively compact with $s(U) = s(V)$, $s|_U$ and $s|_V$ are homeomorphisms, and F is a compact subset of $\bigcup_{\alpha\in\Lambda} \overline{U}\alpha \times \overline{V}\alpha$. Since the map $(\mu\alpha, \nu\alpha) \mapsto \mu\alpha$ and the map $\mu\alpha \mapsto (\mu\alpha)(p, p + d(\alpha))$ are continuous, it follows that $F = \{(\mu\alpha, \nu\alpha) : (\mu, \nu) \in U *_s V, \alpha \in K\}$ for some compact $K \subseteq \overline{s(U)}\Lambda$. It suffices to find $(x, p - q, y) \in Z(U *_s V) \cap Z(F)^c$ with $x \neq y$.

If $p = q$, then since $O \cap \mathcal{G}_\Lambda^{(0)} = \emptyset$ and since the groupoid H of Proposition 4.2 is principal, any $(x, 0, y) \in Z(U *_s V) \cap Z(F)^c$ does the job. So we may suppose that $p \neq q$. By Lemma 5.4, there exists $m \in \mathbb{N}^k$ with $m \geq p$ and an open $W_0 \subseteq \Lambda^m$ such that $Z(W_0) \subseteq Z(U) \cap Z(\overline{U}K)^c$. Let $n := m - p$ and let $W := \{\lambda(p, m) : \lambda \in W_0\} \subseteq \Lambda^n$. Then $r(W) \subseteq s(U) = s(V)$ and $Z(UW *_s VW) \subseteq Z(U *_s V) \cap Z(F)^c \subseteq O$. Let $p' := p - (p \wedge q)$ and $q' := q - (p \wedge q)$. Then $p \neq q$ forces $p' \neq q'$. Since the source map in Λ is open, $s(W) = s(W_0)$ is open, so condition (2) implies that there exists $x \in Z(s(W))$ such that either $d(x) \not\geq p' \vee q'$ or $\sigma^{p'}(x) \neq \sigma^{q'}(x)$. Let $\mu \in UW$ and $\nu \in VW$ be the unique elements such that $s(\mu) = s(\nu) = r(x)$. Then $(\mu x, p - q, \nu x) \in O$.

We will show that $\mu x \neq \nu x$; equation (5.2) then follows. We consider two cases. First suppose that $d(x) \not\geq p' \vee q'$. Since $p' \wedge q' = 0$ it follows that there exists $i \leq k$ such that $d(x)_i < \infty$ and $p'_i \neq q'_i$. Thus $p_i \neq q_i$, and since $d(\mu) = p + n$ and $d(\nu) = q + n$, it follows that $d(\mu)_i - d(\nu)_i \neq 0$. Since $d(x)_i < \infty$, we have $d(\mu x)_i - d(\nu x)_i = d(\mu)_i - d(\nu)_i \neq 0$. In particular $d(\mu x) \neq d(\nu x)$, forcing $\mu x \neq \nu x$ as required. Now suppose that $d(x) \geq p' \vee q'$.

Then (2) says that $\sigma^{p'}(x) \neq \sigma^{q'}(x)$. Since $\mu \in UW \subseteq \Lambda^{p+n}$ and $\nu \in VW \subseteq \Lambda^{q+n}$, we have

$$\sigma^{p+n+q'}(\mu x) = \sigma^{q'}(x) \neq \sigma^{p'}(x) = \sigma^{q+n+p'}(\nu x).$$

Since $p + n + q' = p + q - (p \wedge q) + n = q + n + p'$, we deduce that $\mu x \neq \nu x$ as required.

Now suppose that Λ does not satisfy (2). Fix an open set $V \subseteq \Lambda^0$ and distinct $m, n \in \mathbb{N}^k$ such that $d(x) \geq m \vee n$ and $\sigma^m(x) = \sigma^n(x)$ for all $x \in V\partial\Lambda$. Then $Z(V\Lambda^m *_s V\Lambda^n)$ is a nonempty open subset of $\mathcal{G}_\Lambda$ which does not intersect $\mathcal{G}_\Lambda^{(0)}$ whose every element is an isotropy element, and so (5.2) does not hold. This completes the proof of (1) $\iff$ (2)

We now establish (1) $\iff$ (3). As above, it suffices to show that (5.2) is equivalent to (3).

