

# GROUPOID ALGEBRAS AS CUNTZ-PIMSNER ALGEBRAS

ADAM RENNIE, DAVID ROBERTSON, AND AIDAN SIMS

ABSTRACT. We show that if  $G$  is a second countable locally compact Hausdorff étale groupoid carrying a suitable cocycle  $c : G \rightarrow \mathbb{Z}$ , then the reduced  $C^*$ -algebra of  $G$  can be realised naturally as the Cuntz-Pimsner algebra of a correspondence over the reduced  $C^*$ -algebra of the kernel  $G_0$  of  $c$ . If the full and reduced  $C^*$ -algebras of  $G_0$  coincide, we deduce that the full and reduced  $C^*$ -algebras of  $G$  coincide. We obtain a six-term exact sequence describing the  $K$ -theory of  $C_r^*(G)$  in terms of that of  $C_r^*(G_0)$ .

In this short note we provide a sufficient condition for a groupoid  $C^*$ -algebra to have a natural realisation as a Cuntz-Pimsner algebra. The advantage to knowing this is the complementary knowledge one obtains from the two descriptions.

Our main result starts with a second-countable locally compact Hausdorff étale groupoid  $G$  with a continuous cocycle into the integers which is unperforated in the sense that  $G$  is generated by  $c^{-1}(1)$ . We then show that the reduced groupoid  $C^*$ -algebra  $C_r^*(G)$  is the Cuntz-Pimsner algebra of a natural  $C^*$ -correspondence over the reduced  $C^*$ -algebra  $C_r^*(c^{-1}(0))$  of the kernel of  $c$ .

We also show that if  $C^*(c^{-1}(0))$  and  $C_r^*(c^{-1}(0))$  coincide, then  $C^*(G)$  and  $C_r^*(G)$  coincide as well. We finish by applying results of Katsura to present a six-term exact sequence relating the  $K$ -theory of  $C_r^*(G)$  to that of  $C_r^*(c^{-1}(0))$ .

**Notation 1.** For the duration of the paper, we fix a second-countable locally compact Hausdorff étale groupoid  $G$  with unit space  $G^{(0)}$  and a continuous cocycle  $c : G \rightarrow \mathbb{Z}$ . We suppose that  $c$  is *unperforated* in the sense that if  $c(\gamma) = n > 0$ , then there exist composable  $\gamma_1, \dots, \gamma_n$  such that each  $c(\gamma_i) = 1$  and  $\gamma = \gamma_1 \dots \gamma_n$ . For  $n \in \mathbb{Z}$  we write  $G_n := c^{-1}(n)$ . Recall from [6] that for  $u \in G^0$ , we write  $G_u = s^{-1}(u)$  and  $G^u = r^{-1}(u)$ . The convolution multiplication of functions is denoted by juxtaposition, as is multiplication in the  $C^*$ -completions.

## Remarks.

- (i) Observe that  $G^{(0)} \subseteq G_0$ , and since  $c$  is continuous,  $G_0$  is a clopen subgroupoid of  $G$ .
- (ii) If  $c$  is strongly surjective in the sense that  $c(r^{-1}(u)) = \mathbb{Z}$  for all  $u \in G^{(0)}$  [1, Definition 5.3.7], and  $\gamma \in G_n$ , then there exists  $\alpha \in r^{-1}(r(\gamma)) \cap G_1$ , and then  $\gamma = \alpha(\alpha^{-1}\gamma) \in G_1 G_{n-1}$ . So an induction shows that if  $c$  is strongly surjective then it is unperforated.

