# THE NONCOMMUTATIVE GEOMETRY OF $k$-GRAPH $C^*$-ALGEBRAS

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Abstract. This paper is comprised of two related parts. First we discuss which $k$-graph algebras have faithful gauge invariant traces, where the gauge action of $\mathbf{T}^k$ is the canonical one. We give a sufficient condition for the existence of such a trace, identify the $C^*$-algebras of $k$-graphs satisfying this condition up to Morita equivalence, and compute their $K$-theory.

For $k$-graphs with faithful gauge invariant trace, we construct a smooth $(k,\infty)$-summable semifinite spectral triple. We use the semifinite local index theorem to compute the pairing with $K$-theory. This numerical pairing can be obtained by applying the trace to a $KK$-pairing with values in the $K$-theory of the fixed point algebra of the $\mathbf{T}^k$ action. As with graph algebras, the index pairing is an invariant for a finer structure than the isomorphism class of the algebra.

## 1. Introduction

This paper generalises the construction of semifinite spectral triples for graph algebras, [PRen], to the $C^*$-algebras of higher rank graphs, or $k$-graphs.

Also, the question of existence of faithful traces on $k$-graph algebras has led to a classification up to Morita equivalence of a particular class of $k$-graph algebras possessing such a trace.

Experience with $k$-graph algebras has shown that from a $C^*$-algebraic point of view they tend to behave very much like graph $C^*$-algebras. Consequently the transition from graph $C^*$-algebras to $k$-graph $C^*$-algebras often appears quite simple. The subtlety generally lies in the added combinatorial complexity of $k$-graphs, and in particular in identifying the right higher-dimensional analogues of the graph-theoretic conditions which arise in the one-dimensional case. This experience is borne out again in the current paper: once the appropriate $k$-graph theoretic conditions have been identified, the generalisations of the constructions in [PRen] to higher-rank graphs turn out to be mostly straightforward.

The pay-off, however, is significant. From the point of view of noncommutative geometry, we construct infinitely many examples of (semifinite) spectral triples of every integer dimension $k \geq 1$ ($k = 1$ is contained in [PRen]). These spectral triples are generically semifinite, and so come from $KK$ classes rather than $K$-homology classes. Computations can be made very explicitly with these algebras, and we use this to relate the semifinite index pairing to the

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$KK$-index for these examples. This has led to a general picture of the relationship between semifinite index theory and $KK$-theory, [KNR].

From the $k$-graph algebra point of view, the need for the existence of semifinite traces on $k$-graph algebras has led to advances in $k$-graph theory as well. For example, in Corollary 3.18, we classify a certain class of $k$-graph algebras up to Morita equivalence and calculate their $K$-theory. For graph algebras this is not new because it is already known that all simple graph algebras are either AF or purely infinite [KPR], and in all cases the $K$-theory of graph algebras is completely understood [RSz]. However, there are no such structural results for $k$-graphs, and only for 2-graphs have general $K$-theory computations recently emerged [E]. Consequently any advances on structure theory and $K$-theory for general $k$-graphs are significant from the $k$-graph algebra point of view.

The paper is arranged a follows. In Section 2 we review the basic definitions of $k$-graphs and $k$-graph algebras. In Section 3 we examine the question of existence of certain kinds of traces on a $k$-graph $C^*$-algebra. As in [PRen] we can formulate this question in terms of the underlying $k$-graph, and this leads to some necessary conditions for the existence of traces, and one sufficiency condition. We classify the $k$-graph $C^*$-algebras satisfying this sufficiency condition, up to Morita equivalence, and compute their $K$-theory.

Section 4 reviews the definitions we require pertaining to semifinite spectral triples. In Section 5 we construct a Kasparov module for the $C^*$-algebra of any locally finite, locally convex $k$-graph with no sinks. This is a very general constrction, and the resulting Kasparov module is even iff $k$ is an even integer. In Section 6 we construct $(k, \infty)$-summable spectral triples for $k$-graph algebras with faithful trace. In Section 7 we use these spectral triples to compute index pairings and compare them with the Kasparov product as in [PRen].

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## 2. $k$-Graph $C^*$-Algebras

2.1. **Higher-rank graphs and their $C^*$-algebras.** In this subsection we outline the basic notation and definitions of $k$-graphs and their $C^*$-algebras. We refer the reader to [RSY] for a more thorough account.

**Higher-rank graphs.** Throughout this paper, we regard $\mathbf{N}^k$ as a monoid under pointwise addition. We denote the usual generators of $\mathbf{N}^k$ by $e_1, \ldots, e_k$, and for $n \in \mathbf{N}^k$ and $1 \leq i \leq k$, we denote the $i^{\text{th}}$ coordinate of $n$ by $n_i \in \mathbf{N}$; so $n = \sum n_i e_i$. For $m, n \in \mathbf{N}^k$, we write $m \leq n$ if $m_i \leq n_i$ for all $i$. By $m < n$, we mean $m \leq n$ and $m \neq n$. We use $m \vee n$ and $m \wedge n$ to denote, respectively, the coordinate-wise maximum and coordinate-wise minimum of $m$ and $n$; so that $m \wedge n \leq m, n \leq m \vee n$ and these are respectively the greatest lower bound and least upper bound of $m, n$ in $\mathbf{N}^k$.

**Definition 2.1** (Kumjian-Pask [KP])**.** *A* graph of rank $k$ *or* $k$-graph *is a pair* $(\Lambda, d)$ *consisting of a countable category* $\Lambda$ *and a* degree *functor* $d : \Lambda \to \mathbf{N}^k$ *which satisfy the following* factorisation property*: if* $\lambda \in \mathrm{Mor}(\Lambda)$ *satisfies* $d(\lambda) = m + n$*, then there are unique morphisms* $\mu, \nu \in \mathrm{Mor}(\Lambda)$ *such that* $d(\mu) = m$*,* $d(\nu) = n$*, and* $\lambda = \mu \circ \nu$*.*

The factorisation property ensures (see [KP]) that the identity morphisms of $\Lambda$ are precisely the morphisms of degree 0; that is $\{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\} = d^{-1}(0)$. This means that we may identify each object with its identity morphism, and we do this henceforth. This done, we can regard $\Lambda$ as consisting only of its morphisms, and we write $\lambda \in \Lambda$ to mean $\lambda \in \mathrm{Mor}(\Lambda)$.

Since we are thinking of $\Lambda$ as a kind of graph, we write $r$ and $s$ for the codomain and domain maps of $\Lambda$ respectively. We refer to elements of $\Lambda$ as *paths*, and to the paths of degree 0 (which correspond to the objects of $\Lambda$ as above) as *vertices*. Extending these conventions, we refer to the elements of $\Lambda$ with minimal nonzero degree (that is $d^{-1}(\{e_1, \dots, e_k\})$ as *edges*.

**Notation 2.2.** To try to minimise confusion, we will always use $u, v, w$ to denote vertices, $e, f$ to denote edges, and lower-case Greek letters $\lambda, \mu, \nu$, etc. for arbitrary paths. We will also drop the composition symbol, and simply write $\mu\nu$ for $\mu \circ \nu$ when the two are composable.

WARNING: *because* $\Lambda$ *is a category, composition of morphisms reads from right to left. Hence paths* $\mu$ *and* $\nu$ *in* $\Lambda$ *can be composed to form* $\mu\nu$ *if and only if* $r(\nu) = s(\mu)$*, and in this case,* $r(\mu\nu) = r(\mu)$ *and* $s(\mu\nu) = s(\nu)$*. This is the reverse of the convention for directed graphs, used in* [BPRS, KPR, KPRR, PRen]*, so the reader should beware. In particular the roles of sources and sinks, and of ranges and sources, are opposite to those in* [PRen]*.*

**Definition 2.3.** *For each* $n \in \mathbf{N}^k$*, we write* $\Lambda^n$ *for the collection* $\{\lambda \in \Lambda : d(\lambda) = n\}$ *of paths of degree* $n$*.*

The range and source $r, s$ are thus maps from $\Lambda$ to $\Lambda^0$, and if $v \in \Lambda^0$, then $r(v) = v = s(v)$.

Given $\lambda \in \Lambda$ and $S \subset \Lambda$, it makes sense to write $\lambda S$ for $\{\lambda\sigma : \sigma \in S, r(\sigma) = s(\lambda)\}$, and likewise $S\lambda = \{\sigma\lambda : \sigma \in S, s(\sigma) = r(\lambda)\}$. In particular, if $v \in \Lambda^0$, then $vS$ is the collection of elements of $S$ with range $v$, and $Sv$ is the collection of elements of $S$ with source $v$.

**Definition 2.4.** *Let* $(\Lambda, d)$ *be a* $k$*-graph. We say that* $\Lambda$ *is* row-finite *if* $|v\Lambda^n| < \infty$ *for each* $v \in \Lambda^0$ *and* $n \in \mathbf{N}^k$*. We say that* $\Lambda$ *is* locally-finite *if it is row-finite and also satisfies* $|\Lambda^n v| < \infty$ *for all* $v \in \Lambda^0$ *and* $n \in \mathbf{N}^k$*. We say that* $\Lambda$ *has* no sources *(resp.* no sinks*) if* $v\Lambda^n$ *(resp.* $\Lambda^n v$*) is nonempty for each* $v \in \Lambda^0$ *and* $n \in \mathbf{N}^k$*. Finally, we say that* $\Lambda$ *is* locally convex *if, for each edge* $e \in \Lambda^{e_i}$*, and each* $j \neq i$ *in* $\{1, \dots, k\}$*, we have* $s(e)\Lambda^{e_j} = \emptyset$ *only if* $r(e)\Lambda^{e_j} = \emptyset$*.*

As in [RSY], for locally convex $k$-graphs, we use the notation $\Lambda^{\leq n}$ to denote the collection

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n, \mu \in s(\lambda)\Lambda \text{ and } d(\lambda\mu) \leq n \text{ implies } \mu = s(\lambda)\}.$$

Intuitively, $\Lambda^{\leq n}$ is the collection of paths whose degree is "as large as possible" subject to being dominated by $n$. In a 1-graph, $\Lambda^{\leq n}$ is the set of paths $\lambda \in \Lambda$ whose length is at most $n$ and is less than $n$ only if $s(\lambda)$ receives no edges. The significance of this is that the partial isometries associated to distinct paths in $\Lambda^{\leq n}$ have orthogonal range projections (cf. relation (CK4) below). For more on the importance of $\Lambda^{\leq n}$, see [RSY].

**$\Omega_{k,m}$ and boundary paths.** For $k \geq 1$ and $m \in (\mathbf{N} \cup \{\infty\})^k$, we define a $k$-graph $\Omega_{k,m}$ as follows:

$$\Omega_{k,m}^0 = \{n \in \mathbf{N}^k : n \leq m\} \qquad \Omega_{k,m}^n = \{(p,q) \in \mathbf{N}^k : p, q \in \Omega_{k,m}^0, q - p = n\}$$
$$r(p,q) = p, \qquad s(p,q) = q, \qquad (p,q) \circ (q,n) = (p,n).$$

See Figure 1 for a "picture" of $\Omega_{3,(\infty,2,1)}$.

Each path $\lambda$ of degree $p$ in a $k$-graph $\Lambda$ determines a degree-preserving functor $\hat{\lambda}$ from $\Omega_{k,p}$ to $\Lambda$ as follows: the image $\hat{\lambda}(m,n)$ of the morphism $(m,n) \in \Omega_{k,p}$ is the unique morphism in $\Lambda^{n-m}$ such that there exist $\mu \in \Lambda^m$ and $\nu \in \Lambda^{p-n}$ satisfying $\lambda = \mu\hat{\lambda}(m,n)\nu$. (The existence and uniqueness of $\hat{\lambda}(m,n)$ is guaranteed by the factorisation property).

In fact for each $p \in \mathbf{N}^k$, the map $\lambda \mapsto \hat{\lambda}$ is a bijection between $\Lambda^p$ and the set of degree-preserving functors from $\Omega_{k,p}$ to $\Lambda$. In practise, we just write $\lambda(m,n)$ for the segment $\hat{\lambda}(m,n)$ of $\lambda$ starting at position $n$ and terminating at position $m$, and write $\lambda(m)$ for the vertex at position $m$. If $\lambda \in \Lambda^p$ and $0 \leq m \leq n \leq p$, then

$$\lambda = \lambda(0,m)\lambda(m,n)\lambda(n,p), \quad s(\lambda(m,n)) = \lambda(n) \quad \text{and} \quad r(\lambda(m,n)) = \lambda(m).$$

We extend this correspondence between paths and degree-preserving functors to define the notion of a boundary path in a $k$-graph.

**Definition 2.5.** *A* boundary path *of a $k$-graph $\Lambda$ is a degree-preserving functor $x : \Omega_{k,m} \to \Lambda$ such that*

$$\textit{if } m_i < \infty,\ n \in \mathbf{N}^k,\ n \leq m \textit{ and } n_i = m_i, \textit{ then } x(n)\Lambda^{e_i} = \emptyset;$$

*so the directions in which $x$ is finite are those in which it cannot be extended. If $x : \Omega_{k,m} \to \Lambda$ is a boundary path, we denote $m$ by $d(x)$, and $x(0)$ by $r(x)$. We write $\Lambda^{\leq\infty}$ for the set of all boundary paths of $\Lambda$.*

Note that if $\lambda \in \Lambda$ satisfies $s(\lambda)\Lambda^n = \emptyset$ for all $n > 0$ (that is, if $s(\lambda)$ is a source in $\Lambda$), then the graph morphism $\hat{\lambda} : \Omega_{k,d(\lambda)} \to \Lambda$ discussed above belongs to $\Lambda^{\leq\infty}$; we think of $\lambda$ itself as a boundary path of $\Lambda$.

**Definition 2.6.** *An* end *of $\Lambda$ is a boundary path $x \in \Lambda^{\leq\infty}$ such that for all $n \leq d(x)$, $r(x)\Lambda^n = \{x(0,n)\}$. We denote the set of ends of $\Lambda$ by $Ends(\Lambda)$.*

**Remarks 2.7.** If $x$ is an end of $\Lambda$, then $r(x)\Lambda^{\leq n} = \{x(0, n \wedge d(x))\}$ for all $n \in \mathbf{N}^k$.

**Skeletons.** To draw a $k$-graph, we use its *skeleton*. The skeleton of a $k$-graph $\Lambda$ is the directed graph whose vertices and edges are those of $\Lambda$, but with the $k$ different types of edges distinguished using $k$ different colours. In this paper, we use solid lines for edges of degree $e_1$, dashed lines for edges of degree $e_2$, and dotted lines for edges of degree $e_3$. For example, the skeleton of $\Omega_{3,(\infty,2,1)}$ is presented in Figure 1

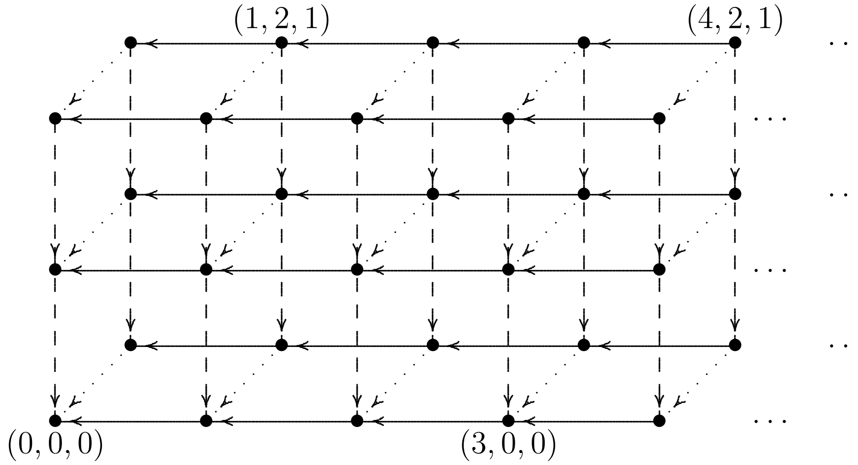


FIGURE 1. The skeleton of $\Omega_{3,(\infty,2,1)}$

The factorisation property says that if $e$ and $f$ are edges of degree $e_i$ and $e_j$ respectively such that $s(e) = r(f)$, then the path $ef$ can be expressed in the form $f'e'$ where $d(f') = e_j$ and $d(e') = e_i$. In the skeleton for $\Omega_{3,(\infty,2,1)}$ there is just one way this can happen; so the skeleton is actually a commuting diagram in the category, and although there appear to be many ways to get from $(1, 2, 1)$ to $(0, 0, 0)$, for example, each of these paths yields the same morphism in the category, so there is really just one path in $\Omega_{3,(\infty,2,1)}$ from $(1, 2, 1)$ to $(0, 0, 0)$.

The information determining the factorisation property is not always included in the skeleton, and it must then be specified separately as a set of *factorisation rules*. The uniqueness of factorisations ensures that amongst the factorisation rules for the skeleton of a $k$-graph, each composition $ef$ where $e$ and $f$ are composable edges of different colours will appear exactly once. A set of factorisation rules for a skeleton with this property is referred to as an *allowable* factorisation regime.

For example, in the 1-skeleton of Figure 2 the allowable factorisation regimes are: $\{ef = he, kf = hk\}$ and $\{ef = hk, kf = he\}$).

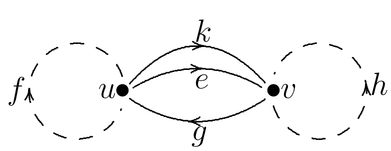


FIGURE 2. A skeleton that admits two distinct factorisation regimes

A skeleton together with an allowable factorisation regime determines at most one $k$-graph. When $k = 2$, each skeleton and allowable factorisation regime determines a unique $k$-graph. For $k \geq 3$, there is an additional associativity condition on the factorisation rules which must be verified [FS]; but the issue does not arise in the examples we give in this paper.