First suppose that Λ satisfies (5.1). Fix an open subset B of $\mathcal{G}_\Lambda \setminus \mathcal{G}_\Lambda^{(0)}$. As above there exist $m, n \in \mathbb{N}^k$ and open sets $U \subseteq \Lambda^m$ and $V \subseteq \Lambda^n$ such that $s(U) = s(V)$, $s|_U$ and $s|_V$ are homeomorphisms and $Z(U *_s V) \subseteq B$. By (5.1), there exists $\tau \in s(U)\Lambda$ such that $\operatorname{MCE}(U\tau, V\tau) = \emptyset$. Let $\alpha \in U$ and $\beta \in V$ be the unique elements such that $s(\alpha) = s(\beta) = r(\tau)$ and fix $x \in s(\tau)\partial\Lambda$. Then $g := (\alpha\tau x, m - n, \beta\tau x) \in Z(U *_s V) \subseteq B$, and since $\operatorname{MCE}(\alpha\tau, \beta\tau) = \emptyset$, we have $\alpha\tau x \neq \beta\tau x$.

Now suppose that Λ does not satisfy (5.1). So there exist $m, n \in \mathbb{N}^k$ and open $U \subseteq \Lambda^m$ and $V \subseteq \Lambda^n$ such that: (1) $s(U) = s(V) = W$, say; (2) the source map restricts to homeomorphisms of U and V onto W; and (3) $\operatorname{MCE}(U\tau, V\tau) \neq \emptyset$ for all $\tau \in W\Lambda$. By passing to subneighbourhoods, we may assume that $\overline{U}$ and $\overline{V}$ are compact and contained in sets on which s is a homeomorphism, and that $\operatorname{MCE}(\overline{U}\tau, \overline{V}\tau) \neq \emptyset$ for all $\tau \in s(\overline{U})$. Fix $\mu \in U$ and $\nu \in V$ with $s(\mu) = s(\nu)$. Then $\operatorname{MCE}(\mu, \nu) \neq \emptyset$ (consider $\tau = s(\mu)$), so $\mu(0, m \wedge n) = \nu(0, m \wedge n)$ and for each $\tau \in s(\mu)\Lambda$, we have

$$\mu(0, m \wedge n) \operatorname{MCE}(\mu(m \wedge n, m)\tau, \nu(m \wedge n, n)\tau) = \operatorname{MCE}(\mu\tau, \nu\tau) \neq \emptyset.$$

So by replacing U with $\{\mu(m \wedge n, m) : \mu \in U\}$ and V with $\{\nu(m \wedge n, n) : \nu \in V\}$, we may assume that $m \wedge n = 0$. We will show that $Z(U *_s V)$ consists entirely of isotropy. We first establish the following claim.

Claim. *For each $p \in \mathbb{N}$, the set $\overline{W\Lambda^{pm}}$ is compact exhaustive for each $v \in s(U)$.* The claim is trivial for $p = 0$, so suppose as an inductive hypothesis that $\overline{W}\Lambda^{pm}$ is compact exhaustive for each $v \in s(U) = W$. Since $m \wedge n = 0$ and hence $(p+1)m \wedge n = 0$, we have $\operatorname{MCE}(\overline{U\Lambda^{pm}}, \overline{V}) \subseteq \overline{V}\Lambda^{(p+1)m}$. Furthermore, for $\nu \in \overline{V}$ and $\tau \in s(\mu)\Lambda^{(p+1)m}$, the element $\mu \in \overline{U}$ with $s(\mu) = r(\tau)$ satisfies $\operatorname{MCE}(\mu, \nu\tau) \neq \emptyset$, so $\nu\tau \in \operatorname{MCE}(\overline{U\Lambda^{pm}}, \overline{V})$. Hence

$$\operatorname{MCE}(\overline{U\Lambda^{pm}}, \overline{V}) = \overline{V}\Lambda^{(p+1)m} \quad \text{for all } p \in \mathbb{N}.$$