**Lemma 2.** *Let  $A_0 \subseteq C_r^*(G)$  be the completion of  $\{f \in C_c(G) : \text{supp}(f) \subseteq G_0\}$ . Then there is an isomorphism  $I_0 : C_r^*(G_0) \rightarrow A_0$  extending the identity map on  $C_c(G_0)$ .*

*Proof.* Fix  $u \in G^{(0)}$ , let  $\pi_u^0$  be the regular representation of  $C^*(G_0)$  on  $\mathcal{H}_u^0 := \ell^2((G_0)_u)$  and let  $\pi_u$  be the regular representation of  $C^*(G)$  on  $\mathcal{H}_u := \ell^2(G_u)$ . For  $a \in C_c(G_0)$ , the subspace  $\mathcal{H}_u^0 \subseteq \mathcal{H}_u$  is reducing for the operator  $\pi_u(a)$ . Let  $P_0 : \mathcal{H}_u \rightarrow \mathcal{H}_u^0$  be the orthonormal projection. We have

$$\|\pi_u^0(a)\| = \|P_0 \pi_u(a) P_0\| \leq \|\pi_u(a)\|.$$

Taking the supremum over all  $u$ , we deduce that  $\|a\|_{C_r^*(G_0)} \leq \|a\|_{C_r^*(G)}$ . So there is a  $C^*$ -homomorphism  $\pi : A_0 \rightarrow C_r^*(G_0)$  extending the identity map on  $C_c(G_0)$ .

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2010 *Mathematics Subject Classification.* 46L05.

*Key words and phrases.* Groupoid; Cuntz-Pimsner;  $C^*$ -correspondence; Hilbert module.

Restriction of functions induces a faithful conditional expectation<sup>†</sup>  $\Psi : C_r^*(G) \rightarrow C_0(G^{(0)})$  and a faithful conditional expectation  $\Psi_0 : C_r^*(G_0) \rightarrow C_0(G_0^{(0)}) = C_0(G^{(0)})$ . We clearly have  $\pi \circ \Psi = \Psi_0 \circ \pi$ , and so a standard argument (see [7, Lemma 3.13]) shows that  $\pi$  is faithful and hence isometric. It follows immediately that  $I_0 := \pi^{-1} : C_r^*(G_0) \rightarrow A_0$  is an isometric embedding of  $C_r^*(G_0)$  in  $C_r^*(G)$  as claimed.  $\square$

**Lemma 3.** *Let  $X(G)$  be the completion in  $C_r^*(G)$  of the subspace  $\{f \in C_c(G) : \text{supp}(f) \subseteq G_1\}$ . Then  $X(G)$  is a  $C^*$ -correspondence over  $C_r^*(G_0)$  with module actions  $a \cdot \xi = I_0(a)\xi$  and  $\xi \cdot a = \xi I_0(a)$  and inner product  $\langle \cdot, \cdot \rangle_{C_r^*(G_0)} : X(G) \times X(G) \rightarrow C_r^*(G)$  given by  $\langle \xi, \eta \rangle_{C_r^*(G_0)} = I_0^{-1}(\xi^* \eta)$ .*

*Proof.* Since  $c$  is a cocycle, we have  $G_1^{-1} = G_{-1}$ . That  $c$  is a cocycle also implies that if  $f \in C_c(G_m)$  and  $g \in C_c(G_n)$  then  $fg \in C_c(G_{m+n})$  and  $f^* \in C_c(G_{-m})$ , and it follows that  $C_c(G_1)C_c(G_0), C_c(G_0)C_c(G_1) \subseteq C_c(G_1)$  and that  $C_c(G_1)^*C_c(G_1) \subseteq C_c(G_0)$ . The  $C^*$ -identity and Lemma 2 show that  $\langle \cdot, \cdot \rangle_{C_r^*(G)}$  is isometric in the sense that  $\|\langle \xi, \xi \rangle_{C_r^*(G_0)}\| = \|\xi\|_{C_r^*(G)}$ , and in particular is positive definite. Continuity therefore guarantees that the operations described determine a Hilbert-module structure on  $X(G)$ . The left action of  $C_r^*(G_0)$  on  $X(G)$  is adjointable since  $\langle a \cdot \xi, \eta \rangle_{C_r^*(G)} = (a\xi)^* \eta = \xi^* (a^* \eta) = \langle \xi, a^* \cdot \eta \rangle_{C_r^*(G)}$ .  $\square$