**Cuntz-Krieger families and $C^*(\Lambda)$.** As with directed graphs, we are interested in higher-rank graphs because we can associate to each one a $C^*$-algebra of Cuntz-Krieger type.

**Definition 2.8.** *Let $(\Lambda, d)$ be a row-finite locally convex $k$-graph. A Cuntz-Krieger $\Lambda$-family is a collection $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying*

(CK1) $\{s_v : v \in \Lambda^0\}$ *is a collection of mutually orthogonal projections;*
(CK2) $s_\mu s_\nu = s_{\mu\nu}$ *for all* $\mu, \nu \in \Lambda$ *with* $s(\mu) = r(\nu)$*;*
(CK3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ *for all* $\lambda \in \Lambda$*; and*
(CK4) $s_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^*$ *for all* $v \in \Lambda^0$ *and* $n \in \mathbf{N}^k$*.*

*As a point of notation, we will henceforth denote the vertex projection $s_v$ by $p_v$ to remind ourselves that it is a projection.*

The Cuntz-Krieger algebra of $\Lambda$, denoted $C^*(\Lambda)$, is the universal $C^*$-algebra generated by a Cuntz-Krieger family $\{s_\lambda : \lambda \in \Lambda\}$. By this we mean that given any other Cuntz-Krieger $\Lambda$-family $\{t_\lambda : \lambda \in \Lambda\}$, there is a homomorphism $\pi_t$ satisfying $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$. By [RSY, Proposition 3.5], if $\mu, \nu \in \Lambda$, then $s_\mu^* s_\nu = \sum_{\mu\alpha=\nu\beta, d(\mu\alpha)=d(\mu)\vee d(\nu)} s_\alpha s_\beta^*$, and hence ([RSY, Remarks 3.8(1)]),

$$C^*(\Lambda) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : s(\alpha) = s(\beta)\}. \tag{1}$$

For the details of the next two paragraphs, see [RSY, page 109].

The universal property of $C^*(\Lambda)$ guarantees that there is an action $\gamma : \mathbf{T}^k \to \operatorname{Aut}(C^*(\Lambda))$ satisfying $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda := z_1^{d(\lambda)_1} \cdot z_2^{d(\lambda)_2} \cdots z_k^{d(\lambda)_k} s_\lambda$ and hence $\gamma_z(p_v) = p_v$. We denote the fixed point algebra for $\gamma$ by $F$, and $\Phi$ denotes the faithful conditional expectation $\Phi : C^*(\Lambda) \to F$ determined by $\Phi(a) = \int_{\mathbf{T}} \gamma_z(a)\, d\mu(z)$.

We have $F = \overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu), s(\mu) = s(\nu)\}$ and $\Phi$ is determined by $\Phi(s_\mu s_\nu^*) = \delta_{d(\mu),d(\nu)} s_\mu s_\nu^*$. For each $n \in \mathbf{N}^k$, we write $F_n := \overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu), \mu, \nu \in \Lambda^{\leq n}, s(\mu) = s(\nu)\}$. Then each $F_n$ is isomorphic to a direct sum of matrix algebras and algebras of compact operators, and $F = \overline{\bigcup F_n}$ is an AF algebra.

## 3. $k$-graph traces and faithful traces on $C^*(\Lambda)$

In this section we investigate conditions which give rise to faithful traces on $C^*(\Lambda)$ for a locally convex locally finite $k$-graph $\Lambda$. As with the $C^*$-algebras of directed graphs, necessary and sufficient conditions for the existence of faithful traces on a $k$-graph algebra are hard to come by. We denote by $A^+$ the positive cone in a $C^*$-algebra $A$, and we use extended arithmetic on $[0, \infty]$ so that $0 \times \infty = 0$. From [PhR] we take the basic definition:

**Definition 3.1.** *A trace on a $C^*$-algebra $A$ is a map $\tau : A^+ \to [0, \infty]$ satisfying*

*1)* $\tau(a + b) = \tau(a) + \tau(b)$ *for all* $a, b \in A^+$

*2)* $\tau(\lambda a) = \lambda \tau(a)$ *for all* $a \in A^+$ *and* $\lambda \geq 0$

*3)* $\tau(a^*a) = \tau(aa^*)$ *for all* $a \in A$

*We say: that* $\tau$ *is faithful if* $\tau(a^*a) = 0 \Rightarrow a = 0$*; that* $\tau$ *is semifinite if* $\{a \in A^+ : \tau(a) < \infty\}$ *is norm dense in* $A^+$ *(or that* $\tau$ *is densely defined); that* $\tau$ *is lower semicontinuous if whenever* $a = \lim_{n\to\infty} a_n$ *in norm in* $A^+$ *we have* $\tau(a) \leq \liminf_{n\to\infty} \tau(a_n)$*.*

We may extend a (semifinite) trace $\tau$ by linearity to a linear functional on (a dense subspace of) $A$. Observe that the domain of definition of a densely defined trace is a two-sided ideal $I_\tau \subset A$. The proof of the following Lemma is identical to that of the analogous result for graph algebras [PRen, Lemma 3.2].

**Lemma 3.2.** *Let* $(\Lambda, d)$ *be a row-finite locally convex k-graph and let* $\tau : C^*(\Lambda) \to \mathbf{C}$ *be a semifinite trace. Then the dense subalgebra*

$$A_c := \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$$

*is contained in the domain* $I_\tau$ *of* $\tau$*.*

Recall from [Si] that a loop with an entrance is a path $\lambda \in \Lambda$ with $r(\lambda) = s(\lambda)$ such that $d(\lambda) \geq e_i$ for some $1 \leq i \leq k$, together with an $e \in \Lambda^{e_i}$ with $r(e) = r(\lambda)$ but $\lambda(0, e_i) \neq e$.

**Lemma 3.3.** *Let* $(\Lambda, d)$ *be a row-finite locally convex k-graph.*

**(i)** *If* $C^*(\Lambda)$ *has a faithful semifinite trace then no loop can have an entrance.*

**(ii)** *If* $C^*(\Lambda)$ *has a gauge-invariant, semifinite, lower semicontinuous trace* $\tau$ *then* $\tau \circ \Phi = \tau$ *and*

$$\tau(s_\mu s_\nu^*) = \delta_{\mu,\nu} \tau(p_{s(\mu)}).$$

*In particular, if* $\tau|_{C^*(\{s_\mu s_\mu^* : \mu \in \Lambda\})} = 0$ *then* $\tau = 0$*.*

*Proof.* The entrance condition implies that $\lambda(0, e_i)$ and the entrance $e$ are distinct paths of degree $e_i$ with the same range, and it follows from (CK3) and (CK4) that

$$s_\lambda^* s_\lambda = p_{s(\lambda)} = p_{r(\lambda)} \geq s_\lambda s_\lambda^* + s_e s_e^*.$$

If $\tau$ is a trace on $C^*(\Lambda)$, we therefore have $\tau(s_\lambda^* s_\lambda) \geq \tau(s_\lambda s_\lambda^*) + \tau(s_e s_e^*)$, and it follows from Lemma 3.2 and the trace property that $\tau(s_e^* s_e) = \tau(s_e s_e^*) = 0$. Theorem 3.15 of [RSY] implies that $s_e^* s_e \neq 0$ so $\tau$ is not faithful.

The proof of the second part is the same as [PRen, Lemma 3.3], but for clarity we remind the reader how the final statement arises. If $\tau$ is gauge invariant we have

$$\tau(s_\mu s_\nu^*) = \tau(\gamma_z(s_\mu s_\nu^*)) = z^{d(\mu)-d(\nu)} \tau(s_\mu s_\nu^*)$$

for all $z \in \mathbf{T}^k$. Hence $\tau(s_\mu s_\nu^*)$ is zero unless $d(\mu) = d(\nu)$, and so $\tau = \tau \circ \Phi$. Moreover if $d(\mu) = d(\nu)$, then using the trace property,

$$\tau(s_\mu s_\nu^*) = \tau(s_\nu^* s_\mu) = \delta_{\nu,\mu} \tau(p_{s(\nu)}) = \delta_{\nu,\mu} \tau(s_\nu^* s_\nu).$$

This proves that if $\tau|_{\operatorname{span}\{s_\mu s_\mu^* : \mu \in \Lambda\}} = 0$ then $\tau|A_c = 0$. The details of extending this to the $C^*$-completion are as in [PRen]. □

Whilst the condition that no loop has an entrance is necessary for the existence of a faithful semifinite trace, it is not sufficient. For example, let $\Lambda$ be any 2-graph whose skeleton is the one illustrated in Figure 3 (there are many allowable factorisation regimes to choose from). Then $\Lambda$ is locally convex and locally finite, contains no sinks or sources, and contains no cycles at all, so certainly no cycles with entrances, yet $C^*(\Lambda)$ does not admit a faithful semifinite trace. To see why note that (CK4) forces $s_g s_g^* = p_v = s_e s_e^* + s_f s_f^*$ so if $\tau$ is a trace on $C^*(\Lambda)$ then the

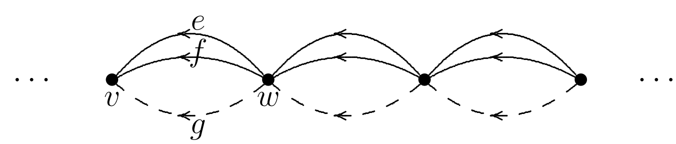


FIGURE 3. A 2-graph whose $C^*$-algebra does not admit a faithful semifinite trace

trace property forces

$$\tau(p_v) = \tau(s_g s_g^*) = \tau(p_w) \quad \text{and} \quad \tau(p_v) = \tau(s_e s_e^*) + \tau(s_f s_f^*) = 2\tau(p_w),$$

and hence $\tau(p_w) = 0$.

The situation illustrated in Figure 3 is more subtle than those which can arise for graph $C^*$-algebras. However, as with directed graphs, the obstructions to the existence of a faithful semifinite trace on a $k$-graph algebra can be expressed most naturally for general $k$-graphs $\Lambda$ in terms of a function $g_\tau : \Lambda^0 \to \mathbf{R}^+$ which arises naturally from each trace $\tau$ on $C^*(\Lambda)$.

**Lemma 3.4.** *Let $\Lambda$ be a locally convex row-finite $k$-graph, and suppose that $\tau$ is a semifinite trace on $C^*(\Lambda)$. Then the function $g_\tau : \Lambda^0 \to \mathbf{R}^+$ defined by $g_\tau(v) := \tau(p_v)$ satisfies $g_\tau(v) = \sum_{\lambda \in v\Lambda^{\leq n}} g_\tau(s(\lambda))$ for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$.*

*Proof.* Fix $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. By (CK4), we have $p_v = \sum_{\lambda \in v\Lambda^{\leq n}} s_\lambda s_\lambda^*$. Hence

$$\tau(p_v) = \sum_{\lambda \in v\Lambda^{\leq n}} \tau(s_\lambda s_\lambda^*) = \sum_{\lambda \in v\Lambda^{\leq n}} \tau(s_\lambda^* s_\lambda) = \sum_{\lambda \in v\Lambda^{\leq n}} \tau(p_{s(\lambda)}),$$

and the result follows from the definition of $g_\tau$. □

Returning to the example of Figure 3 we can see that if $\tau$ is a trace on $C^*(\Lambda)$ then $g_\tau(v)$ must simultaneously be equal to $g_\tau(w)$ and $2g_\tau(w)$, forcing $g_\tau(w)$ and hence $\tau(p_w)$ to be equal to zero.

Motivated by Lemma 3.4, we make the following definition (see [T] for the origins of this definition):

**Definition 3.5.** *Let $\Lambda$ be a locally convex row-finite $k$-graph. A function $g : \Lambda^0 \to \mathbf{R}^+$ is called a $k$-*graph trace *on $\Lambda$ if it satisfies*

$$g(v) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda)) \text{ for all } v \in \Lambda^0 \text{ and } n \in \mathbf{N}^k. \tag{2}$$

*We say that $g$ is* faithful *if $g(v) \neq 0$ for all $v \in \Lambda^0$.*

**Remarks 3.6.** Notice that if $x$ is an end of $\Lambda$, then $x(0)\Lambda^{\leq n} = \{x(0,n)\}$ for any $n \leq d(x)$. It follows that each $k$-graph trace on $\Lambda$ is constant on the vertices of $x$.

We want to be able to construct semifinite lower semicontinuous gauge-invariant traces on $C^*(\Lambda)$ from $k$-graph traces on $\Lambda$. The idea is to use (2) to define a trace on $C^*(\Lambda)$ by $\tau_g\Big(\sum_{\mu,\nu\in F} a_{\mu,\nu} s_\mu s_\nu^*\Big) = \sum_{\mu\in F} a_{\mu,\mu} g(s(\mu))$. There are two problems to overcome: is $\tau_g$ well-defined in the first place, and when is $\tau_g$ faithful? To address these problems, we establish in the next three results that there is a faithful conditional expectation $\Psi$ on $C^*(\Lambda)$ satisfying $\Psi(s_\mu s_\nu^*) = \delta_{\mu,\nu} s_\mu s_\mu^*$. The difficulty of establishing the existence of this expectation is one feature of the greater complexity of $k$-graph algebras compared with graph algebras.

**Proposition 3.7.** *Let $I$ be a countable index set, and let $\mathcal{H}$ be the separable Hilbert space $\ell^2(I)$. For each $i \in I$ let $p_i$ denote the rank-one projection $e_i \otimes \overline{e_i}$ onto the subspace $\mathbf{C}e_i$. For each $S \in \mathcal{B}(\mathcal{H})$, the sum $\sum_{i\in I} p_i S p_i$ converges in the strong operator topology to an operator $\Psi(S) \in \mathcal{B}(\mathcal{H})$. The map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ obtained from this formula is a faithful expectation.*

*Proof.* For the strong-operator convergence, fix $h \in \mathcal{H}$. Then $h = \sum_{j\in I}(h|e_j)e_j$, so for a finite subset $F \subset I$, we have

$$\Big\| \sum_{i\in I\setminus F} (p_i S p_i) h \Big\|^2 = \Big\| \sum_{i\in I\setminus F} \Big( p_i S p_i \big( \sum_{j\in I} (h|e_j) e_j \big) \Big) \Big\|^2 = \Big\| \sum_{i\in I\setminus F} (h|e_j) p_i S e_i \Big\|^2 \leq \sum_{i\in I\setminus F} \|S\|^2 |(h|e_i)|^2,$$

which approaches 0 as $F$ increases over finite subsets of $I$. It follows easily from this that the sequence $\sum_{i\in F} p_i S p_i h$ is Cauchy in $\mathcal{H}$ and hence converges.

The linearity of the map $\Psi$ obtained in this way is clear. To see that $\Psi$ is norm decreasing, we first calculate:

$$\begin{aligned} \|\Psi(S)\|^2 &= \sup_{\|h\|=1} \|\Psi(S)h\|^2 = \sup_{\|h\|=1} \| \sum_{i\in I} (h|e_i) p_i S e_i \|^2 \\ &\leq \sup_{\|h\|=1} \sum_{i\in I} |(h|e_i)|^2 (\sup_{j\in I} \|p_j S e_j\|)^2 = (\sup_{j\in I} \|p_j S e_j\|)^2. \end{aligned}$$

Hence, if we write $S_i := (Se_i|e_i)$ for each $i \in I$, we have $\|\Psi(S)\| \leq \sup_{i\in I} \|S_i\|$. On the other hand, since $\|\Psi(S)e_i\| = \|p_i S p_i e_i\| = \|S_i\|$ for each $I$, we have $\|\Psi(S)\| = \sup_{i\in I} \|S_i\|$ for all $S \in \mathcal{B}(\mathcal{H})$. Now each $p_i$ is a projection, so we have $\|S_i\| = \|p_i S p_i\| \leq \|S\|$ for all $i \in I$, and hence

$$\|\Psi(S)\| = \sup_{i\in I} |S_i| \leq \|S\|,$$

and $\Psi$ is norm decreasing as required.