Since each of $\overline{U}$ and $\overline{W}\Lambda^{pm}$ is compact, continuity of composition implies that $\overline{U\Lambda^{pm}}$ is compact. Since $\overline{V}$ is compact also, and Λ is compactly aligned, it follows that $\overline{V}\Lambda^{(p+1)m}$ is compact. Since $\lambda \mapsto \lambda(m, d(\lambda))$ is continuous on $\Lambda^{n+(p+1)m}$, we deduce that $\overline{W}\Lambda^{(p+1)m}$ is compact. It remains to show that it is exhaustive for each $v \in W$. For this fix $\tau \in W\Lambda$. The inductive hypothesis supplies an element η of $\operatorname{MCE}(\overline{W}\Lambda^{pm}, \tau)$. By choice of U and V, we have $\operatorname{MCE}(U\eta, V\eta) \neq \emptyset$, say $\mu\eta\xi = \nu\eta\zeta \in \operatorname{MCE}(U\eta, V\eta)$ with $\mu \in U$ and $\nu \in V$. By definition of η, we have $d(\eta) = (pm) \vee d(\tau) \geq pm$, so $d(\nu\eta\zeta) = d(\mu\eta\xi) \geq d(\mu) + d(\eta) \geq (p+1)m$. Since $d(\nu) \wedge m = 0$, it follows that $d(\eta\zeta) \geq (p+1)m$. Since $\eta(0, d(\tau)) = \tau$, we have $(\eta\zeta)(0, d(\tau) \vee (p+1)m) \in \operatorname{MCE}(\tau, \overline{W}\Lambda^{(p+1)m})$. So $\overline{W}\Lambda^{(p+1)m}$ is exhaustive for v. This proves the claim.

Now fix $(\mu, \nu) \in U *_s V$ and $x \in s(\mu)\partial\Lambda$ so that $(\mu x, m - n, \nu x)$ is a typical element of $Z(U *_s V)$. We must show that $\mu x = \nu x$. The claim and the definition of $\partial\Lambda$ imply that for each $p \in \mathbb{N}$ there exists $\mu \in \overline{W}\Lambda^{pm}$ such that $d(x) \geq d(\mu)$ and $x(0, pm) = \mu$. In particular, $d(x)_i = \infty$ whenever $m_i > 0$, and similarly $d(x)_i = \infty$ whenever $n_i > 0$. So $d(\mu x) = d(\nu x) = d(x)$, and $p \leq d(x)$ if and only if $p \leq d(\mu x)$. By choice of U and V, we have $\mathrm{MCE}(\mu x(0,p), \nu x(0,p)) \neq \emptyset$ for all $p \leq d(x)$. Hence $(\mu x)(0,p) = (\nu x)(0,p)$ for all $p \leq d(\mu x) = d(\nu x)$. That is, $\mu x = \nu x$ as required. □

We use Anantharaman-Delaroche's criterion for pure infiniteness of a groupoid C^*-algebra [1, Proposition 2.4] to provide a criterion under which $C^*(\Lambda)$ is simple and purely infinite. Recall from [1, Definition 2.1] that a groupoid $\mathcal{G}$ is *locally contracting* if, for every open $U \subseteq \mathcal{G}^{(0)}$ there exist an open subset V of U and an open bisection B such that $\overline{V} \subseteq s(B)$ and $r(B\overline{V}) \subsetneq V$.