**Notation 4.** We write  $I_1$  for the inclusion map  $X(G) \hookrightarrow C_r^*(G)$ .

**Lemma 5.** *The embeddings  $I_1 : X(G) \hookrightarrow C_r^*(G)$  and  $I_0 : C_r^*(G_0) \hookrightarrow C_r^*(G)$  form a Toeplitz representation  $(I_1, I_0)$  of  $X(G)$  in  $C_r^*(G)$ .*

*Proof.* For  $\xi \in X(G)$  and  $a \in C_r^*(G_0)$ , we have  $I_1(\xi)I_0(a) = \xi a = I_1(\xi \cdot a)$  by definition of the actions on  $X(G)$ . For  $\xi, \eta \in C_c(G_1)$ , we have

$$I_0(\langle \xi, \eta \rangle_{A_0}) = I_0(\xi^* \eta) = \xi^* \eta = I_1(\xi)^* I_1(\eta).$$

So  $(I_1, I_0)$  is a Toeplitz representation as claimed.  $\square$

We aim to show that  $(I_1, I_0)$  is in fact a Cuntz-Pimsner covariant representation, but we have a little work to do first.

**Lemma 6.** *For each integer  $n > 0$ , the space*

$$\text{span}\{f_1 f_2 \cdots f_n : f_i \in C_c(G_1) \text{ for all } i\}$$

*is dense in  $C_c(G_n)$ , and we have  $C_c(G_n)^* = C_c(G_{-n})$ .*

*Proof.* Fix  $x \neq y \in G_n$ . Since  $G_n = c^{-1}(n)$  is locally compact, by the Stone-Weierstrass theorem it suffices to construct functions  $f_1, \dots, f_n \in C_c(G_1)$  such that  $(f_1 \cdots f_n)(x) = 1$  and  $(f_1 \cdots f_n)(y) = 0$ .

Since  $c$  is unperforated, there exist  $x_1, \dots, x_n \in G_1$  such that  $x = x_1 \cdots x_n$ . Choose a precompact open bisection  $U_n \subseteq G_1$  such that  $x_n \in U_n$  and such that if  $s(x) \neq s(y)$  then  $s(y) \notin s(U_n)$ ; this is possible because  $s$  is a local homeomorphism. For each  $1 \leq i \leq n-1$  choose a precompact open bisection  $U_i \subseteq G_1$  with  $x_i \in U_i$ . Then  $U_1 \cdots U_n$  is a precompact open bisection containing  $x$  because multiplication in  $G$  is continuous and open. If  $s(x) = s(y)$ , then  $y \notin U_1 \cdots U_n$  because the latter is a bisection; and if  $s(x) \neq s(y)$  then  $y \notin U_1 \cdots U_n$  by choice of  $U_n$ .

By Urysohn's Lemma, for each  $i$  there exists  $f_i \in C_c(U_i)$  with  $f_i(x_i) = 1$ . Then  $(f_1 \cdots f_n)(x) = 1$  by construction, and the convolution formula says that

$$\text{supp}(f_1 \cdots f_n) = \text{supp}(f_1) \text{supp}(f_2) \cdots \text{supp}(f_n) \subseteq U_1 \cdots U_n,$$

which yields  $(f_1 \cdots f_n)(y) = 0$ .

<sup>†</sup>To see this observe that for  $f \in C_c(G)$ , we have  $f(u) = (\pi_u(f)\delta_u|\delta_u)$ , giving  $\|f|_{G^{(0)}}\| = \sup_u (\pi_u(f)\delta_u|\delta_u) \leq \sup_u \|\pi_u(f)\| = \|f\|_{C_r^*(G)}$ , so restriction induces an idempotent map  $\Psi : C_r^*(G) \rightarrow C_0(G^{(0)})$  of norm one. Hence [2, Theorem II.6.10.2] implies that  $\Psi$  is a conditional expectation. Given  $a \in C_r^*(G) \setminus \{0\}$ , [6, Proposition II.4.2] shows that  $a \in C_0(G)$ , so  $a(\gamma) \neq 0$  for some  $\gamma$ . Then  $\Psi(a^*a)(s(\gamma)) = \sum_{\alpha \in G_{s(\gamma)}} \bar{a}(\alpha^{-1})a(\alpha) \geq |a(\gamma)|^2 > 0$ , so  $\Psi$  is faithful.