It remains to show that $\Psi$ takes nonzero positive elements to nonzero positive elements. Fix $S \neq 0$. To see that $\Psi(S^*S) \neq 0$, note that $Se_i \neq 0$ for some $i \in I$, and we have

$$\|\Psi(S^*S)\|^2 \geq \|\Psi(S^*S)e_i\|^2 = \|p_i S^* S p_i e_i\|^2 = (Sp_i e_i | S p_i e_i) = \|Se_i\|^2 \geq 0.$$

To see that $\Psi(S^*S)$ is positive, define $A \in \mathcal{B}(\mathcal{H})$ by $Ae_i := \|Se_i\| e_i$ for all $i$. Then $A^*Ae_i = \|Se_i\|^2 e_i = p_i S^* S p_i e_i = \Psi(S^*S)e_i$ for all $i$, and so $\Psi(S^*S) = A^*A$ is positive. $\square$

Let $\mathcal{H} := \ell^2(\Lambda^{\leq\infty})$. There is a Cuntz-Krieger $\Lambda$-family in $\mathcal{B}(\mathcal{H})$ given by $S_\lambda e_x := \delta_{s(\lambda),r(x)} e_{\lambda x}$ [RSY, Theorem 3.15] and all the partial isometries $S_\lambda$ are nonzero. The associated representation $\pi_S$ of $C^*(\Lambda)$ is faithful on $F$, but generally not on $C^*(\Lambda)$ entire. To obtain a faithful representation of $C^*(\Lambda)$, we augment the boundary-path representation as follows: for each $n \in \mathbf{Z}^k$, let $L_n \in \mathcal{B}(\ell^2(\mathbf{Z}^k))$ be the shift operator $L_n e_m := e_{n+m}$, and then define operators $\{(S \otimes L)_\lambda : \lambda \in \Lambda\}$ by $(S \otimes L)_\lambda := S_\lambda \otimes L_{d(\lambda)} \in \mathcal{B}(\mathcal{H}) \otimes C^*(\mathbf{Z}^k)$. These operators form a Cuntz-Krieger family for $\Lambda$ and the corresponding representation of $C^*(\Lambda)$ is faithful [Si, Theorem 4.76]. We call this representation the *augmented boundary path representation.*

**Proposition 3.8.** *Let $\Lambda$ be a locally convex row-finite $k$-graph, and let $\{(S \otimes L)_\lambda : \lambda \in \Lambda\}$ be the augmented boundary path representation on $\mathcal{H} = \ell^2(\Lambda^{\leq\infty} \times \mathbf{Z}^k)$. Let $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the faithful conditional expectation onto the diagonal from Proposition 3.7. Then for $\lambda, \mu \in \Lambda$, we have*

$$\Psi((S \otimes L)_\lambda (S \otimes L)^*_\mu) = \delta_{\lambda,\mu} (S \otimes L)_\lambda (S \otimes L)^*_\lambda$$

*Proof.* Fix $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$. For $(x, n) \in \Lambda^{\leq\infty} \times \mathbf{Z}^k$, we have

$$(S \otimes L)_\lambda (S \otimes L)^*_\mu e_{(x,n)} = \begin{cases} e_{(\lambda x(d(\mu),d(x)),n+d(\lambda)-d(\mu))} & \text{if } x(0, d(\mu)) = \mu \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Suppose first that $\lambda \neq \mu$. There are two cases:
**Case 1:** $d(\lambda) \neq d(\mu)$. Then $n + d(\lambda) - d(\mu) \neq n$ for all $n \in \mathbf{Z}^k$, and hence

$$p_{(x,n)} e_{(\lambda x(d(\mu),d(x)),n+d(\lambda)-d(\mu))} = 0 \text{ for all } (x, n) \in \Lambda^\infty \times \mathbf{Z}^k.$$

Consequently $\Psi((S \otimes L)_\lambda (S \otimes L)^*_\mu)$ annihilates all basis elements in $\mathcal{H}$, and hence is equal to zero.
**Case 2:** $d(\lambda) = d(\mu)$. Then for $y \in s(\lambda)\Lambda^{\leq\infty}$, we have $\lambda y \neq \mu y$ because these two boundary paths have different initial segments of degree $d(\lambda)$. So if $x(0, d(\mu)) = \mu$, we have $\lambda x(0, d(\mu)) \neq x$, and so once again $p_{(x,n)} e_{(\lambda x(d(\mu),d(x)),n+d(\lambda)-d(\mu))} = 0$ for all $(x, n) \in \Lambda^\infty \times \mathbf{Z}^k$. Again, we conclude that $\Psi((S \otimes L)_\lambda (S \otimes L)^*_\mu) = 0$.

Now suppose that $\lambda = \mu$. Then

$$(S \otimes L)_\lambda (S \otimes L)^*_\mu = (S \otimes L)_\lambda (S \otimes L)^*_\lambda = \sum_{x \in s(\lambda)\Lambda^{\leq\infty}, n \in \mathbf{Z}^k} p_{(\lambda x,n)},$$

and this is clearly fixed under $\Psi$. □

**Corollary 3.9.** *Let $\Lambda$ be a locally convex row-finite $k$-graph. There is a faithful conditional expectation $\Psi : C^*(\Lambda) \to X := \overline{\operatorname{span}}\{s_\lambda s^*_\lambda : \lambda \in \Lambda\}$ which satisfies $\Psi(s_\lambda s^*_\mu) = \delta_{\lambda,\mu} s_\lambda s^*_\lambda$ for all $\lambda, \mu \in \Lambda$.*

*Proof.* By [Si, Theorem 4.7.6], the augmented boundary path representation is faithful on $C^*(\Lambda)$. The result now follows from Propositions 3.7 and 3.8. □

**Proposition 3.10.** *Let $\Lambda$ be a row-finite locally convex $k$-graph. Then there is a one-to-one correspondence between faithful graph traces on $\Lambda$ and faithful, semifinite, lower semicontinuous, gauge invariant traces on $C^*(\Lambda)$.*

*Proof.* Given a faithful $k$-graph trace, the existence of $\Psi : C^*(\Lambda) \to \overline{span}\{S_\mu S_\mu^*\}$ given by Corollary 3.9 shows that the functional $\tau_g : A_c \to \mathbf{C}$ defined by

$$\tau_g(S_\mu S_\nu^*) := \delta_{\mu,\nu} g(s(\mu))$$

is well-defined. As in [PRen], one checks that $\tau_g$ is a gauge invariant trace on $A_c$ and is faithful because for $a = \sum_{i=1}^n c_{\mu_i,\nu_i} S_{\mu_i} S_{\nu_i}^* \in A_c$ we have $a^*a \geq \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 S_{\nu_i} S_{\nu_i}^*$, so

$$\begin{aligned} \langle a, a\rangle_g := \tau_g(a^*a) &= \tau_g(\sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 S_{\nu_i} S_{\nu_i}^*) \\ &= \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 \tau_g(S_{\nu_i} S_{\nu_i}^*) = \sum_{i=1}^n |c_{\mu_i,\nu_i}|^2 g(s(\nu_i)) > 0. \end{aligned} \tag{4}$$

by definition of $\tau_g$.

Then $\langle a, b\rangle_g = \tau_g(b^*a)$ defines a positive definite inner product on $A_c$ which makes it a Hilbert algebra (that the left regular representation of $A_c$ is nondegenerate follows from $A_c^2 = A_c$).

The rest of the proof is the same as [PRen, Proposition 3.9], except that we use the gauge invariant uniqueness theorem for $k$-graphs, [RSY, Theorem 4.1], to show that we obtain faithful representation of $A = C^*(\Lambda)$ on the Hilbert space completion of $A_c$. □

In addition to this result, we can also use $k$-graph traces to formulate other necessary conditions, and later one sufficient condition, on a $k$-graph for the existence of a trace on $C^*(\Lambda)$.

For the purposes of the following two lemmas we say that paths $\mu$ and $\nu$ in a $k$-graph $\Lambda$ are *orthogonal* if the range projections $s_\mu s_\mu^*$ and $s_\nu s_\nu^*$ are orthogonal in $C^*(\Lambda)$. By [RSY, Proposition 3.5], $\mu$ and $\nu$ are orthogonal if and only if they have no common extensions.

**Lemma 3.11.** *Suppose that $(\Lambda, d)$ is a row-finite $k$-graph and there exist vertices $v, w \in \Lambda^0$ with an infinite number of mutually orthogonal paths from $w$ to $v$. Then there is no faithful $k$-graph trace on $\Lambda^0$.*

*Proof.* Let $(\lambda_n)_{n\in\mathbf{N}}$ be the infinite set of orthogonal paths from $w$ to $v$. Suppose that $\tau$ is a trace on $C^*(\Lambda)$. For each $n$, $\tau(s_{\lambda_n} s_{\lambda_n}^*) = \tau(s_{\lambda_n}^* s_{\lambda_n}) = \tau(p_w)$. It follows that for any $N$, we have $\tau(p_v) \geq \sum_{n=1}^N \tau(s_{\lambda_n} s_{\lambda_n}^*) = N\tau(p_w)$, and it follows that $\tau(p_w) = 0$. Hence $g_\tau(w) = 0$, and it follows from Proposition 3.10 that no $k$-graph trace on $\Lambda^0$ is faithful. □

**Corollary 3.12.** *Suppose that $(\Lambda, d)$ is a row-finite $k$-graph and there exists a vertex $v \in \Lambda^0$ with an infinite number of mutually orthogonal paths from an end to $v$. Then there is no faithful $k$-graph trace on $\Lambda^0$.*

*Proof.* Since $k$-graph traces are constant on ends by Remarks 3.6, the proof is identical to that of Lemma 3.11. □

We now aim to provide a sufficient condition for a $k$-graph to admit a faithful $k$-graph trace.

**Notation.** Let $\Lambda$ be a locally convex row-finite $k$-graph. Define a relation $\sim$ on the ends of $\Lambda$ by $x \sim y$ if and only if $x(n) = y(m)$ for some $n \leq d(x)$ and $m \leq d(y)$ (so $x \sim y$ if they have vertices in common). It is easy to check that $\sim$ is an equivalence relation. We write $[x]$ for the equivalence class of an end $x$ under $\sim$.

If a vertex $v$ lies on an end of $\Lambda$, then $v\Lambda^{\leq\infty}$ consists of a single boundary path which we denote $x_v$, and this boundary path is itself an end of $\Lambda$.

**Proposition 3.13.** *Let $\Lambda$ be a locally convex row-finite $k$-graph. Suppose that there is a function $v \mapsto n_v$ from $\Lambda^0$ to $\mathbf{N}^k$ such that for each $v \in \Lambda^0$ and each $\lambda \in v\Lambda^{\leq n_v}$, $s(\lambda)$ lies on an end of $\Lambda$.*

(a) *If $g : \Lambda^0 \to \mathbf{R}^+$ is a $k$-graph trace, then there is a well-defined function from* $\operatorname{Ends}(\Lambda)/\sim$ *to $\mathbf{R}^+$ satisfying $g([x]) := g(x(0))$, and*

$$g(v) = \sum_{\lambda \in v\Lambda^{\leq n_v}} g([x_{s(\lambda)}]) \quad \text{for every } v \in \Lambda^0. \tag{5}$$

(b) *Conversely, given any function $g$ from* $\operatorname{Ends}(\Lambda)/\sim$ *to $\mathbf{R}^+$, there is a unique graph-trace $\overline{g}$ on $\Lambda$ satisfying $\overline{g}(x(0)) = g([x])$ for all $x \in \operatorname{Ends}(\Lambda)$.*

*Proof of Proposition 3.13(a).* If $x$ is an end of $\Lambda$ then $g$ is constant on $x$. Hence if $x(m) = y(n)$ for some $m, n$, we have $g(x(0)) = g(x(m)) = g(y(n)) = g(y(0))$ and it follows that $g(x(p)) = g(y(q))$ for all $p, q$. The formula (5) holds by definition of a $k$-graph trace. □

To prove the second part of the Proposition we need to know that for a fixed function $g$ from $\operatorname{Ends}(\Lambda)/\sim$ to $\mathbf{R}^+$, the formula (5) is independent of the choice of function $v \mapsto n_v$.

**Lemma 3.14.** *Suppose that $\Lambda$ satisfies the hypotheses of Propopoition 3.13, and let $g$ be a function from* $\operatorname{Ends}(\Lambda)/\sim$ *to $\mathbf{R}^+$. Define $g(v) := g([x_v])$ for each vertex $v$ that lies on an end of $\Lambda$. Fix $v \in \Lambda^0$ and suppose $n_1, n_2 \in \mathbf{N}^k$ each have the property that $s(\lambda)$ lies on an end of $\Lambda$ for each $\lambda \in v\Lambda^{\leq n_i}$. Then*

$$\sum_{\mu \in v\Lambda^{\leq n_1}} g(s(\mu)) = \sum_{\nu \in v\Lambda^{\leq n_2}} g(s(\nu)).$$

*Proof.* Let $n := n_1 \vee n_2$. For each $\mu \in \Lambda^{\leq n_1}$, since $s(\mu)$ lies on an end, we have that $s(\mu)\Lambda^{\leq n - n_1}$ is a singleton by Remark 2.7; say $s(\mu)\Lambda^{\leq n-n_1} = \{\nu_\mu\}$. If $x$ is the end such that $x(0) = s(\mu)$, then $\nu_\mu$ is the initial segment of $x$, and since $g$ is constant on ends, it follows that $g(s(\mu)) = g(s(\nu))$. Applying this argument to each $\mu \in v\Lambda^{\leq n_1}$ we obtain

$$\sum_{\mu \in v\Lambda^{\leq n_1}} g(s(\mu)) = \sum_{\mu \in v\Lambda^{\leq n_1}} g(s(\mu\nu_\mu)) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda))$$

But an identical argument shows that

$$\sum_{\nu \in v\Lambda^{\leq n_2}} g(s(\nu)) = \sum_{\lambda \in v\Lambda^{\leq n}} g(s(\lambda)),$$

and the result follows. □

*Proof of Proposition 3.13(b).* Define $\overline{g}(v) : \Lambda^0 \to \mathbf{R}^+$ by

$$\overline{g}(v) := \sum_{\lambda \in v\Lambda^{\leq n_v}} g([x_{s(\lambda)}])$$

Note that if $x$ is an end of $\Lambda$, then $n_{x(0)} := 0$ has the property that $s(\lambda)$ lies on an end of $\Lambda$ for each $\lambda \in x(0)\Lambda^{n_{x(0)}}$. Hence Lemma 3.14 shows that $\overline{g}(x(0)) = g([x])$.

Fix $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. We must show that

$$\overline{g}(v) = \sum_{\lambda \in v\Lambda^{\leq n}} \overline{g}(s(\lambda)). \tag{6}$$

We may assume without loss of generality that $n \leq n_v$ because if it is not, then $n'_v := n \vee n_v$ can be used in place of $n_v$ by Lemma 3.14 and satisfies $n \leq n'_v$. Since $\Lambda^{\leq n_v} = \Lambda^{\leq n}\Lambda^{\leq n_v - n}$ [RSY, Lemma 3.6], we then have that for each $\lambda \in v\Lambda^{\leq n}$, the element $n_v - n$ has the property that for each $\alpha \in s(\lambda)\Lambda^{\leq n_v - n}$, the source of $\alpha$ is on an end of $\Lambda$ and hence

$$\overline{g}(s(\lambda)) = \sum_{\alpha \in s(\lambda)\Lambda^{\leq n_v - n}} g(s(\alpha)) \tag{7}$$

by Lemma 3.14. But now

$$\begin{aligned}
\sum_{\lambda \in v\Lambda^{\leq n}} \overline{g}(s(\lambda)) &= \sum_{\lambda \in v\Lambda^{\leq n}} \Big( \sum_{\alpha \in s(\lambda)\Lambda^{\leq n_v - n}} g(s(\alpha)) \Big) \quad \text{by (7)} \\
&= \sum_{\lambda\alpha \in v\Lambda^{\leq n_v}} g(s(\lambda\alpha)) \quad \text{by [RSY, Lemma 3.6]} \\
&= \overline{g}(v)
\end{aligned}$$

by definition of $\overline{g}$. □

Finally we show that we can check that a given function is a graph trace just by considering edges and vertices in the skeleton of $\Lambda$. This is useful as it simplifies the task of checking that a given function is a $k$-graph trace.

**Lemma 3.15.** *Let $\Lambda$ be a locally-convex row-finite $k$-graph. Suppose that $g : \Lambda^0 \to \mathbf{R}^+$ satisfies $g(v) = \sum_{e \in v\Lambda^{e_i}} g(s(e))$ for all $v \in \Lambda^0$ and all $1 \leq i \leq k$ such that $v\Lambda^{e_i} \neq \emptyset$. Then $g$ is a $k$-graph trace.*

*Proof.* We establish (2) by induction on $\ell(n) = \sum_{i=1}^{k} n_i$. For any $v \in \Lambda^0$, if $\ell(n) = 0$ then $n = 0$ and $v\Lambda^{\leq n} = \{v\}$; thus the right-hand side of (2) collapses to $g(v)$, so (2) holds trivially.

Now suppose as an inductive hypothesis that (2) holds for $\ell(n) \leq L$, and fix $v \in \Lambda^0$ and $n \in \mathbf{N}^k$ with $\ell(n) = L+1$. Since $\ell(n) = L+1 \geq 1$, there is some $1 \leq i \leq k$ such that $n_i \neq 0$. By the inductive hypothesis, we have that

$$g(v) = \sum_{\lambda \in v\Lambda^{\leq n - e_i}} g(s(\lambda)). \tag{8}$$

For each $\lambda \in \Lambda$, there are two possibilities: either (a) $s(\lambda)\Lambda^{e_i}$ is empty so that $s(\lambda)\Lambda^{\leq e_i} = \{s(\lambda)\}$; or (b) $s(\lambda)\Lambda^{e_i}$ is nonempty so that $s(\lambda)\Lambda^{\leq e_i} = s(\lambda)\Lambda^{e_i}$. In case (a), we have $g(s(\lambda)) = \sum_{e \in s(\lambda)\Lambda^{\leq e_i}} g(s(e))$ by our basis case; in case (b), we have $g(s(\lambda)) = \sum_{e \in s(\lambda)\Lambda^{e_i}} g(s(e))$ by assumption on $g$. So for each $\lambda \in v\Lambda^{\leq n}$ we have

$$g(s(\lambda)) = \sum_{\alpha \in s(\lambda)\Lambda^{\leq e_i}} g(s(\alpha)). \tag{9}$$

By [RSY, Lemma 3.6], composition of morphisms is a bijective map from $\{(\lambda, e) : \lambda \in v\Lambda^{\leq n - e_i}, e \in \Lambda^{\leq e_i}, s(\lambda) = r(e)\}$ to $v\Lambda^{\leq n}$. Using this in the first equality, we calculate:

$$\begin{aligned}
\sum_{\mu \in v\Lambda^{\leq n}} g(s(\mu)) &= \sum_{\lambda \in v\Lambda^{\leq n-e_i}} \sum_{\alpha \in s(\lambda)\Lambda^{\leq e_i}} g(s(\lambda\alpha)) \\
&= \sum_{\lambda \in v\Lambda^{\leq n-e_i}} \sum_{\alpha \in s(\lambda)\Lambda^{\leq e_i}} g(s(\alpha)) \\
&= \sum_{\lambda \in v\Lambda^{\leq n-e_i}} g(s(\lambda)) \quad \text{by (9) applied to each } \lambda \\
&= g(v) \quad \text{by (8).}
\end{aligned}$$

□

3.1. **The $C^*$-algebras of $k$-graphs which admit $k$-graph traces.** In this subsection we give some structural results and $K$-theory calculations for $C^*(\Lambda)$ when $\Lambda$ is a $k$-graph which satisfies the hypotheses of Proposition 3.13.