Definition 5.7. Given a compactly aligned topological k-graph Λ, we say that a precompact open subset U of Λ^0 is *contracting* if there exist $m \neq n \in \mathbb{N}^k$ and nonempty precompact open sets $Y_m \subseteq \Lambda^m$ and $Y_n \subseteq \Lambda^n$ such that all of the following hold: $s(Y_m) = s(Y_n)$; $\overline{r(Y_m)} \subseteq r(Y_n) = U$; the source map restricts to a homeomorphism on each of Y_m and Y_n; for every $\mu \in Y_m$ and $\nu \in Y_n$ such that $r(\mu) = r(\nu)$, we have $\mathrm{MCE}(\mu\tau, \nu) \neq \emptyset$ for all $\tau \in s(\mu)\Lambda$; and there exists an open subset W of $Y_n\Lambda$ such that $\{\zeta(0,n) : \zeta \in W\} = Y_n$ and $\mathrm{MCE}(\mu, \zeta) = \emptyset$ for all $\mu \in Y_m$ and $\zeta \in W$.

Proposition 5.8. *Let Λ be a compactly aligned topological k-graph. Suppose that for every $v \in \Lambda^0$ there exist $p \in \mathbb{N}^k$ and an open set $V \subseteq \Lambda^p$ such that $v \in r(V)$ and $s(V)$ is contracting. Then $\mathcal{G}_\Lambda$ is locally contracting. If Λ also satisfies the hypotheses of Theorem 5.3, then $C^*(\Lambda)$ is simple and purely infinite.*

To prove the proposition, we first prove that contracting neighbourhoods in Λ^0 give rise to contracting bisections in $\mathcal{G}_\Lambda$.

Lemma 5.9. *Let Λ be a compactly aligned topological k-graph. Suppose that $U \subseteq \Lambda^0$ is contracting, and let m, n, $Y_m \subseteq \Lambda^m$, $Y_n \subseteq \Lambda^n$ and $W \subseteq Y_n\Lambda$ be as in Definition 5.7. Let Y_n' be a nonempty open set with $\overline{Y_n'} \subseteq Y_n$, and let $Y_m' := s^{-1}(Y_n') \cap Y_m$. Then $\overline{r(Z(Y_m' *_s Y_n'))} \subsetneq s(Z(Y_m' *_s Y_n'))$.*

Proof. We first claim that $\mathrm{MCE}(Y_m, Y_n) = Y_m\Lambda^{(m\vee n)-m}$, and $\overline{s(Y_m)}\Lambda^{(m\vee n)-m}$ is compact exhaustive for each $v \in s(Y_m)$.

The containment $\mathrm{MCE}(Y_m, Y_n) \subseteq Y_m\Lambda^{(m\vee n)-m}$ is clear. For the reverse containment, fix $\tau \in s(Y_m)\Lambda^{(m\vee n)-m}$, let μ be the unique element of $Y_m r(\tau)$, and fix $\nu \in Y_n$ such that $r(\nu) = r(\mu)$. By hypothesis, $\mathrm{MCE}(\mu\tau, \nu) \neq \emptyset$, and since $d(\mu\tau) = m \vee n$, it follows that $\mu\tau \in \mathrm{MCE}(Y_m, Y_n)$.

To prove the claim, it remains to show that $\overline{s(Y_m)}\Lambda^{(m\vee n)-m}$ is compact exhaustive for each $v \in s(Y_m)$. First observe that $\mathrm{MCE}(\overline{Y_m}, \overline{Y_n})$ is compact because Λ is compactly aligned. Since $\lambda \mapsto \lambda(m, m \vee n)$ is continuous, it follows that $\{\lambda(m, m \vee n) : \lambda \in \mathrm{MCE}(\overline{Y_m}, \overline{Y_n})\}$ is compact, so the first statement of the lemma shows that $\overline{s(Y_m)}\Lambda^{(m\vee n)-m}$ is compact. To see that it is exhaustive for each $v \in s(Y_m)$, fix $\tau \in s(Y_m)\Lambda$. Let $\mu \in Y_m$ and $\nu \in Y_n$ be the unique elements whose sources are equal to $r(\tau)$. By hypothesis, we have $\mathrm{MCE}(\mu\tau, \nu) \neq \emptyset$, say $\mu\tau\alpha \in \mathrm{MCE}(\mu\tau, \nu)$. Then $d(\mu\tau\alpha) \geq m \vee n$, and so $\eta := (\tau\alpha)(0, (m \vee n) - m)$ belongs to $\overline{s(Y_m)}\Lambda^{(m\vee n)-m}$. In particular $\tau\alpha \in \mathrm{MCE}(\tau, \eta) \subseteq \mathrm{MCE}(\tau, \overline{s(Y_m)}\Lambda^{(m\vee n)-m})$. This proves the claim.