The involution formula  $f^*(\gamma) = \overline{f}(\gamma^{-1})$  and the cocycle property  $c(\gamma^{-1}) = -c(\gamma)$  show that  $f \in C_c(G_n)$  if and only if  $f^* \in C_c(G_{-n})$ .  $\square$

**Corollary 7.** *There is an injection  $\psi : \mathcal{K}(X(G)) \rightarrow C_r^*(G_0)$  such that  $\psi(\theta_{\xi,\eta}) = \xi\eta^*$  for all  $\xi, \eta \in C_c(G_1)$ .*

*Proof.* The discussion on page 202 of [5] (see also [3, Section 1]) shows that there is a homomorphism  $I_1^{(1)} : \mathcal{K}(X(G)) \rightarrow C_r^*(G)$  such that  $I_1^{(1)}(\theta_{\xi,\eta}) = I_1(\xi)I_1(\eta)^* = \xi\eta^* = I_0(\xi\eta^*)$ . Since  $I_0$  is isometric, the composition  $\psi := I_0^{-1} \circ I_1^{(1)}$  is a homomorphism satisfying the desired formula, and we need only show that  $\psi$  is injective.

If  $T \in \mathcal{K}(X(G))$  and  $\psi(T) = 0$ , then  $I_1^{(1)}(T) = 0$  as well. For  $\eta, \xi, \zeta \in X(G)$ , we have

$$I_1^{(1)}(\theta_{\eta,\xi})I_1(\zeta) = I_1(\eta)I_1(\xi)^*I_1(\zeta) = I_1(\eta \cdot \langle \xi, \zeta \rangle_{A_0}) = I_1(\theta_{\eta,\xi}(\zeta)),$$

and linearity and continuity show that for all  $S \in \mathcal{K}(X(G))$  and  $\zeta \in X(G)$ , we have the formula  $I_1^{(1)}(S)I_1(\zeta) = I_1(S\zeta)$ . In particular,  $0 = I_1^{(1)}(T)I_1(\zeta) = I_1(T\zeta)$  for all  $\zeta \in X(G)$ . Since  $I_1$  is isometric, we deduce that  $T\zeta = 0$  for all  $\zeta \in C_c(G_1)$ , and so  $T = 0$ . So  $\psi$  is injective.  $\square$

**Lemma 8.** *Let  $\phi = \phi_{X(G)} : C_r^*(G_0) \rightarrow \mathcal{L}(X(G))$  be the homomorphism implementing the left action. Then*

- (1)  $\ker(\phi) = C_r^*(G_0) \cap C_0(\{g \in G_0 : s(g) \notin r(G_1)\})$ ;
- (2)  $\ker(\phi)^\perp = C_r^*(G_0) \cap C_0(\{g \in G_0 : s(g) \in r(G_1)\}) = \overline{\text{span}}\{fg^* : f, g \in C_c(X_1)\}$ ;
- (3)  $\phi_{X(G)}(a) \in \mathcal{K}(X(G))$  for all  $a \in C_r^*(G)$ , and the Katsura ideal  $J_{X(G)}$  is

$$J_{X(G)} = \overline{\text{span}}\{fg^* : f, g \in C_c(G_1)\};$$

- (4) the Toeplitz representation  $(I_1, I_0)$  of Lemma 5 is Cuntz-Pimsner covariant.

*Proof.* (1) Choose  $a \in C_r^*(G_0)$ ; by [6, Proposition II.4.2],  $a \in C_0(G)$ . Suppose that  $a \notin \ker \phi$ . Then there exists  $\xi \in X(G)$  such that  $a\xi = \phi(a)\xi$  is nonzero, so there exist composable  $\gamma, \gamma'$  with  $a(\gamma) \neq 0$  and  $\xi(\gamma') \neq 0$ . We have  $r(\gamma') \in r(G_1)$  by definition of  $X(G)$ , and so  $a \notin C_0(\{g \in G_0 : s(g) \notin r(G_1)\})$ .