**Proposition 3.16.** *Let $\Lambda$ be a $k$-graph, and suppose that the boundary path $x : \Omega_{k,m} \to \Lambda$ is surjective. Let $v$ denote the vertex $x(0) \in \Lambda^0$. Then*

(a) *the collection $G := \{p - q : p, q \leq d(x), x(p) = x(q)\}$ is a subgroup of $\mathbf{Z}^k$;*
(b) *the projection $p_v$ is full in $C^*(\Lambda)$; and*
(c) *there is an isomorphism $\phi$ of the full corner $p_v C^*(\Lambda) p_v$ onto the subalgebra $C^*(\{L_n : n \in G\}) \subset C^*(\mathbf{Z}^k)$ which satisfies $\phi(s_{x(0,p)} s^*_{x(0,q)}) = L_{p-q}$ whenever $x(p) = x(q)$.*

*In particular, $C^*(\Lambda)$ is Morita equivalent to $C^*(G)$ which is isomorphic to $C(\mathbf{T}^l)$ for some $0 \leq l \leq k$.*

*Proof.* (a) If $n \in G$, then $n = p - q$ where $x(p) = x(q)$. But then $-n = q - p$ also belongs to $G$, and $G$ is closed under inverses. To see that it is closed under addition, suppose that $x(p) = x(q)$ and $x(p') = x(q')$, so $n = p - q$ and $n' - p' - q'$ belong to $G$. We must show that $n + n' \in G$. Since $q, p' \leq d(x)$, we have $q \vee p' \leq d(x)$. Let $\alpha := x(q, q \vee p')$ and let $\beta := x(p', q \vee p')$. Clearly $s(\alpha) = s(\beta)$. But $r(\alpha) = x(q) = x(p)$, and since $x$ is surjective, it follows that $x(0,p)\alpha = x(0, p+(q\vee p')-q)$; and similarly, we have $x(0,q')\beta = x(0, q'+(q\vee p')-p')$. Hence $x(p+(q\vee p')-q) = x(q'+(q\vee p')-p')$, and so $n+n' = p-q+p'-q' = (p+(q\vee p')-q)-(q'+(q\vee p')-p')$ belongs to $G$ as required.

(b) Since $x$ is surjective, the hereditary subset of $C^*(\Lambda)$ generated by $v$ as in [RSY, §5] is all of $\Lambda^0$, so the ideal generated by $p_v$ is $C^*(\Lambda)$ as required.

(c) We have that $p_v C^*(\Lambda) p_v = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda,\mu \in v\Lambda, s(\lambda) = s(\mu)\} = \overline{\operatorname{span}}\{s_{x(0,p)} s_{x(0,q)}^* : x(p) = x(q)\}$ because $x$ is surjective. Suppose that $x(p) = x(q)$ and $x(p') = x(q')$, and that $p-q = p'-q'$. Then

$$(*)\quad \begin{aligned} s_{x(0,p)} s_{x(0,q)}^* &= s_{x(0,p)} \Big( \textstyle\sum_{\alpha\in x(p)\Lambda^{\leq(p\vee p')-p}} s_\alpha s_\alpha^* \Big) s_{x(0,q)}^* \quad \text{by (CK4)} \\ &= s_{x(0,p)} \big( s_{x(p,p\vee p')} s_{x(p,p\vee p')}^* \big) s_{x(0,q)}^* \quad \text{as } x \text{ is surjective} \\ &= s_{x(0,p\vee p')} s_{x(0,q+(p\vee p')-p)}^* \\ &= s_{x(0,p')} \big( s_{x(p',p\vee p')} s_{x(p',p\vee p')}^* \big) s_{x(0,(q+(p\vee p')-p)-((p\vee p')-p'))} \\ &= s_{x(0,p')} s_{x(0,(q-p+p'))} \quad \text{as in } (*) \\ &= s_{x(0,p')} s_{x(0,q')}^* \quad \text{since } p-q = p'-q'. \end{aligned}$$

We can therefore sensibly define operators $\{U_n : n \in G\} \subset C^*(\Lambda)$ by $U_n := s_{x(0,p)} s_{x(0,q)}^*$ for any $p,q$ such that $x(p) = x(q)$ and $p-q = n$.

We claim that the $U_n$ are unitaries. To see this, fix $n \in G$ and fix $p,q \leq d(x)$ with $x(p) = x(q)$ and $p-q = n$. Then

$$U_n U_n^* = (s_{x(0,p)} s_{x(0,q)}^*)(s_{x(0,p)} s_{x(0,q)}^*)^* = s_{x(0,p)} s_{x(0,q)}^* s_{x(0,q)} s_{x(0,p)}^* = s_{x(0,p)} s_{x(0,p)}^* = p_v$$

because $x$ is surjective. Reversing the roles of $p$ and $q$ gives $U_n^* U_n = p_v$ as well.

Next we claim that $n \mapsto U_n$ is a representation of $G$. We must show that $U_{-n} = U_n^*$ and $U_n U_{n'} = U_{n+n'}$ for $n, n' \in G$. For the first of these, fix $n \in G$ and $p,q$ with $x(p) = x(q)$, and $p-q = n$. Then

$$U_{-n} = U_{q-p} = s_{x(0,q)} s_{x(0,p)}^* = (s_{x(0,p)} s_{x(0,q)}^*)^* = U_n^*.$$

For the second, first note that since $x$ is surjective,

(i) if $x(a) = x(b)$, then $\Lambda^{\min}(x(0,a), x(0,b)) = \{(x(a, a\vee b), x(b, a\vee b))\}$, and it follows from [RSY, Proposition 3.5] that $s_{x(0,a)}^* s_{x(0,b)} = s_{x(a,a\vee b)} s_{x(b,a\vee b)}^*$.

(ii) if $a,b,c \leq d(x)$ with $b \leq c$ and $x(a) = x(b)$, then $x(0,a)x(b,c) = x(0, a+c-b)$.

Now fix $n, n' \in G$, and $p, q, p', q'$ such that $x(p) = x(q)$, $x(p') = x(q')$, $p - q = n$ and $p' - q' = n'$, and calculate:

$$\begin{aligned} U_n U_{n'} &= s_{x(0,p)} s^*_{x(0,q)} s_{x(0,p')} s^*_{x(0,q')} \\ &= s_{x(0,p)} s_{x(q,q\vee p')} s^*_{x(p',q\vee p')} s^*_{x(0,q')} \quad \text{by (i)} \\ &= s_{x(0,p+(q\vee p')-q)} s^*_{x(0,q'+(q\vee p')-p')} \quad \text{by (ii)} \\ &= U_{(p+(q\vee p')-q)-(q'+(q\vee p')-p')} \\ &= U_{n+n'} \end{aligned}$$

as required. So $n \mapsto U_n$ is a representation of $G$ as claimed, and it follows that there is a surjective $C^*$-homomorphism $\phi : C^*(G) \to C^*(\{U_n : n \in G\}) = p_v C^*(\Lambda) p_v$ which satisfies $\phi(\chi_n) = U_n$ for all $n \in G$.

It remains only to show that $\phi$ is injective. For this, we need only show that $U_m \neq U_n$ for $m \neq n$ and that each $U_n$ where $n \neq 0$ has full spectrum. First fix $m \neq n \in \mathbf{Z}^k$. Fix any $z \in \mathbf{T}^k$ such that $z^{m-n} \neq 1$, and note that

$$\gamma_z(U_m U_n^*) = z^{m-n} U_m U_n^* \neq U_m U_n^*,$$

which implies that $U_m U_n^* \neq p_v$, so $U_m \neq U_n$. Now fix $n \in G \setminus \{0\}$ and $w \in \mathbf{T}$. We know that $\sigma(U_n)$ is nonempty, and since $U_n$ is a unitary, its spectrum is contained in $\mathbf{T}$. Fix $z \in \sigma(U_n)$. Since $z\overline{w} \in \mathbf{T}$, and since $n \neq 0$ we may choose $z' \in \mathbf{T}^k$ such that $(z')^n = z\overline{w}$. But now

$$\begin{aligned} z \in \sigma(U_n) &\implies s_{x(0,p)} s^*_{x(0,q)} - z p_v \notin (p_v C^*(\Lambda) p_v)^{-1} \\ &\implies \gamma_{z'}(s_{x(0,p)} s^*_{x(0,q)} - z p_v) \notin (p_v C^*(\Lambda) p_v)^{-1} \\ &\implies z\overline{w}(s_{x(0,p)} s^*_{x(0,q)} - w p_v) \notin (p_v C^*(\Lambda) p_v)^{-1} \\ &\implies w \in \sigma(U_n). \end{aligned}$$

Since $w \in \mathbf{T}$ was arbitrary, it follows that $\sigma(U_n) = \mathbf{T}$ as required. This establishes (c).

For the final statement of the theorem, observe that the Morita equivalence of $C^*(\Lambda)$ with $C^*(G)$ follows immediately from (2) and (3), and since $G$ is a subgroup of $\mathbf{Z}^k$, we must have $G \cong \mathbf{Z}^l$ for some $0 \leq l \leq k$. □

**For the remainder of the section, $\Lambda$ will be a fixed $k$-graph which satisfies the hypotheses of Proposition 3.13.**

A vertex $v \in \Lambda^0$ lies on an end of $\Lambda$ if and only if $|v\Lambda^{\leq n}| = 1$ for all $n \in \mathbf{N}^k$. Let $\mathrm{Ends}(\Lambda)^0$ denote the collection of all such vertices, and enumerate it $\mathrm{Ends}(\Lambda) = \{v_1, v_2, \dots\}$. Since an end of $\Lambda$ is completely determined by its initial vertex, this gives an enumeration $\{x_i : i \in \mathbf{N}\}$ of $\mathrm{Ends}(\Lambda)$ itself. Select a subsequence $(j_i)_{i=1}^\infty$ of $\mathbf{N}$ inductively as follows:

$$j_1 = 1 \quad \text{and} \quad j_{n+1} = \min\{i \in \mathbf{N} : x_i \not\sim x_{j_l} \text{ for } 1 \leq l \leq n\}.$$

Then $V := \{v_{j_l} : l \in \mathbf{N}\} \subset \Lambda^0$ has the property that for each $x \in \mathrm{Ends}(\Lambda)$ there is a unique $v \in V$ such that $x \sim x(v)$. That is $\{x(v) : v \in V\}$ is a complete set of representatives for the equivalence classes under $\sim$. We fix this collection for the remainder of the section.

**Proposition 3.17.** *For each $v \in V$, let $\Lambda(v)$ be the image of $x(v)$ which is a subcategory of $\Lambda$. Then each $(\Lambda(v), d|_{\Lambda(v)})$ is itself a $k$-graph, and $C^*(\Lambda)$ is Morita equivalent to $\bigoplus_{v\in V} C^*(\Lambda(v))$.*

*Proof.* Since each $x(v)$ is an end, each $\Lambda(v)^0$ is a hereditary subset of $\Lambda^0$. For distinct $v, w \in V$, we have $\Lambda(v) \cap \Lambda(w) = \emptyset$ because otherwise $x(v) \sim x(w)$ contradicting our choice of $V$. By [RSY, Theorem 5.2], for each $v \in \Lambda^0$, the projection $P_v := \sum_{w\in\Lambda(v)^0} p_w$ determines an ideal $I_v := C^*(\Lambda)P_vC^*(\Lambda)$ which is Morita equivalent to $C^*(\Lambda(v)^0\Lambda) = C^*(\Lambda(v))$. Each $I_v = \overline{\text{span}}\{s_\lambda s_\mu^* : s(\lambda) = s(\mu) \in \Lambda(v)^0\}$, and since the distinct $\Lambda(v)$ do not intersect, it follows that, for distinct $v, w \in V$, we have $I_vI_w = \{0\}$, so that the ideal generated by all the $P_v$ is isomorphic to $\bigoplus_{v\in V} I_v$.

If $x$ is an end, and some vertex of $x$, say $x(n)$ lies in $\Lambda(v)^0$, then since $x(0)\Lambda^n = \{x(0, n)\}$, we have that $x(0)$ belongs to the saturation of $\bigcup_{v\in V} \Lambda(v)^0$. By assumption on $\Lambda$, for each vertex $w \in \Lambda^0$ there is an element $n_w \in \mathbf{N}^k$ such that $s(w\Lambda^{\le n_w}) \subset \text{Ends}(\Lambda)^0$, and it follows that every vertex of $\Lambda$ belongs to the saturation of the set $\bigcup_{v\in V} \Lambda(v)^0$. Another application of [RSY, Theorem 5.2] shows that the ideal generated by all the $I_v$ is all of $C^*(\Lambda)$. It follows from the previous paragraph that $C^*(\Lambda) = \bigoplus I_v$, which is Morita equivalent to $\bigoplus C^*(\Lambda(v))$ as required. □

**Corollary 3.18.** *For each $v \in V$, let $G_v := \{p - q : x(v)(p) = x(v)(q)\} \subset \mathbf{Z}^k$. Then $C^*(\Lambda)$ is Morita equivalent to $\bigoplus_{v\in V} C^*(G_v) \cong \bigoplus_{v\in V} C(\mathbf{T}^{l_v})$ where $0 \le l_v \le k$ for each $v$. In particular $K_*(C^*(\Lambda))$ is isomorphic to $\bigoplus_{v\in V} K_*(C(\mathbf{T}^{l_v}))$.*

Whilst all of the general results on semifinite spectral triples in this paper apply to any $k$-graph $C^*$-algebra with a faithful gauge invariant trace, those algebras satisfying the extra conditions of Proposition 3.13 have a particularly simple form, and computations with them are more tractable.

## 4. Semifinite Spectral Triples

We begin with some semifinite versions of standard definitions and results. Let $\tau$ be a fixed faithful, normal, semifinite trace on the von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_\mathcal{N}$ be the $\tau$-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$).

**Definition 4.1.** *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that*

*1) $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$*

*2) $a(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_\mathcal{N}$ for all $\lambda \notin \mathbf{R}$ and all $a \in \mathcal{A}$*

*3) The triple is said to be even if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$, and $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$. Otherwise it is odd.*

**Definition 4.2.** *A semifinite spectral triple* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *is* $QC^k$ *for* $k \geq 1$ *(Q for quantum) if for all* $a \in \mathcal{A}$ *the operators* $a$ *and* $[\mathcal{D},a]$ *are in the domain of* $\delta^k$*, where* $\delta(T) = [|\mathcal{D}|,T]$ *is the partial derivation on* $\mathcal{N}$ *defined by* $|\mathcal{D}|$*. We say that* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *is* $QC^\infty$ *if it is* $QC^k$ *for all* $k \geq 1$*.*

**Note**. The notation is meant to be analogous to the classical case, but we introduce the $Q$ so that there is no confusion between quantum differentiability of $a \in \mathcal{A}$ and classical differentiability of functions.

**Remarks concerning derivations and commutators**. By partial derivation we mean that $\delta$ is defined on some subalgebra of $\mathcal{N}$ which need not be (weakly) dense in $\mathcal{N}$. More precisely, dom $\delta = \{T \in \mathcal{N} : \delta(T) \text{ is bounded}\}$. We also note that if $T \in \mathcal{N}$, one can show that $[|\mathcal{D}|,T]$ is bounded if and only if $[(1+\mathcal{D}^2)^{1/2},T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1+\mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{N}$. In fact, writing $|\mathcal{D}|_1 = (1+\mathcal{D}^2)^{1/2}$ and $\delta_1(T) = [|\mathcal{D}|_1,T]$ we have

$$\text{dom } \delta^n = \text{dom } \delta_1^n \quad \text{for all } n.$$

We also observe that if $T \in \mathcal{N}$ and $[\mathcal{D},T]$ is bounded, then $[\mathcal{D},T] \in \mathcal{N}$. Similar comments apply to $[|\mathcal{D}|,T]$, $[(1+\mathcal{D}^2)^{1/2},T]$. The proofs of these statements can be found in [CPRS2].

**Definition 4.3.** *A* $*$*-algebra* $\mathcal{A}$ *is smooth if it is Fréchet and* $*$*-isomorphic to a proper dense subalgebra* $i(\mathcal{A})$ *of a* $C^*$*-algebra* $A$ *which is stable under the holomorphic functional calculus.*

Thus saying that $\mathcal{A}$ is *smooth* means that $\mathcal{A}$ is Fréchet and a pre-$C^*$-algebra. Asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of $A$ immediately implies that the Fréchet topology of $\mathcal{A}$ is finer than the $C^*$-topology of $A$ (since Fréchet means locally convex, metrizable and complete.) We will sometimes speak of $\overline{\mathcal{A}} = A$, particularly when $\mathcal{A}$ is represented on Hilbert space and the norm closure $\overline{\mathcal{A}}$ is unambiguous. At other times we regard $i : \mathcal{A} \hookrightarrow A$ as an embedding of $\mathcal{A}$ in a $C^*$-algebra. We will use both points of view.

It has been shown that if $\mathcal{A}$ is smooth in $A$ then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [GVF, S]. This ensures that the $K$-theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map $i$. This definition ensures that a smooth algebra is a 'good' algebra, [GVF], so these algebras have a sensible spectral theory which agrees with that defined using the $C^*$-closure, and the group of invertibles is open.