It follows from the claim and the definition of $\partial\Lambda$ that

$$\overline{r(Z(Y'_m *_s Y'_n))} = \overline{Z(Y'_m)} \subseteq Z(Y'_n) = s(Z(Y'_m *_s Y'_n)).$$

To see that the containment is strict, observe that $Z(W) \cap Z(Y'_n)$ is a nonempty open subset of $Z(Y'_n) \setminus Z(Y'_m)$. □

Proof of Proposition 5.8. To see that $\mathcal{G}_\Lambda$ is locally contracting, fix a nonempty open subset U of $\mathcal{G}_\Lambda^{(0)} = \partial\Lambda$. By Lemma 5.4 there exist $q \in \mathbb{N}^k$ and a nonempty open set $X \subseteq \Lambda^q$ such that $Z(\overline{X}) \subseteq U$. Since the source map is open, $s(X)$ is nonempty and open. Fix $v \in s(X)$. By hypothesis, there exist $p \in \mathbb{N}^k$ and an open set $V \subseteq \Lambda^p$ such that $v \in r(V)$ and $s(V)$ is contracting. Fix m, n, Y_m, Y_n and W as in Definition 5.7. Since each of Y_m, Y_n, X and V is open and since composition is an open map, each of XVY_m and XVY_n is open. Hence $Z(XVY_m)$ and $Z(XVY_n)$ are open. Let $B := Z(XVY_m *_s XVY_n)$. Then B is a precompact open bisection. Fix $\lambda \in XVY_n$ and an open neighbourhood Y'_n of $\lambda(p+q, p+q+n)$ such that $\overline{Y'_n} \subseteq Y_n$. Let $K := Z(XVY'_n)$. Then $\overline{K} \subseteq s(B)$, and Lemma 5.9 implies that

$$\begin{aligned} r(B\overline{K}) &= \overline{(\sigma^{p+q})^{-1}(r((Z(Y'_m *_s Y'_n)))) \cap Z(XV)} \\ &= (\sigma^{p+q})^{-1}\big(\overline{r((Z(Y'_m *_s Y'_n)))}\big) \cap \overline{Z(XV)} \\ &\subsetneq (\sigma^{p+q})^{-1}(s(Z(Y'_m *_s Y'_n))) \cap Z(XV) = K, \end{aligned}$$

so $\mathcal{G}_\Lambda$ is locally contracting as required.

For the final statement, observe that Theorem 5.1 of [27] implies that $\mathcal{G}_\Lambda$ is topologically principal and Lemma 5.5 implies that $\mathcal{G}_\Lambda$ is minimal. Theorem 4.3 implies that $\mathcal{G}_\Lambda$ is amenable. Hence [1, Proposition 2.4] implies that $C^*(\Lambda) = C^*(\mathcal{G}_\Lambda)$ is simple and purely infinite. □

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Département de Mathématiques, Université d'Orléans, BP 6759, 45067 Orléans Cedex 2, FRANCE
E-mail address: Jean.Renault@univ-orleans.fr

School of Mathematics and Applied Statistics, Austin Keane Building (15), University of Wollongong, NSW 2522, AUSTRALIA
E-mail address: asims@uow.edu.au

Department of Mathematics, Dartmouth College, Hanover, NH 03755-3551, USA
E-mail address: dana.williams@Dartmouth.edu

School of Mathematical and Physical Sciences, Building V, University of Newcastle, Callaghan NSW 2308, AUSTRALIA
E-mail address: Trent.Yeend@ihpa.gov.au