On the other hand, suppose that  $a \notin C_0(\{g \in G_0 : s(g) \notin r(G_1)\})$ . Choose  $g \in G_0$  such that  $s(g) \in r(G_1)$  and  $a(g) \neq 0$ . Choose  $\gamma \in G_1$  with  $r(\gamma) = s(g)$ , fix a precompact open bisection  $U \subseteq G_1$  containing  $\gamma$  and use Urysohn's lemma to choose  $\xi \in C_c(U) \subseteq C_c(G_1)$  such that  $\xi(\gamma) = 1$ . Then  $(\phi(a)\xi)(g\gamma) = \sum_{\alpha\beta=g\gamma} a(\alpha)\xi(\beta)$ . Since  $\xi$  is supported on the bisection  $U$ , and since  $\alpha\beta = g\gamma$  implies  $s(\beta) = s(\gamma)$ , if  $\alpha\beta = g\gamma$  and  $a(\alpha)\xi(\beta) \neq 0$ , we have  $\beta = \gamma$  and then  $\alpha = g$  by cancellation. So  $(\phi(a)\xi)(g\gamma) = a(g) \neq 0$  and in particular  $a \notin \ker \phi$ .

(2) Suppose that  $a \in \ker(\phi)^\perp$ . Fix  $g \in G_0$  with  $s(g) \notin r(G_1)$ . Since  $G_1 = c^{-1}(1)$  is clopen and  $r$  is a local homeomorphism,  $r(G_1)$  is also clopen and so there is an open  $U \subseteq G^{(0)} \setminus r(G_1)$  with  $s(g) \in U$ . By Urysohn, there exists  $f \in C_c(U)$  with  $f(s(g)) = 1$ . Part (1) gives  $g \in \ker(\phi)$  and so  $af = 0$ , which gives  $0 = (af)(g) = a(g)$ . So  $a \in C_0(\{g \in G_0 : s(g) \in r(G_1)\})$ . On the other hand, if  $a \in C_0(\{g \in G_0 : s(g) \in r(G_1)\})$  and  $f \in \ker(\phi)$ , then the preceding paragraph shows that  $s(\{\gamma : a(\gamma) \neq 0\}) \cap r(\{\gamma : f(\gamma) \neq 0\}) = \emptyset$ , and so  $af = 0$  showing that  $a \in \ker(\phi)^\perp$ .

(3) We have seen that  $r(G_1)$  is clopen, and since  $G_0$  is also clopen,  $s^{-1}(r(G_1)) \cap G_0$  is clopen. So for  $a \in C_c(G_0)$  the pointwise products  $a_1 = 1_{s^{-1}(r(G_1))}a$  and  $a_0 = 1_{G_0 \setminus s^{-1}(r(G_1))}a$  belong to  $C_c(G_0)$  and satisfy  $a = a_0 + a_1$ . Since  $a_0 \in \ker(\phi)$ , we have  $\phi(a) = \phi(a_1)$ . So  $\phi(C_r^*(G_0)) = \overline{\phi(C_c(s^{-1}(r(G_1)) \cap G_0))}$ . As observed above, if  $\gamma \in s^{-1}(r(G_1)) \cap G_0$  then for any  $\alpha \in G_1$  with  $r(\alpha) = s(\gamma)$ , we can write  $\gamma = (\gamma\alpha)\alpha^{-1} \in G_1G_1^{-1}$ . Since multiplication in  $G$  is open, it follows that

$$\{UV^{-1} : U, V \subseteq G_1 \text{ are precompact open bisections}\}$$

is a base for the topology on  $s^{-1}(r(G_1)) \cap G_0$ , and so  $C_c(s^{-1}(r(G_1)) \cap G_0) = \overline{\text{span}}\{fg^* : f, g \in C_c(G_1)\}$ . For  $f, g \in C_c(G_1)$  and  $\xi \in C_c(G_1) \subseteq X(G)$ , we have  $\phi(fg^*)(\xi) = fg^*\xi = \theta_{f,g}(\xi)$ . So  $\phi(fg^*) = \theta_{f,g} \in \mathcal{K}(X(G))$ . Hence  $\phi(C_c(s^{-1}(r(G_1)) \cap G_0)) \subseteq \mathcal{K}(X(G))$ , and the result follows.