Stability under the holomorphic functional calculus extends to nonunital algebras, since the spectrum of an element in a nonunital algebra is defined to be the spectrum of this element in the 'one-point' unitization, though we must of course restrict to functions satisfying $f(0) = 0$. Likewise, the definition of a Fréchet algebra does not require a unit. The point of contact between smooth algebras and $QC^\infty$ spectral triples is the following Lemma, proved in [R1].

**Lemma 4.4.** *If* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *is a* $QC^\infty$ *spectral triple, then* $(\mathcal{A}_\delta,\mathcal{H},\mathcal{D})$ *is also a* $QC^\infty$ *spectral triple, where* $\mathcal{A}_\delta$ *is the completion of* $\mathcal{A}$ *in the locally convex topology determined by the seminorms*

$$q_{n,i}(a) = \|\delta^n d^i(a)\|, \quad n \geq 0, \ i = 0, 1,$$

*where* $d(a) = [\mathcal{D},a]$*. Moreover,* $\mathcal{A}_\delta$ *is a smooth algebra.*

We call the topology on $\mathcal{A}$ determined by the seminorms $q_{ni}$ of Lemma 4.4 the $\delta$-topology.

Whilst smoothness does not depend on whether $\mathcal{A}$ is unital or not, many analytical problems arise because of the lack of a unit. As in [R1, R2, GGISV], we make two definitions to address these issues.

**Definition 4.5.** *An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}$, there exists $\phi \in \mathcal{A}$ such that for each $i$*

$$\phi a_i = a_i \phi = a_i.$$

**Definition 4.6.** *Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{A}_c \subseteq \mathcal{A}$ be a dense subalgebra with local units. Then we call $\mathcal{A}$ a quasi-local algebra (when $\mathcal{A}_c$ is understood.) If $\mathcal{A}_c$ is a dense ideal with local units, we call $\mathcal{A}_c \subset \mathcal{A}$ local.*

Quasi-local algebras have an approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n$, [R1].

**Example** For a $k$-graph $C^*$-algebra $A = C^*(\Lambda)$, Equation (1) shows that

$$A_c = \text{span}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } s(\mu) = s(\nu)\}$$

is a dense subalgebra. It has local units because

$$p_v S_\mu S_\nu^* = \begin{cases} S_\mu S_\nu^* & v = r(\mu) \\ 0 & \text{otherwise} \end{cases}.$$

Similar comments apply to right multiplication by $p_{r(\nu)}$. By summing the source and range projections (without repetitions) of all $S_{\mu_i} S_{\nu_i}^*$ appearing in a finite sum

$$a = \sum_i c_{\mu_i,\nu_i} S_{\mu_i} S_{\nu_i}^*$$

we obtain a local unit for $a \in A_c$. By repeating this process for any finite collection of such $a \in A_c$ we see that $A_c$ has local units.

We also require that when we have a spectral triple the operator $\mathcal{D}$ is compatible with the quasi-local structure of the algebra, in the following sense.

**Definition 4.7.** *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, then we define $\Omega_\mathcal{D}^*(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.*

**Definition 4.8.** *A local spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with $\mathcal{A}$ quasi-local such that there exists an approximate unit $\{\phi_n\} \subset \mathcal{A}_c$ for $\mathcal{A}$ satisfying*

$$\Omega_\mathcal{D}^*(\mathcal{A}_c) = \bigcup_n \Omega_\mathcal{D}^*(\mathcal{A})_n,$$

$$\Omega_\mathcal{D}^*(\mathcal{A})_n = \{\omega \in \Omega_\mathcal{D}^*(\mathcal{A}) : \phi_n \omega = \omega \phi_n = \omega\}.$$

**Remark** A local spectral triple has a local approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n\phi_{n+1} = \phi_n$ and $\phi_{n+1}[\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n]\phi_{n+1} = [\mathcal{D}, \phi_n]$. This is the crucial property we require to prove our summability results for nonunital spectral triples, to which we now turn.

4.1. **Summability.** In the following, let $\mathcal{N}$ be a semifinite von Neumann algebra with faithful normal trace $\tau$. Recall from [FK] that if $S \in \mathcal{N}$, the $t^{\text{th}}$ *generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{\|SE\| \ : \ E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1-E) \le t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N})$ consists of those operators $T \in \mathcal{N}$ such that $\|T\|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will simply write $\mathcal{L}^1$ for this ideal in order to simplify the notation, and denote the norm on $\mathcal{L}^1$ by $\|\cdot\|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1$ if $\|T\|_1 := \int_0^\infty \mu_t(T)dt < \infty$.

Note that in the case where $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1$ is not complete in this norm but it is complete in the norm $\|\cdot\|_1 + \|\cdot\|_\infty$. (where $\|\cdot\|_\infty$ is the uniform norm). Another important ideal for us is the domain of the Dixmier trace:

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \ : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds < \infty \right\}.$$

There are related ideals for $p > 1$: to describe them first set

$$\psi_p(t) = \begin{cases} t & \text{for } 0 \le t \le 1 \\ t^{1-\frac{1}{p}} & \text{for } 1 \le t. \end{cases}$$

Then define

$$\mathcal{L}^{(p,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \ : \|T\|_{\mathcal{L}^{(p,\infty)}} := \sup_{t>0} \frac{1}{\psi_p(t)} \int_0^t \mu_s(T)ds < \infty \right\}.$$

For $p > 1$ there is also the equivalent definition

$$\mathcal{L}^{(p,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \ : \sup_{t>0} \frac{t}{\psi_p(t)} \mu_t(T) < \infty \right\}.$$

If $T \in \mathcal{L}^{(p,\infty)}(\mathcal{N})$, then $T^p \in \mathcal{L}^{(1,\infty)}(\mathcal{N})$.

We will suppress the $(\mathcal{N})$ in our notation for these ideals, as $\mathcal{N}$ will always be clear from context. The reader should note that $\mathcal{L}^{(1,\infty)}$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of $\tau$-measurable operators affiliated to $\mathcal{N}$. Our notation is however consistent with that of [C] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of $\tau$-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that

$$\mu_\infty(T) := \lim_{t\to\infty} \mu_t(T) = 0.$$

**Definition 4.9.** *A semifinite local spectral triple is* $(k, \infty)$*-summable if*

$$a(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(k,\infty)} \quad \textit{for all } a \in \mathcal{A}_c, \quad \lambda \in \mathbf{C} \setminus \mathbf{R}.$$

**Remark** If $\mathcal{A}$ is unital, $\ker \mathcal{D}$ is $\tau$-finite dimensional. Note that the summability requirements are only for $a \in \mathcal{A}_c$. We do not assume that elements of the algebra $\mathcal{A}$ are all integrable in the nonunital case. Strictly speaking, this definition describes *local* $(k, \infty)$-summability, however we use the terminology $(k, \infty)$-summable to be consistent with the unital case.

We need to briefly discuss the Dixmier trace, but fortunately we will usually be applying it in reasonably simple situations. For more information on semifinite Dixmier traces, see [CPS2]. For $T \in \mathcal{L}^{(1,\infty)}$, $T \geq 0$, the function

$$F_T : t \mapsto \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds$$

is bounded. For certain generalised limits $\omega \in L^\infty(\mathbf{R}_*^+)^*$, we obtain a positive functional on $\mathcal{L}^{(1,\infty)}$ by setting

$$\tau_\omega(T) = \omega(F_T).$$

This is the Dixmier trace associated to the semifinite normal trace $\tau$, denoted $\tau_\omega$, and we extend it to all of $\mathcal{L}^{(1,\infty)}$ by linearity, where of course it is a trace. The Dixmier trace $\tau_\omega$ is defined on the ideal $\mathcal{L}^{(1,\infty)}$, and vanishes on the ideal of trace class operators. Whenever the function $F_T$ has a limit at infinity, all Dixmier traces return the value of the limit. We denote the common value of all Dixmier traces on measurable operators by $⨍$. So if $T \in \mathcal{L}^{(1,\infty)}$ is measurable, for any allowed functional $\omega \in L^\infty(\mathbf{R}_*^+)^*$ we have

$$\tau_\omega(T) = \omega(F_T) = ⨍ T.$$

**Example** The Dirac operator on the $k$-torus. Let $\gamma^j$, $j = 1, \ldots, k$, be generators of the Clifford algebra of $\mathbf{R}^k$ with the usual Euclidean inner product. Form the Dirac operator on spinors $\mathcal{D} = \sum_{j=1}^k \gamma^j \frac{\partial}{\partial \theta^j}$, which acts on $L^2(\mathbf{T}^k) \otimes \mathbf{C}^{2^{[k/2]}}$, and for $n \in \mathbf{Z}^k$, let $n^2 \in \mathbf{N}$ denote the sum $n^2 = \sum_{i=1}^k n_i^2$ of the squares of the coordinates of $n$. Then it is well known that the spectrum of $\mathcal{D}^2$ consists of eigenvalues $\{n^2 \in \mathbf{N}\}$, where each $n \in \mathbf{Z}^k$ is counted once. A careful calculation taking account of the multiplicities, [La], shows that using the standard operator trace, the function $F_{(1+\mathcal{D}^2)^{-k/2}}$ is

$$\frac{1}{\log(|\{n : |n| \leq N\}|)} \sum_{|n|=0}^{N} (1+n^2)^{-k/2} = \frac{2^{[k/2]} vol(S^{k-1})}{k \log N} \sum_{m=0}^{N} (1+m^2)^{-1/2} + o(1)$$

and this is bounded. Hence $(1+\mathcal{D}^2)^{-k/2} \in \mathcal{L}^{(1,\infty)}$ and

$$\text{Trace}_\omega((1+\mathcal{D}^2)^{-k/2}) = ⨍ (1+\mathcal{D}^2)^{-k/2} = \frac{2^{[k/2]} vol(S^{k-1})}{k} = \frac{2^{[k/2]} vol(S^{k-1})}{(2\pi)^k k} vol(\mathbf{T}^k).$$

Numerous properties of local algebras are established in [R1, R2]. The introduction of quasi-local algebras in [GGISV] led to a review of the validity of many of these results for quasi-local algebras. Most of the summability results of [R2] are valid in the quasi-local setting. In addition, the summability results of [R2] are also valid for general semifinite spectral triples since they rely only on properties of the ideals $\mathcal{L}^{(p,\infty)}$, $p \geq 1$, [C, CPS2], and the trace property. We quote the version of the summability results from [R2] that we require below.

**Proposition 4.10** ([R2]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $QC^\infty$, local $(k,\infty)$-summable semifinite spectral triple. Let $T \in \mathcal{N}$ satisfy $T\phi = \phi T = T$ for some $\phi \in \mathcal{A}_c$. Then*

$$T(1+\mathcal{D}^2)^{-k/2} \in \mathcal{L}^{(1,\infty)}.$$

*For $Re(s) > k$, $T(1+\mathcal{D}^2)^{-s/2}$ is trace class. If the limit*

$$\lim_{s\to k/2^+}(s-k/2)\tau(T(1+\mathcal{D}^2)^{-s}) \tag{10}$$

*exists, then it is equal to*

$$\frac{k}{2}⨍ T(1+\mathcal{D}^2)^{-k/2}.$$

*In addition, for any Dixmier trace $\tau_\omega$, the function*

$$a \mapsto \tau_\omega(a(1+\mathcal{D}^2)^{-k/2})$$

*defines a trace on $\mathcal{A}_c \subset \mathcal{A}$.*

## 5. Constructing a $C^*$-module and a Kasparov module

Let $A = C^*(\Lambda)$ where $\Lambda$ is a locally finite locally convex $k$-graph. Let $F = C^*(\Lambda)^\gamma$ be the fixed point subalgebra for the gauge action. Finally, let $A_c = \text{span}\{s_\mu s_\nu^* : \mu,\nu \in \Lambda\}$ and let $F_c = \text{span}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\} = F \cap A_c$ so that $A$ and $F$ are the $C^*$-completions of $A_c$ and $F_c$. Note that the expectation $\Phi : A \to F$ outlined at the end of Section 2 restricts to an expectation, also denoted $\Phi$ of $A_c$ onto $F_c$.

We make $A^{2^{[k/2]}} = \mathbf{C}^{2^{[k/2]}} \otimes A$ a right inner product-$F$-module. The right action of $F$ on $A$ is by right multiplication. The inner product is defined by

$$(x|y)_R := \sum_{j=1}^{2^{[k/2]}} \Phi(x_j^* y_j) \in F.$$

It is simple to check the requirements that $(\cdot|\cdot)_R$ defines an $F$-valued inner product on $A^{2^{[k/2]}}$. The requirement $(x|x)_R = 0 \Rightarrow x = 0$ follows from the faithfulness of $\Phi$.

**Definition 5.1.** *Define $X$ to be the completion of $A^{2^{[k/2]}}$ to a $C^*$-module over $F$ for the $C^*$-module norm*

$$\|x\|_X^2 := \|(x|x)_R\|_A = \|(x|x)_R\|_F = \|\sum_{i=1}^{2^{[k/2]}} \Phi(x_i^* x_i)\|_F.$$

*Define $X_c$ to be the pre-$C^*$-module over $F_c$ with linear space $A_c^{2^{[k/2]}}$ and the inner product $(\cdot|\cdot)_R$.*

**Remark** Typically, the action of $F$ does not map $X_c$ to itself, so we may only consider $X_c$ as an $F_c$ module. This is a reflection of the fact that $F_c$ and $A_c$ are quasilocal not local.

**Remark** Frequently we will define an operator $T$ on the $F$ module $A$, and implicitly extend $T$ to $X$ by $\text{id}_{2^{[k/2]}} \otimes T$, where $\text{id}_{2^{[k/2]}}$ is the identity operator in the matrix algebra $M_{2^{[k/2]}}(\mathbf{C})$.

**Remark** There is an irreducible representation $\gamma$ of the complex Clifford algebra $\mathbf{C}\mathit{liff}_k = \mathit{Cliff}(\mathbf{C}^k)$ on $X$ as adjointable operators. We employ the convention that

$$\gamma^l\gamma^j + \gamma^j\gamma^l := \gamma(e^l)\gamma(e^j) + \gamma(e^j)\gamma(e^l) = -2\delta^{lj} Id_{2^{[k/2]}}.$$

When $k$ is even the operator $\omega_{\mathbf{C}} := i^{[(k+1)/2]}\gamma^1\cdots\gamma^k$ is self-adjoint, has $\omega_{\mathbf{C}}^2 = Id_{2^{[k/2]}}$ and $\gamma^j\omega_{\mathbf{C}} = -\omega_{\mathbf{C}}\gamma^j$ for $j = 1,\ldots,k$. When $k$ is odd, $\omega_{\mathbf{C}}$ is central in the Clifford algebra, and we choose the representation with $\omega_{\mathbf{C}} = 1$.

The inclusion map $\iota : A \to X$ is continuous since

$$\|a\|_X^2 = \|\Phi(a^*a)\|_F \le \|a^*a\|_A = \|a\|_A^2.$$

We can also define the gauge action $\gamma$ on $A \subset X$, and as

$$\begin{aligned}\|\gamma_z(a)\|_X^2 &= \|\Phi((\gamma_z(a))^*(\gamma_z(a)))\|_F = \|\Phi(\gamma_z(a^*)\gamma_z(a))\|_F \\ &= \|\Phi(\gamma_z(a^*a))\|_F = \|\Phi(a^*a)\|_F = \|a\|_X^2,\end{aligned}$$

for each $z \in \mathbf{T}^k$, the action of $\gamma_z$ is isometric on $A \subset X$ and so extends to a unitary $U_z$ on $X$. This unitary is $F$-linear and adjointable, and we obtain a strongly continuous action of $\mathbf{T}^k$ on $X$, which we still denote by $\gamma$.

For each $n \in \mathbf{Z}^k$, define an operator $\Phi_n$ on $X$ by

$$\Phi_n(x) = \frac{1}{(2\pi)^k}\int_{\mathbf{T}^k} z^{-n}\gamma_z(x)d^k\theta, \quad z_j = e^{i\theta_j}, \quad x \in X.$$

Observe that on generators we have

$$\Phi_n(S_\alpha S_\beta^*) = \begin{cases} S_\alpha S_\beta^* & d(\alpha) - d(\beta) = n \\ 0 & d(\alpha) - d(\beta) \neq n \end{cases}. \tag{11}$$

**Remark** If $(\Lambda, d)$ is a finite $k$-graph with no cycles, then for $n$ sufficiently large there are no paths of degree $n$ and so $\Phi_n = 0$. This will obviously simplify many of the convergence issues below.

The proof of the following Lemma is identical to that of [PRen, Lemma 4.2].

**Lemma 5.2.** *The operators $\Phi_n$ are adjointable endomorphisms of the $F$-module $X$ such that $\Phi_n^* = \Phi_n = \Phi_n^2$ and $\Phi_n\Phi_m = \delta_{n,m}\Phi_n$. If $K \subset \mathbf{Z}^k$ then the sum $\sum_{n\in K}\Phi_n$ converges strictly to a projection in the endomorphism algebra. The sum $\sum_{n\in\mathbf{Z}^k}\Phi_n$ converges to the identity operator on $X$.*

**Corollary 5.3.** *Let $x \in X$. Then with $x_n = \Phi_n x$ the sum $\sum_{n\in\mathbf{Z}^k} x_n$ converges in $X$ to $x$.*

5.1. **The Kasparov Module.** As we did in Section 4, for $n \in \mathbf{Z}^k$, we write $n^2 = \sum_{j=1}^k n_j^2$ and $|n| = \sqrt{n^2}$.

The theory of unbounded operators on $C^*$-modules that we require is all contained in Lance's book, [L, Chapters 9,10]. We quote the following definitions (adapted to our situation).

**Definition 5.4.** *Let $Y$ be a right $C^*$-$B$-module. A densely defined unbounded operator $\mathcal{D}$ :* dom $\mathcal{D} \subset Y \to Y$ *is a $B$-linear operator defined on a dense $B$-submodule* dom $\mathcal{D} \subset Y$. *The operator $\mathcal{D}$ is closed if the graph*

$$G(\mathcal{D}) = \{(x, \mathcal{D}x)_R : x \in \text{dom } \mathcal{D}\}$$

*is a closed submodule of $Y \oplus Y$.*

Given a densely defined unbounded operator $\mathcal{D} : \text{dom } \mathcal{D} \subset Y \to Y$, define a submodule

$$\text{dom } \mathcal{D}^* := \{y \in Y : \exists z \in Y \text{ such that } \forall x \in \text{dom } \mathcal{D}, (\mathcal{D}x|y)_R = (x|z)_R\}.$$

Then for $y \in \text{dom } \mathcal{D}^*$ define $\mathcal{D}^* y = z$. Given $y \in \text{dom } \mathcal{D}^*$, the element $z$ is unique, so $\mathcal{D}^*$ is well-defined, and moreover is closed.

**Definition 5.5.** *Let $Y$ be a right $C^*$-$B$-module. A densely defined unbounded operator $\mathcal{D}$ : $\text{dom } \mathcal{D} \subset Y \to Y$ is symmetric if for all $x, y \in \text{dom } \mathcal{D}$*

$$(\mathcal{D}x|y)_R = (x|\mathcal{D}y)_R.$$

*A symmetric operator $\mathcal{D}$ is self-adjoint if $\text{dom } \mathcal{D} = \text{dom } \mathcal{D}^*$ (and so $\mathcal{D}$ is necessarily closed). A densely defined unbounded operator $\mathcal{D}$ is regular if $\mathcal{D}$ is closed, $\mathcal{D}^*$ is densely defined, and $(1 + \mathcal{D}^*\mathcal{D})$ has dense range.*

The extra requirement of regularity is necessary in the $C^*$-module context for the continuous functional calculus, and is not automatically satisfied, [L, Chapter 9].

With these definitions in hand, we return to our $C^*$-module $X$. The proof of the following Proposition is an exact analogue of [PRen, Proposition 4.6].

**Proposition 5.6.** *Let $X$ be the right $C^*$-$F$-module of Definition 5.1. Define $X_{\mathcal{D}} \subset X$ to be the linear space*

$$X_{\mathcal{D}} = \{x = \sum_{n \in \mathbf{Z}^k} x_n : \| \sum_{n \in \mathbf{Z}^k} n^2 (x_n|x_n)_R \| < \infty\}.$$

*For $x \in X_{\mathcal{D}}$ define*

$$\mathcal{D}x = \sum_{n \in \mathbf{Z}^k} \gamma(in) x_n = i \sum_{n \in \mathbf{Z}^k} \sum_{j=1}^{k} \gamma^j n_j x_n.$$

*Then $\mathcal{D} : X_{\mathcal{D}} \to X$ is self-adjoint and regular.*

**Remark** The map $\mathcal{D}$ is Clifford multiplication by the vector $in \in \mathbf{C}^k$. Any $S_\alpha S_\beta^* \in A_c$ is in $X_{\mathcal{D}}$ and

$$\mathcal{D} S_\alpha S_\beta^* = i \sum_{j=1}^{k} \gamma^j (d(\alpha)_j - d(\beta)_j) S_\alpha S_\beta^*$$

as the reader will easily verify. Thus we have

$$\mathcal{D}^2 \Phi_n x = \sum_{j=1}^{k} n_j^2 \Phi_n x = n^2 \Phi_n x.$$

There is a continuous functional calculus for self-adjoint regular operators, [L, Theorem 10.9], and we use this to obtain spectral projections for $\mathcal{D}^2$ at the $C^*$-module level. Let $f_m \in C_c(\mathbf{R})$ be 1 in a small neighbourhood of $m \in \mathbf{Z}$ and zero on $(-\infty, m - 1/2] \cup [m + 1/2, \infty)$. Then it is clear that

$$\sum_{n \in \mathbf{Z}^k,\ n^2 = m} \Phi_n = f_m(\mathcal{D}^2).$$

The next Lemma is the first place where we need our $k$-graph to be locally finite and have no sinks. It is also the point where the generalisation from the graph case differs the most.

**Lemma 5.7.** *Assume that the $k$-graph $(\Lambda, d)$ is locally finite, locally convex and has no sinks. For all $a \in A$ and $n \in \mathbf{Z}^k$, $a\Phi_n \in End^0_F(X)$, the compact endomorphisms of the right $F$-module $X$. If $a \in A_c$ then $a\Phi_n$ is finite rank.*

*Proof.* Let $n \in \mathbf{Z}^k$ and write $n = n_1 + n_2$ with $n_1 \geq 0$ and $n_2 < 0$. We will see that the precise choice of $n_1, n_2$ is largely irrelevant. For $v \in \Lambda^0$, let $|v|_n$ denote the number of paths $\rho \in \Lambda$ with $d(\rho) = n$ and $s(\rho) = v$, i.e. $|v|_n = |\Lambda^n v|$. Since $\Lambda$ has no sinks and is locally finite, for all $n$ and all $v$ we have $0 < |v|_n < \infty$ .

Now define, for $n = n_1 + n_2$ as above,

$$T_{v,n_1,n_2} = \sum_{d(\alpha)=n_1, d(\beta)=-n_2, s(\alpha)=s(\beta), r(\alpha)=v} \frac{1}{|s(\beta)|_{-n_2}} \Theta^R_{S_\alpha S^*_\beta, S_\alpha S^*_\beta},$$

where for $x, y, z \in X$

$$\Theta^R_{x,y} z := x(y|z)_R,$$

defines a rank one operator. Observe that since $\Lambda$ is locally finite this is a finite sum of rank one operators and so finite rank. We claim that $T_{v,n_1,n_2} = p_v\Phi_n$. It suffices to prove that the difference $p_v\Phi_n - T_{v,n_1,n_2}$ vanishes on $X_c \subset X$. That is, we just need to show that $(p_v\Phi_n - T_{v,n_1,n_2})S_\mu S^*_\nu = 0$ for all $\mu, \nu$. So first we compute, with $q = d(\alpha) \vee d(\mu)$,

$$S^*_\alpha S_\mu = \sum_{\alpha\sigma=\mu\rho, \alpha\sigma \in \Lambda^q} S_\sigma S^*_\rho$$

by [RSY, Proposition 3.5 and Remarks 3.8(2)]. Next consider

$$\Phi(S_\beta S^*_\alpha S_\mu S^*_\nu) = \Phi(S_\beta \sum_{\alpha\sigma=\mu\rho, \alpha\sigma \in \Lambda^q} S_\sigma S^*_\rho S^*_\nu).$$

This is zero unless $d(\beta) + d(\sigma) - d(\rho) - d(\nu) = 0$. Now $d(\sigma) - d(\rho) = d(\mu) - d(\alpha)$ so

$$\Phi(S_\beta S^*_\alpha S_\mu S^*_\nu) = \delta_{d(\mu)-d(\nu), d(\alpha)-d(\beta)} S_\beta \sum_{\alpha\sigma=\mu\rho, \alpha\sigma \in \Lambda^q} S_\sigma S^*_\rho S^*_\nu.$$

Of course, $d(\alpha) - d(\beta) = n$. Since each $S^*_\beta S_\beta = p_{s(\beta)}$, we can perform the sum over $\beta$:

$$\begin{aligned}\sum_{\alpha,\beta} \frac{1}{|s(\beta)|_{-n_2}} p_v \Theta_{S_\alpha S^*_\beta, S_\alpha S^*_\beta} S_\mu S^*_\nu &= \sum_{\alpha,\beta} \frac{1}{|s(\beta)|_{-n_2}} \delta_{d(\mu)-d(\nu),n} p_v S_\alpha S^*_\beta S_\beta \sum_{\alpha\sigma=\mu\rho, \alpha\sigma \in \Lambda^q} S_\sigma S^*_\rho S^*_\nu \\ &= \sum_\alpha \delta_{d(\mu)-d(\nu),n} p_v S_\alpha \sum_{\alpha\sigma=\mu\rho, \alpha\sigma \in \Lambda^q} S_\sigma S^*_\rho S^*_\nu \\ &= \sum_\alpha \delta_{d(\mu)-d(\nu),n} p_v \sum_{\alpha\sigma=\mu\rho, \alpha\sigma \in \Lambda^q} S_\mu S_\rho S^*_\rho S^*_\nu.\end{aligned}$$

If we suppose that a given $\alpha$ has no common extensions with $\mu$, then this particular term in the sum contributes zero. Summing over all $\alpha$ (of fixed length $n_1$) with common extensions with $\mu$

yields

$$\sum_\alpha \delta_{d(\mu)-d(\nu),n} p_v \sum_{\alpha\sigma=\mu\rho,\alpha\sigma\in\Lambda^q} S_\mu S_\rho S_\rho^* S_\nu^* = p_v \sum_{\rho\in\Lambda^{q-d(\mu)},r(\rho)=s(\mu)} S_\mu S_\rho S_\rho^* S_\nu^* = S_\mu S_\nu^*.$$

Hence we conclude that

$$\begin{aligned}\sum_{\alpha,\beta} \frac{1}{|s(\beta)|_{-n_2}} p_v \Theta_{S_\alpha S_\beta^*, S_\alpha S_\beta^*} S_\mu S_\nu^* &= \delta_{d(\mu)-d(\nu),n} p_v S_\mu S_\nu^* \\ &= p_v \Phi_n S_\mu S_\nu^*.\end{aligned}$$

As $\mu, \nu$ were arbitrary paths, this shows that $p_v\Phi_n$ is a finite rank endomorphism. For arbitrary $a = \sum c_j S_{\mu_j} S_{\nu_j}^*$, where the sum is finite, we may apply the same reasoning to each $p_{s(\nu_j)}$ to see that $a\Phi_n$ is finite rank for all $a \in A_c$.

To see that $a\Phi_k$ is compact for all $a \in A$, recall that every $a \in A$ is a norm limit of a sequence $\{a_i\}_{i\geq 0} \subset A_c$. Thus for any $n \in \mathbf{Z}^k$ $a\Phi_n = \lim_{i\to\infty} a_i\Phi_n$ and so is compact. □

**Lemma 5.8.** *Assume that the $k$-graph $(\Lambda, d)$ is locally finite and has no sinks. For all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of the $F$-module $X$.*

*Proof.* First let $a = p_v$ for $v \in \Lambda^0$. Then the sum

$$R_{v,N} := p_v \sum_{|n|=0}^{N} \Phi_n (1+n^2)^{-1/2}$$

is finite rank, by Lemma 5.7. We will show that the sequence $\{R_{v,N}\}_{N\geq 0}$ is convergent with respect to the operator norm $\|\cdot\|_{End}$ of endomorphisms of $X$. Indeed, assuming that $M > N$,

$$\begin{aligned}\|R_{v,N} - R_{v,M}\|_{End} &= \|p_v \sum_{|n|=N+1}^{M} \Phi_n (1+n^2)^{-1/2}\|_{End} \\ &\leq (1+(N+1)^2)^{-1/2} \to 0, \qquad (12)\end{aligned}$$

since the ranges of the $p_v\Phi_n$ are orthogonal for different $n$. Thus, using the argument from Lemma 5.7, $a(1+\mathcal{D}^2)^{-1/2} \in End_F^0(X)$ for all $a \in A_c$. Letting $\{a_i\}$ be a Cauchy sequence from $A_c$, we have

$$\|a_i(1+\mathcal{D}^2)^{-1/2} - a_j(1+\mathcal{D}^2)^{-1/2}\|_{End} \leq \|a_i - a_j\|_{End} = \|a_i - a_j\|_A \to 0,$$

since $\|(1+\mathcal{D}^2)^{-1/2}\| \leq 1$. Thus the sequence $a_i(1+\mathcal{D}^2)^{-1/2}$ is Cauchy in norm and we see that $a(1+\mathcal{D}^2)^{-1/2}$ is compact for all $a \in A$. □

**Proposition 5.9.** *Assume that the $k$-graph $(\Lambda, d)$ is locally finite and has no sinks. Let $V = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then $(X, V)$ defines a class in $KK^{k \bmod 2}(A, F)$.*

*Proof.* We refer to [K] for more information. We need to show that various operators belong to $End_F^0(X)$. First, $V - V^* = 0$, so $a(V - V^*)$ is compact for all $a \in A$. Also $a(1-V^2) = a(1+\mathcal{D}^2)^{-1}$

which is compact from Lemma 5.8 and the boundedness of $(1+\mathcal{D}^2)^{-1/2}$. Finally, we need to show that $[V,a]$ is compact for all $a\in A$. First we suppose that $a\in A_c$. Then we have

$$\begin{aligned}[V,a] &= [\mathcal{D},a](1+\mathcal{D}^2)^{-1/2}-\mathcal{D}(1+\mathcal{D}^2)^{-1/2}[(1+\mathcal{D}^2)^{1/2},a](1+\mathcal{D}^2)^{-1/2}\\ &= b_1(1+\mathcal{D}^2)^{-1/2}+Vb_2(1+\mathcal{D}^2)^{-1/2},\end{aligned}$$

where $b_1=[\mathcal{D},a]\in A_c$ and $b_2=[(1+\mathcal{D}^2)^{1/2},a]$. Provided that $b_2(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism, Lemma 5.8 will show that $[V,a]$ is compact for all $a\in A_c$. So consider the action of $[(1+\mathcal{D}^2)^{1/2},S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2}$ on $x=\sum_{n\in\mathbf{Z}^k}x_n$. We find

$$\begin{aligned}&\sum_{n\in\mathbf{Z}^k}[(1+\mathcal{D}^2)^{1/2},S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2}x_n\\ &= \sum_{n\in\mathbf{Z}^k}\left((1+(d(\mu)-d(\nu)+n)^2)^{1/2}-(1+n^2)^{1/2}\right)(1+n^2)^{-1/2}S_\mu S_\nu^* x_n\\ &= \sum_{n\in\mathbf{Z}^k}f_{\mu,\nu}(n)S_\mu S_\nu^*\Phi_n x. \qquad (13)\end{aligned}$$

The function

$$f_{\mu,\nu}(n)=\left((1+(d(\mu)-d(\nu)+n)^2)^{1/2}-(1+n^2)^{1/2}\right)(1+n^2)^{-1/2}$$

goes to zero as $n^2\to\infty$. As the $S_\mu S_\nu^*\Phi_n$ are finite rank with orthogonal ranges (for different $n$), the sum in (13) converges in the endomorphism norm, and so converges to a compact endomorphism. For general $a\in A_c$ we write $a$ as a finite linear combination of generators $S_\mu S_\nu^*$, and apply the above reasoning to each term in the sum to find that $[(1+\mathcal{D}^2)^{1/2},a]$ is a compact endomorphism for all $a\in A_c$.

Now let $a\in A$ be the norm limit of a Cauchy sequence $\{a_i\}_{i\geq 0}\subset A_c$. Then

$$\|[V,a_i-a_j]\|_{End}\leq 2\|a_i-a_j\|_{End}\to 0,$$

so the sequence $[V,a_i]$ is also Cauchy in norm, and so the limit is compact.

It is also clear from the construction that if $k$ is even, the Kasparov module is even (with grading given by $\omega_{\mathbf{C}}$) and so belongs to $KK^0(A,F)$, while when $k$ is odd, the Kasparov module belongs to $KK^1(A,F)$. □

## 6. The Gauge Spectral Triple of a $k$-Graph Algebra

In this section we will construct a semifinite spectral triple for those locally convex $k$-graph $C^*$-algebras which possess a faithful, semifinite, lower-semicontinuous, gauge invariant trace, $\tau$. Recall from Proposition 3.10 that such traces arise from faithful $k$-graph traces.

We will begin with the right $F_c$ module $X_c$. In order to deal with the spectral projections of $\mathcal{D}$ we will also assume throughout this section that $(\Lambda,d)$ is locally finite and has no sinks. This ensures, by Lemma 5.7 that for all $a\in A$ and $n\in\mathbf{Z}^k$ the endomorphisms $a\Phi_n$ of $X$ are compact endomorphisms.

We define a **C**-valued inner product on $X_c$ by

$$\langle x, y\rangle := \tau((x|y)_R) = \sum_{j=1}^{2^{[k/2]}} \tau(\Phi(x_j^* y_j)) = \sum_{j=1}^{2^{[k/2]}} \tau(x_j^* y_j).$$

Observe that this inner product is linear in the second variable. We define the Hilbert space $\mathcal{H} = L^2(X, \tau)$ to be the completion of $X_c$ in the norm coming from the inner product.