(4) Part (3) shows that  $\text{span}\{fg^* : f, g \in C_c(G_1)\}$  is dense in the Katsura ideal  $J_{X(G)}$ . For  $f, g \in C_c(G_1)$  we have  $I_0(fg^*) = fg^*$  because  $I_0$  extends the inclusion  $C_c(G_0) \subseteq C_c(G)$  by definition, and we have just seen that  $\phi(fg^*) = \theta_{f,g}$ , giving  $I_1^{(1)}(\phi(fg^*)) = I_1^{(1)}(\theta_{f,g}) = I_1(f)I_1(g)^* = fg^*$  as well.  $\square$

**Notation 9.** Recall that if  $(\psi, \pi)$  is a Toeplitz representation of a  $C^*$ -correspondence  $X$  in a  $C^*$ -algebra  $B$ , then for  $n \geq 2$  we write  $\psi_n$  for the continuous linear map from  $X^{\otimes n}$  to  $B$  such that  $\psi(x_1 \otimes \cdots \otimes x_n) = \psi(x_1) \cdots \psi(x_n)$  for all  $x_i \in X$ .

**Proposition 10.** *The integrated form  $I_1 \times I_0$  of the representation in part (4) of Lemma 8 is an isomorphism of  $\mathcal{O}_{X(G)}$  onto  $C_r^*(G)$ .*

*Proof.* The grading  $c : C_r^*(G) \rightarrow \mathbb{Z}$  induces an action  $\beta$  of  $\mathbb{T}$  on  $C_r^*(G)$  such that

$$\beta_z(a) = z^{c(a)}a \quad \text{for all } z \in \mathbb{T} \text{ and } a \in \bigcup_n C_c(G_n).$$

Lemma 6 implies that for  $n \geq 1$ , the image  $I_n(C_c(G_1)^{\odot n})$  is dense in  $C_c(G_n)$ . This implies that  $I_1 \times I_0$  is equivariant for the gauge action  $\alpha$  on  $\mathcal{O}_{X(G)}$  and  $\beta$ . The gauge-invariant uniqueness theorem [4, Theorem 6.4] implies that  $I_1 \times I_0$  is injective. Moreover, since each  $I_n(C_c(G_1)^{\odot n})^*$  is dense in  $C_c(G_n)^*$ , which is  $C_c(G_{-n})$  by Lemma 6,  $I_1 \times I_0$  has dense range, and so is surjective since it is a homomorphism between  $C^*$ -algebras.  $\square$

**Corollary 11.** *Suppose that  $C^*(G_0) = C_r^*(G_0)$ . Then  $C^*(G) = C_r^*(G)$  and so Proposition 10 describes an isomorphism  $\mathcal{O}_{X(G)} \cong C^*(G)$ .*

*Proof.* We saw in Lemma 2 that the completion  $A_0$  of  $C_c(G_0)$  in  $C_r^*(G)$  coincides with  $C_r^*(G_0)$ , and therefore, by hypothesis, with  $C^*(G_0)$ . So  $\|I_0(a)\| = \|a\|_{C^*(G_0)}$  for all  $a \in C_c(G_0)$ . For  $\xi \in C_c(G_1)$ , we have

$$\|\xi^*\xi\|_{C^*(G)} \geq \|\xi^*\xi\|_{C_r^*(G)}.$$

The function  $\xi^*\xi$  belongs to  $C_c(G_0)$  and so  $\|\xi^*\xi\|_{C^*(G)} = \|\xi^*\xi\|_{C_r^*(G_0)}$  by Lemma 2, and this last is equal to  $\|\xi^*\xi\|_{C^*(G_0)}$  by hypothesis. Since the  $I$ -norm on  $C_c(G)$  agrees with the  $I$ -norm on  $C_c(G_0)$ , the canonical inclusion  $C_c(G_0) \hookrightarrow C_c(G)$  is  $I$ -norm bounded, and so extends to a  $C^*$ -homomorphism  $C^*(G_0) \rightarrow C^*(G)$ , which forces  $\|\xi^*\xi\|_{C^*(G_0)} \geq \|\xi^*\xi\|_{C^*(G)}$ . So we have equality throughout, giving

$$\|\xi^*\xi\|_{C^*(G)} = \|\xi^*\xi\|_{C_r^*(G_0)} = \|\langle I_1(\xi), I_1(\xi) \rangle_{C_r^*(G_0)}\|.$$