**Lemma 6.1.** *The $C^*$-algebra $A = C^*(E)$ acts on $\mathcal{H}$ by an extension of left multiplication. This defines a faithful nondegenerate $*$-representation of $A$. Moreover, any endomorphism of $X$ leaving $X_c$ invariant extends uniquely to a bounded linear operator on $\mathcal{H}$.*

*Proof.* The first statement follows from the proof of Proposition 3.10. Now let $T$ be an endomorphism of $X$ leaving $X_c$ invariant. Then [RW, Cor 2.22],

$$(Tx|Ty)_R \leq \|T\|_{End}^2 (x|y)_R$$

in the algebra $F$. Now the norm of $T$ as an operator on $\mathcal{H}$, denoted $\|T\|_\infty$, can be computed in terms of the endomorphism norm of $T$ by

$$\begin{aligned} \|T\|_\infty^2 &:= \sup_{\|x\|_{\mathcal{H}} \leq 1} \langle Tx, Tx\rangle = \sup_{\|x\|_{\mathcal{H}} \leq 1} \tau((Tx|Tx)_R) \\ &\leq \sup_{\|x\|_{\mathcal{H}} \leq 1} \|T\|_{End}^2 \tau((x|x)_R) = \|T\|_{End}^2. \end{aligned} \tag{14}$$

□

**Corollary 6.2.** *The endomorphisms $\{\Phi_n\}_{n\in\mathbf{Z}^k}$ define mutually orthogonal projections on $\mathcal{H}$. For any $K \subset \mathbf{Z}^k$ the sum $\sum_{n\in K} \Phi_n$ converges strongly to a projection in $\mathcal{B}(\mathcal{H})$. In particular, $\sum_{n\in\mathbf{Z}^k} \Phi_n = Id_{\mathcal{H}}$, and for all $x \in \mathcal{H}$ the sum $\sum_n \Phi_n x$ converges in norm to $x$.*

*Proof.* As in Lemma 5.2, we can use the continuity of the $\Phi_n$ on $\mathcal{H}$, which follows from Lemma 6.1, to see that the relation $\Phi_n\Phi_m = \delta_{n,m}\Phi_n$ extends from $X_c \subset \mathcal{H}$ to $\mathcal{H}$. The proof of the strong convergence of sums of $\Phi_n$'s is just as in Lemma 5.2 after replacing the $C^*$-module norm with the Hilbert space norm. □

**Lemma 6.3.** *The operator $\mathcal{D}$ extends to a closed unbounded self-adjoint operator on $\mathcal{H}$. The closure of the operator $\mathcal{D}|_{X_c}$ is $\mathcal{D}$.*

*Proof.* The proof is essentially the same as the $C^*$-module version, Lemma 5.6. By replacing the $C^*$-module norm and the $C^*$-Cauchy-Schwartz inequality with the Hilbert space analogues, the proof that $\mathcal{D}$ is closed goes through as before. We then define dom $\mathcal{D}$ to be the completion of $X_c$ in the norm

$$x \to \|x\|_{\mathcal{H},\mathcal{D}} := \|x\|_{\mathcal{H}} + \|\mathcal{D}x\|_{\mathcal{H}}.$$

The proofs of symmetry and self-adjointness now follow just as in the $C^*$-module case. The last statement follows from the definition of dom $\mathcal{D}$. □

The Hilbert space $\mathcal{H}$ and operator $\mathcal{D}$ are two of the ingredients of our spectral triple. We also need a $*$-algebra. In fact $A_c$ will do the job, but it also has a natural completion $\mathcal{A}$ which is useful too. To prove both these assertions we need the following lemma. The proof is the same as [PRen, Lemma 5.4].

**Lemma 6.4.** *Let $\mathcal{H}, \mathcal{D}$ be as above and let $|\mathcal{D}| = \sqrt{\mathcal{D}^*\mathcal{D}} = \sqrt{\mathcal{D}^2}$ be the absolute value of $\mathcal{D}$. Then for $S_\alpha S_\beta^* \in A_c$, the operator $[|\mathcal{D}|, S_\alpha S_\beta^*]$ is well-defined on $X_c$, and extends to a bounded operator on $\mathcal{H}$ with*

$$\|[|\mathcal{D}|, S_\alpha S_\beta^*]\|_{\mathcal{H}} \leq \Big| d(\alpha) - d(\beta) \Big|.$$

*Similarly,* $\|[\mathcal{D}, S_\alpha S_\beta^*]\|_{\mathcal{H}} = \Big| d(\alpha) - (\beta) \Big|$.

**Corollary 6.5.** *The algebra $A_c$ is contained in the smooth domain of the derivation $\delta$ where for $T \in \mathcal{B}(\mathcal{H})$, $\delta(T) = [|\mathcal{D}|, T]$. That is*

$$A_c \subseteq \bigcap_{n \geq 0} \mathrm{dom}\ \delta^n.$$

**Definition 6.6.** *Define the $*$-algebra $\mathcal{A} \subset A$ to be the completion of $A_c$ in the $\delta$-topology. By Lemma 4.4, $\mathcal{A}$ is Fréchet and stable under the holomorphic functional calculus.*

**Lemma 6.7.** *If $a \in \mathcal{A}$ then $[\mathcal{D}, a] \in \mathcal{A}$ and the operators $\delta^k(a)$, $\delta^k([\mathcal{D}, a])$ are bounded for all $k \geq 0$. If $\phi \in \mathcal{A}$ satisfies $\phi a = a = a\phi$, then $\phi[\mathcal{D}, a] = [\mathcal{D}, a] = [\mathcal{D}, a]\phi$. The norm closed algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ is $A \otimes M_{2^{[k/2]}}(\mathbf{C})$. In particular, $\mathcal{A}$ is quasi-local.*

We leave the straightforward proofs of these statements to the reader.

At this point we have most of the structure required to define a semifinite local spectral triple. The one remaining piece of information we require is the compactness of $a(\lambda - \mathcal{D})^{-1}$, $\lambda \in \mathbf{C} \setminus \mathbf{R}$, $a \in \mathcal{A}$, relative to some trace on some von Neumann algebra to which $\mathcal{D}$ is affiliated. There is a canonical choice of von Neumann algebra and trace, and for this choice $a(1 + \mathcal{D}^2)^{-k/2}$ is in the domain of the Dixmier trace for all $a \in \mathcal{A}$.

6.1. **Traces and Compactness Criteria.** We continue to assume that $(\Lambda, d)$ is a locally convex locally finite $k$-graph with no sinks and that $\tau$ is a faithful, semifinite, lower-semicontinuous, gauge invariant trace on $C^*(\Lambda)$. We will define a von Neumann algebra $\mathcal{N}$ with a faithful semifinite normal trace $\tilde{\tau}$ so that $\mathcal{A} \subset \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ and $\mathcal{H}$ are as defined in the last subsection. Moreover the operator $\mathcal{D}$ will be affiliated to $\mathcal{N}$. The aim of this subsection is to prove the following result.

**Theorem 6.8.** *Let $(\Lambda, d)$ be a locally convex locally finite $k$-graph with no sinks, and let $\tau$ be a faithful semifinite trace on $C^*(\Lambda)$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a $QC^\infty$ $(k, \infty)$-summable odd local semifinite spectral triple (relative to $(\mathcal{N}, \tilde{\tau})$). For all $a \in \mathcal{A}$, the operator $a(1 + \mathcal{D}^2)^{-1/2}$ is not trace class. Suppose that $v\Lambda^{\leq n} = v\Lambda^n$ for all $n \in \mathbf{N}^k$. Then*

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-k/2}) = C_k \tau(p_v),$$

*where $\tilde\tau_\omega$ is any Dixmier trace associated to $\tilde\tau$, and*

$$C_k = \frac{2^{[k/2]} vol(S^{k-1})}{k}$$

**Remarks 6.9.** The hypothesis that $v\Lambda^{\leq n} = v\Lambda^n$ for all $n \in \mathbf{N}^k$ is perhaps somewhat opaque. This theorem generalises [PRen, Theorem 5.8], which requires that the vertex $v$ has "no sinks downstream" to ensure that $p_v = \sum_{s(\alpha)=v, |\alpha|=n} s_\alpha s_\alpha^*$ for all $n \in \mathbf{N}$. The hypothesis that $v\Lambda^{\leq n} = v\Lambda^n$ for all $n \in \mathbf{N}^k$ has precisely the same effect (consider relation (CK4)). Indeed this is precisely the notion that the $\Lambda^{\leq n}$ notation was developed to capture: $\Lambda^{\leq n}$ is supposed to consist of all paths of degree $n$ together with all paths whose degree is less than $n$ because they originate at a source in direction $n$ [RSY].

To prove the theorem, we need some preliminary definitions and results.

**Definition 6.10.** *Let $End_F^{00}(X_c)$ denote the algebra of finite rank operators on $X_c$ acting on $\mathcal{H}$. Define $\mathcal{N} = (End_F^{00}(X_c))''$, and let $\mathcal{N}_+$ denote the positive cone in $\mathcal{N}$.*

**Definition 6.11.** *Let $T \in \mathcal{N}$. For $n \in \mathbf{N}^k$ and $v \in \Lambda^0$, let $|v|_n$ denote the number of paths of degree $n$ with source $v$. Let $\Lambda \times_s^{\min} \Lambda$ denote the set of pairs*

$$\{(\alpha, \beta) \in \Lambda : s(\alpha) = s(\beta), d(\alpha) \wedge d(\beta) = 0\}.$$

*For $(\alpha, \beta) \in \Lambda \times_s^{\min} \Lambda$, define*

$$\omega_{\alpha,\beta}(T) = \frac{1}{|s(\alpha)|_{d(\beta)}} \langle s_\alpha s_\beta^*, T s_\alpha s_\beta^* \rangle.$$

*Note that if $d(\alpha) = d(\beta) = 0$, then $\alpha = \beta = v$ for some $v \in \Lambda^0$, and since $s_v = p_v$ by convention, we have $\omega_{v,v}(T) = \langle p_v, T p_v \rangle$. Define*

$$\tilde\tau : \mathcal{N}_+ \to [0, \infty], \quad by \quad \tilde\tau(T) = \lim_{L \nearrow \Lambda \times_s^{\min} \Lambda} \sum_{(\alpha,\beta) \in L} \omega_{\alpha,\beta}(T) \tag{15}$$

*where $L$ increases over the net of finite subsets of $\Lambda \times_s^{\min} \Lambda$.*

**Remarks**

(1) For $T, S \in \mathcal{N}_+$ and $\lambda \geq 0$ we have

$$\tilde\tau(T + S) = \tilde\tau(T) + \tilde\tau(S) \quad \text{and} \quad \tilde\tau(\lambda T) = \lambda \tilde\tau(T) \quad \text{where} \quad 0 \times \infty = 0.$$

(2) Note that for $\mu \in \Lambda$, we have $\omega_{\mu,s(\mu)}(T) = \langle s_\mu, T s_\mu \rangle$ and $\omega_{s(\mu),\mu}(T) = \frac{1}{|s(\mu)|_{d(\mu)}} \langle s_\mu^*, T s_\mu^* \rangle$. Consequently, if $\Lambda$ is a 1-graph then for $\mu \in \Lambda \backslash \Lambda^0$, the map $\omega_\mu$ of [PRen, Definition 5.10] is precisely $\omega_{\mu,s(\mu)} + \omega_{s(\mu),\mu}$, while for $v \in \Lambda^0$, $\omega_v = \omega_{v,v}$. In particular, for a 1-graph, (15) is just a slightly more efficient expression for the definition of $\tilde\tau$ of [PRen, Definition 5.10].

**Proposition 6.12.** *The function $\tilde\tau : \mathcal{N}_+ \to [0, \infty]$ defines a faithful normal semifinite trace on $\mathcal{N}$. Moreover,*

$$End_F^{00}(X_c) \subset \mathcal{N}_{\tilde\tau} := \text{span}\{T \in \mathcal{N}_+ : \tilde\tau(T) < \infty\},$$

*the domain of definition of* $\tilde{\tau}$, *and*

$$\tilde{\tau}(\Theta^R_{x,y}) = \langle y, x\rangle = \tau(y^*x), \quad x, y \in X_c.$$

The proof of this important, but technical, result is extremely similar to that of [PRen, Proposition 5.11], differing only in the details of the calculations establishing the analogue of [PRen, Equation (18)] and showing that $\tilde{\tau}(\Theta^R_{x,y}) = \tau(y^*x)$ for all $x, y$.

**Lemma 6.13.** *Let* $(\Lambda, d)$ *be a locally convex locally finite* $k$*-graph with no sinks and a faithful gauge invariant trace* $\tau$ *on* $C^*(\Lambda)$. *Let* $v \in \Lambda^0$ *and* $n \in \mathbf{Z}^k$. *Then*

$$\tilde{\tau}(p_v\Phi_n) \leq \tau(p_v)$$

*with equality when* $v\Lambda^{\leq p} = v\Lambda^p$.

*Proof.* Let $n = n_+ + n_-$ where $n_+ \geq 0$, $n_- \leq 0$, and $n_+ \vee -n_- = n_+ - n_-$. By Lemma 5.7 and Proposition 6.12 we have

$$\begin{aligned}
\tilde{\tau}\left(p_v\Phi_n\right) &= \tilde{\tau}\left(p_v \sum_{d(\alpha)=n_+, d(\beta)=-n_-} \frac{1}{|s(\beta)|_{-n_-}} \Theta_{S_\alpha S_\beta^*, S_\alpha S_\beta^*}\right) \\
&= \tau\left(\sum_{d(\alpha)=n_+, d(\beta)=-n_-} \frac{1}{|s(\beta)|_{-n_-}} (S_\alpha S_\beta^* | p_v S_\alpha S_\beta^*)_R\right) \\
&= \tau\left(\sum_{d(\alpha)=n_+, d(\beta)=-n_-} \frac{1}{|s(\beta)|_{-n_-}} \Phi(S_\beta S_\alpha^* p_v S_\alpha S_\beta^*)\right) \\
&= \tau\left(\sum_{d(\alpha)=n_+, d(\beta)=-n_-, r(\alpha)=v} \frac{1}{|s(\beta)|_{-n_-}} S_\alpha S_\beta^* S_\beta S_\alpha^* p_v\right) \\
&= \tau\left(\sum_{d(\alpha)=n_+, r(\alpha)=v} S_\alpha S_\alpha^* p_v\right).
\end{aligned}$$

If there are no sources within $|n_+|$ of $v$, then $\sum_{d(\alpha)=n_+, r(\alpha)=v} S_\alpha S_\alpha^* = p_{r(\alpha)} = p_v$. Otherwise the sum on the right is strictly less than $p_v$. So

$$\tilde{\tau}(p_v\Phi_n) \leq \tau(p_v)$$

with equality when there are no sources within $|n_+|$ of $v$. □

**Proposition 6.14.** *Assume that the locally convex* $k$*-graph* $(\Lambda, d)$ *is locally finite, has no sinks and has a faithful gauge invariant trace on* $C^*(\Lambda)$. *For all* $a \in \mathcal{A}_c$ *the operator* $a(1 + \mathcal{D}^2)^{-k/2}$ *is in the ideal* $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tilde{\tau})$. *When* $v \in \Lambda^0$ *satisfies* $v\Lambda^{\leq n} = v\Lambda^n$ *for all* $n \in \mathbf{N}^k$, *we have*

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-k/2}) = \frac{2^{[k/2]} vol(S^{k-1})}{k}\tau(p_v).$$

*Proof.* It suffices to show this for a projection $a = p_v$ for $v \in \Lambda^0$, and extending to more general $a \in A_c$ using the arguments of Lemma 5.7. We compute the partial sums defining the trace of $p_v(1+\mathcal{D}^2)^{-k/2}$. Lemma 6.13 gives us

$$\tilde{\tau}\left(p_v \sum_{|n|\le N}(1+n^2)^{-k/2}\Phi_n\right) \le \sum_{|n|\le N}(1+n^2)^{-k/2}\tau(p_v). \tag{16}$$

We have equality when $v\Lambda^{\le n} = v\Lambda^n$ whenever $|n| \le N$. Since $\Lambda$ has no sinks, the sequence

$$\frac{1}{\log|\{n : |n| \le N\}|}\sum_{|n|\le N}(1+n^2)^{-k/2}\tilde{\tau}(p_v\Phi_k)$$

is bounded (there is at least one 'direction' in which $n$ can increase indefinitely, so the sequence does not go to zero). Hence $p_v(1+\mathcal{D}^2)^{-k/2} \in \mathcal{L}^{(1,\infty)}$ and for any $\omega$-limit we have

$$\tilde{\tau}_\omega(p_v(1+\mathcal{D}^2)^{-k/2}) \le \omega\text{-lim}\,\frac{2^{[k/2]}vol(S^{k-1})}{k\log m}\sum_{m=0}^{N}(1+m^2)^{-1/2}\tilde{\tau}(p_v\Phi_k).$$

When there are no sources in $\Lambda$, we have equality in Equation (16) for any $v \in \Lambda^0$ and so

$$\tilde{\tau}_\omega(p_v(1+\mathcal{D}^2)^{-k/2}) = \frac{2^{[k/2]}vol(S^{k-1})}{k}\tau(p_v).$$

□

Computing the Dixmier trace when $v\Lambda^{\le n}$ may be strictly larger than $v\Lambda^n$ for some $n$ is harder.

**Remark** Using Proposition 4.10, one can check that

$$res_{s=0}\tilde{\tau}(p_v(1+\mathcal{D}^2)^{-k/2-s}) = \frac{k}{2}\tilde{\tau}_\omega(p_v(1+\mathcal{D}^2)^{-k/2}). \tag{17}$$

We will require this formula when we apply the local index theorem.

**Corollary 6.15.** *Assume $\Lambda$ is locally finite, has no sources and has a faithful $k$-graph trace. Then for all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$.*

*Proof.* (of Theorem 6.8.) That we have a $QC^\infty$ spectral triple follows from Corollary 6.5, Lemma 6.7 and Corollary 6.15. The properties of the von Neumann algebra $\mathcal{N}$ and the trace $\tilde{\tau}$ follow from Proposition 6.12. The $(k,\infty)$-summability and the value of the Dixmier trace comes from Proposition 6.14. The locality of the spectral triple follows from Lemma 6.7. □

## 7. The Local Index Theorem for the Gauge Spectral Triple

The local index theorem for semifinite spectral triples described in [CPRS2, CPRS3] is relatively simple for the spectral triples constructed here. This is because of the simple way in which the triples are built using the Clifford algebra.