In particular,  $\|\xi\|_{C^*(G)} = \|I_1(\xi)\|$ . We deduce that  $(I_0, I_1)$  extend to a Cuntz-Pimsner covariant Toeplitz representation  $(\tilde{I}_0, \tilde{I}_1)$  of  $X(G)$  in  $C^*(G)$ . Let  $q : C^*(G) \rightarrow C_r^*(G)$  denote the quotient map. Then  $q \circ (\tilde{I}_1 \times \tilde{I}_0) = I_1 \times I_0$ , which is injective by Proposition 10, and we deduce that  $q$  is also injective.  $\square$

### Remark.

If  $G_0$  is an amenable groupoid (we don't have to specify which flavour because topological amenability and measurewise amenability coincide for second-countable locally compact Hausdorff étale groupoids [1, Theorem 3.3.7]), then [8, Proposition 9.3] shows that  $G$  is amenable as well, and then the preceding corollary follows. So the corollary has independent content only if  $G_0$  is not amenable but nevertheless has identical full and reduced norms.

Let

$$L_0 = \{a \in M_2(C_c(G)) : a_{11} \in J_{X(G)}, a_{22} \in C_c(G_0), a_{12} \in C_c(G_1) \text{ and } a_{21} \in C_c(G_{-1})\}.$$

Then the completion of  $L_0$  in the norm induced by the reduced norm on  $C_r^*(G)$  is

$$L = \begin{pmatrix} J_{X(G)} & X(G) \\ X(G)^* & C_r^*(G_0) \end{pmatrix}.$$

Part (3) of Lemma 8 shows that  $L$  is a  $C^*$ -subalgebra of  $M_2(C_r^*(G))$ , and that the corners  $\begin{pmatrix} J_{X(G)} & 0 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 \\ 0 & C_r^*(G_0) \end{pmatrix}$  are full. So the inclusion maps  $i^{11} : J_{X(G)} \rightarrow L$  into the top-left corner and  $i^{22} : C_r^*(G_0) \rightarrow L$  into the bottom-right corner determine isomorphisms  $i_*^{11} : K_*(J_{X(G)}) \rightarrow K_*(L)$  and  $i_*^{22} : K_*(C_r^*(G_0)) \rightarrow K_*(L)$ . The composite  $[X] := (i_*^{22})^{-1} \circ i_*^{11} : K_*(J_{X(G)}) \rightarrow K_*(C_r^*(G_0))$  is the map appearing in [4, Theorem 8.6] (see [4, Remark B4]). The inclusion map  $\iota : J_{X(G)} \hookrightarrow C_r^*(G_0)$  also induces a homomorphism  $\iota_* : K_*(J_{X(G)}) \rightarrow K_*(C_r^*(G_0))$ , and Theorem 8.6 of [4] gives the following.

**Corollary 12.** *There is an exact sequence in  $K$ -theory as follows:*

$$\begin{array}{ccccc} K_0(J_{X(G)}) & \xrightarrow{\iota_* - [X]} & K_0(C_r^*(G_0)) & \xrightarrow{(I_0)_*} & K_0(C_r^*(G)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(G)) & \xleftarrow{(I_0)_*} & K_1(C_r^*(G_0)) & \xleftarrow{\iota_* - [X]} & K_1(J_{X(G)}) \end{array}$$

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*E-mail address:* renniea@uow.edu.au, droberts@uow.edu.au, asims@uow.edu.au

SCHOOL OF MATHEMATICS AND APPLIED STATISTICS, UNIVERSITY OF WOLLONGONG, WOLLONGONG, NSW, 2522, AUSTRALIA