In the following discussion we assume we have a fixed locally finite locally convex $k$-graph $(\Lambda, d)$ without sinks, and possessing a faithful $k$-graph trace. We let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the associated gauge spectral triple relative to $(\mathcal{N}, \tilde{\tau})$ constructed in the previous section.

Elementary manipulations with the Clifford variables, like those in [BCPRSW, Section 11.1], along with the Dixmier trace results, show that when $k$ is odd we are left with only one term in the local index theorem

$$\phi_k(a_0, a_1, ..., a_k) = -\sqrt{2\pi i}\frac{1}{k!}\frac{1}{\sqrt{\pi}}\Gamma(k/2+1)res_{r=(1-k)/2}\tilde{\tau}(a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_k](1+\mathcal{D}^2)^{-(k-1)/2-r}).$$

Likewise when $k$ is even we are left with only two terms:

$$\phi_k(a_0, a_1, ..., a_k) = \frac{1}{k!}\Gamma(k/2+1)res_{r=(1-k)/2}\tilde{\tau}(\gamma a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_k](1+\mathcal{D}^2)^{-(k-1)/2-r}),$$

$$\phi_0(a_0) = res_{r=(1-k)/2}\frac{1}{(r-(1-k)/2)}\tilde{\tau}(\gamma a_0(1+\mathcal{D}^2)^{-(k-1)/2-r}).$$

The zero component in the even case likewise vanishes for our examples. The reason for this is simply that we have complete symmetry between the $\pm 1$ eigenspaces of $\gamma$, and so for $Re(r)$ large

$$\frac{1}{(r-(1-k)/2)}\tilde{\tau}(\gamma a_0(1+\mathcal{D}^2)^{-(k-1)/2-r}) = 0.$$

Hence in this particular case, the local index theorem is in fact computed using the Hochschild class (top component) of the Chern character, [CPRS1].

For $k$-graphs $\Lambda$ to which Proposition 3.13 applies to ensure the existence of a faithful $k$-graph trace, we have seen that the $K$-theory of $C^*(\Lambda)$ resides entirely in the set of ends. It is therefore only necessary to produce generators corresponding to these ends.

The form of the Chern character given above shows that in odd dimensions we can detect only ends which are odd, whilst in even dimensions we can only detect ends which are even. A simple analysis based on the Clifford algebra then shows that in fact we can only pair with ends which are $k$-tori, $k$-planes, or more generally $k$-cylinders.

In order to produce representatives for generators of $K$-theory coming from these ends, it is necessary to have generators of the $K$-theory of ordinary tori (those for planes are of course well known). In fact we really only need those generators which pair with the Dirac class. These in turn can all be obtained, using the universal coefficient theorem and the fact that the $K$-theory of tori is free abelian, by using iterated products with the circle.

We illustrate this with a specific example; the nontrivial generator of $K^2(\mathbf{T}^2) = \mathbf{Z}^2$ (the other products are simpler). We wish to compute the product of $[u] \in K_1(C(\mathbf{T}^1))$ with itself to obtain the nontrivial projection in $K_0(\mathbf{T}^2)$. Let $\theta, \phi \in [0, 2\pi]$, and set $z = e^{i\theta}$. Then $\theta \to z$ represents the generator $[u]$. Define

$$K(\theta) = \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y(\phi, \theta) = e^{i\phi K(\theta)/4}e^{iS/4}.$$

Then the product $[u] \times [u]$ is the class of the projection

$$P(\phi,\theta) = Y(\phi,\theta)^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y(\phi,\theta).$$

A lengthy computation shows that

$$P(\phi,\theta) = \begin{pmatrix} 1-\sin^2(\phi/2)\cos^2(\theta/2) & \frac{i}{2}\sin(\phi)\cos^2(\theta/2) - \frac{1}{2}\sin(\phi/2)\sin(\theta) \\ -\frac{i}{2}\sin(\phi)\cos^2(\theta/2) - \frac{1}{2}\sin(\phi/2)\sin(\theta) & \sin^2(\phi/2)\cos^2(\theta/2). \end{pmatrix}$$

An even lengthier calculation using the residue cocycle from the local index theorem shows that $P(\phi,\theta)$ has pairing with

$$\mathrm{Dirac}_{\mathbf{T}^2} = \left( C^\infty(\mathbf{T}^2), L^2(\mathbf{T}^2)\otimes \mathbf{C}^2, \begin{pmatrix} 0 & -\partial_\phi + i\partial_\theta \\ \partial_\phi + i\partial_\theta & \end{pmatrix} \right)$$

equal to one. Hence $p$ is the desired generator.

**Example.** Consider the 2-graph $\Lambda_n$ whose skeleton is illustrated in Figure 4

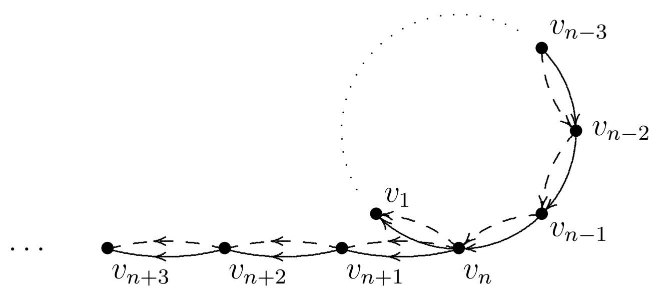


FIGURE 4. The 2-graph $\Lambda_n$

We label the solid edge whose range is $v_i$ by $e_i$ and the dashed edge with the same range is labelled $f_i$ for all $i < n$. Without the infinite tail to the left, we think of this 2-graph as the '$n$-point 2-torus', and this is justified by Proposition 3.16 which implies that $p_{v_n}$ is a full projection in $C^*(\Lambda_n)$ and that $p_{v_n}C^*(\Lambda_n)p_{v_n}$ is isomorphic to $C(\mathbf{T}^2)$.

We can say explicitly what this isomorphism is. It is easy to check that $p_{v_n}C^*(\Lambda)p_{v_n}$ is generated by the unitaries

$$u_1 := s_{e_1}s_{e_2}\dots s_{e_n} \quad \text{and} \quad u_2 = s_{f_1}s_{e_2}\dots s_{e_n}.$$[1]

The aim now is to prove the following analogue of [PRen, Corollary 6.6].

**Lemma 7.1.** *The $C^*$-algebra $C^*(\Lambda_n)$ is isomorphic to $\mathcal{K}\otimes C(\mathbf{T}^2)$, while the core $F_n = C^*(\Lambda_n)^\gamma$ is isomorphic to $\oplus_{j=1}^n \mathcal{K}$.*

[1] Note that while $w = s_{f_1}s_{f_2}\cdots s_{f_n}$ would appear to be a more natural candidate for the second generator than $u_2$, it is easy to see that $u_2$ does not belong to $C^*(\{u_1, w\})$ and so $u_1$ and $w$ do not generate $p_{v_n}C^*(\Lambda)p_{v_n}$. From a $K$-theoretic point of view, however, the distinction is not important: one can check that $u_2$ and $w$ have the same class in $K_1(p_{v_n}C^*(\Lambda)p_{v_n})$.

*Proof.* For $1 \le i < j$ let $\theta_{i,j} := s_{e_j} \dots s_{e_i+1}$, for $i > j$, let $\theta_{i,j} := \theta^*_{j,i} = s^*_{e_{j+1}} \dots s^*_{e_i}$, and for $i = j$, let $\theta_{i,i} := p_{v_i}$. It is easy to see that these elements form a collection of matrix units in $C^*(\Lambda)$ indexed by $\mathbf{N}$, and hence generate a subalgebra isomorphic to $\mathcal{K}$. For $i = 1, 2$ define a unitary $U_i \in \mathcal{M}(C^*(\Lambda_n))$ by $U_i := \sum_{j\in\mathbf{N}} \theta_{n,j} u_i \theta_{j,n}$.

Since the $\theta_{n,j}$ have orthogonal range projections, $C^*(\{U_1, U_2\})$ is canonically isomorphic to $C^*(\{u_1, u_2\})$, which in turn is canonically isomorphic to $C(\mathbf{T}^2)$ just as in Proposition 3.16. It is easy to check that $U_1$ and $U_2$ commute with the matrix units $\theta_{i,j}$ so $C^*(\{U_1, U_2, \theta_{i,j} : i, j \in \mathbf{N}\}) \cong \mathcal{K} \otimes C(\mathbf{T}^2)$ (it is worth noting that compression by $p_{v_n} = \theta_{n,n}$ takes $U_i$ to $u_i$). It now remains to show that this algebra is all of $C^*(\Lambda_n)$.

For $i \ne 1$, we have $s_{e_i} = \theta_{i,i+1}$, and we have $s_{e_1} = u_1\theta_{n,1}$ and $s_{f_2} = u_2\theta_{n,1}$. The only possible factorisation rule for $\Lambda_n$ satisfies $e_{i+1}f_i = f_{i+1}e_i$ for all $i$, and it now follows that $s_{f_i} = \theta_{1,i}s_{f_1}\theta_{i-1,n}$ for all $i > 1$. Hence all the generators of $C^*(\Lambda_n)$ belong to $C^*(\{U_1, U_2, \theta_{i,j} : i, j \in \mathbf{N}\})$ and it follows that $C^*(\Lambda_n)$ is isomorphic to $\mathcal{K} \otimes C(\mathbf{T}^2)$ as required.

To see that $F_n$ is isomorphic to $\oplus_{i=1}^n \mathcal{K}$, first observe that the subalgebra $C^*(\{s_\alpha : \alpha \in \Lambda_n : d(\alpha)_2 = 0\})$ generated by paths consisting only of solid edges is canonically isomorphic to the graph algebra $C^*(E_n)$ described in [PRen, Corollary 6.6], and that this isomorphism intertwines the restriction of the gauge action on $C^*(\Lambda_n)$ to $(\mathbf{T}, 1)$ and the gauge action on $C^*(E_n)$.

It is shown in [PRen] that the core of $C^*(E_n)$ is isomorphic to $\oplus_{i=1}^n \mathcal{K}$: the minimal projections in the $j^{\text{th}}$ copy of $\mathcal{K}$ are the vertex projections $s_{v_i} : i \cong l \bmod n$, and for $i \ge j$, the $(i, j)^{\text{th}}$ matrix unit is $\theta^l_{i,j} := s_\eta s^*_{L(v_j)^{i-j}} s^*_\zeta$ where $\eta$ is the shortest path from $v_l$ to $v_{in+l}$, $\zeta$ is the shortest path from $v_l$ to $v_{jn+l}$, and $L(v_l)$ is the loop of length $n$ based at $v_l$.

Hence it suffices to show here that

$$F_n = \overline{\operatorname{span}}\{s_\mu s^*_\nu : d(\mu) = d(\nu), d(\mu)_2 = d(\nu)_2 = 0, s(\mu) = s(\nu)\}. \tag{18}$$

Recall from [RSY, Section 4.1] that

$$F_n = \overline{\operatorname{span}}\{s_\mu s^*_\nu : \mu, \nu \in \Lambda_n, d(\mu) = d(\nu), s(\mu) = s(\nu)\},$$

so we just need to show that if $\mu, \nu \in \Lambda_n$ satisfy $d(\mu) = d(\nu)$ and $s(\mu) = s(\nu)$, then there exist $\eta$ and $\zeta$ such that $d(\eta) = d(\zeta) = (c, 0)$ for some $c \in \mathbf{N}$, $s(\eta) = s(\zeta)$, and $s_\eta s^*_\zeta = s_\mu s^*_\nu$. Fix $\mu, \nu \in \Lambda_n$ with $d(\mu) = d(\nu)$ and $s(\mu) = s(\nu)$, and write $(a, b)$ for $d(\mu)$. Let $\beta$ be the unique path of degree $(b, 0)$ whose range is equal to the source of $\mu$. By the factorisation property we can express $\mu = \mu_1\mu_2$ and $\nu = \nu_1\nu_2$ where $d(\mu_1) = d(\nu_1) = (a, 0)$ and $d(\mu_2) = d(\nu_2) = (0, b)$. Applying the factorisation property again, we obtain

$$\mu\beta = \mu_1\mu_2\beta = \mu_1\beta'\mu_2' \quad \text{and} \quad \nu\beta = \nu_1\nu_2\beta = \nu_1\beta''\nu_2'$$

where $d(\beta') = d(\beta'') = (b, 0)$ and $d(\mu_2') = d(\nu_2') = (0, b)$. Since $|\mu_1\beta'| = |\mu_1| + |\beta| = a + b = |\mu|$, we have $s(\mu_1\beta') = s(\mu)$, and similarly $s(\nu_1\beta'') = s(\nu) = s(\mu)$. Hence $\mu_2'$ and $\nu_2'$ are two paths with the same degree and same range. Since $v\Lambda_n^p$ is a singleton for each $v$ and $p$, it follows that $\mu_2' = \nu_2'$, so $s_{\mu_2'} s^*_{\nu_2'} = p_{s(\mu)}$ by (CK4). But now

$$s_\mu s^*_\nu = s_\mu p_{s(\mu)} s^*_\nu = s_\mu s_\beta s^*_\beta s^*_\nu = s_{\mu_1\beta'\mu_2'} s^*_{\nu_1\beta''\nu_2'} = s_{\mu_1\beta'} s_{\mu_2'} s^*_{\mu_2'} s_{\nu_1\beta''} = s_{\mu_1\beta'} s_{\nu_1\beta''}.$$

Since $d(\mu_1\beta') = (a+b,0) = d(\nu_1\beta'')$ and since $s(\beta') = s(\beta'') = s(\mu)$, this establishes (18). □

**Corollary 7.2.** *Although the isomorphism class of $C^*(\Lambda_n)$ is independent of $n$, $n$ is an invariant for the pair of algebras $(C^*(\Lambda_n), F_n)$.*

*Proof.* As in [PRen] we will compute first with the '$n$-point 2-torus', the analogous calculation for the 2-graph $\Lambda_n$ will then follow from the isomorphism $K_0(\mathcal{K}^n) \cong K_0(\mathbf{C}^n) = \mathbf{Z}^n$.

Let $\phi : C(\mathbf{T}^2) \to M_n(C(\mathbf{T}^2))$ be given by

$$\phi(f(z_1,z_2)) = \theta_{n,n} f(w_1,w_2)\theta_{n,n} + (1-\theta_{n,n}) = p_{v_n} f(w_1,w_2) p_{v_n} + (1-p_{v_n}).$$

Here we have set $w_1 = u_1$ and $w_2 = w$, as in the proof of Lemma 7.1, and denoted the generating unitaries of $C(\mathbf{T}^2)$ by $z_1, z_2$. Also $\theta_{n,n}$ is the projection $p_{v_n}$. Let $(X,\mathcal{D})$ be the Kasparov module for the $n$-point 2-torus built from the gauge action of $\mathbf{T}^2$. Then $\mathcal{D} = \sum_{j=1}^n p_{v_j}\mathcal{D} = \sum p_{v_j}\mathcal{D}p_{v_j}$, and the pull back of $(X,\mathcal{D})$ by $\phi$ is

$$\phi^*(X,\mathcal{D}) = (p_{v_n}X, p_{v_n}\mathcal{D}) \oplus \text{ degenerate module} \in KK^0(C(\mathbf{T}^2), F)$$

since $1 - p_{v_n}$ commutes with $\mathcal{D}$. The isomorphism $\psi : F \to \mathbf{C}^n$ given by

$$\psi(\sum_{j=1}^n z_j p_{v_j}) = (z_1, \dots, z_n)$$

gives us

$$\psi_*\phi^*(X,\mathcal{D}) = \oplus_{j=1}^n (p_{v_n}Xp_{v_j}, p_{v_n}\mathcal{D}) \in \oplus_{j=1}^n K^0(C(\mathbf{T}^2)).$$

The class of $(p_{v_n}Xp_{v_j}, p_{v_n}\mathcal{D})$ is easily seen to be the Dirac operator on $\mathbf{T}^2$ for the usual flat metric. In the following we will suppress $\psi$ and just identify $F$ with $\mathbf{C}^n$.

Now we can compute the pairing of $\mathcal{D}$ with $P(w_1,w_2) = \phi(P(z_1,z_2))$ where $P(z_1,z_2)$ is the Bott projector of $\mathbf{T}^2$ constructed earlier. We have

$$\begin{aligned}
\langle [P(w_1,w_2)], [(X,\mathcal{D})]\rangle &= \langle \phi_*([P(z_1,z_2)], [(X,\mathcal{D})]\rangle \\
&= \langle [P(z_1,z_2)], \phi^*[(X,\mathcal{D})]\rangle \quad \text{functoriality of Kasparov product} \\
&= \langle [P(z_1,z_2)], [(p_{v_n}X, p_{v_n}\mathcal{D})] \oplus [\text{degenerate module}]\rangle \\
&= \langle P(z_1,z_2), \oplus_{j=1}^n [(p_{v_n}Xp_{v_j}, p_{v_n}\mathcal{D})]\rangle \\
&= \sum_{j=1}^n \langle P(z_1,z_2), [\text{Dirac}_{\mathbf{T}^2}]\rangle \\
&= \sum_{j=1}^n \langle [z_1]\times[z_2], [\text{Dirac}_{\mathbf{T}^1}] \times [\text{Dirac}_{\mathbf{T}^1}]\rangle \quad \text{by [HR, Theorem 10.8.7]} \\
&= -\sum_{j=1}^n \langle [z_1], [\text{Dirac}_{\mathbf{T}^1}]\rangle\langle [z_2], [\text{Dirac}_{\mathbf{T}^1}]\rangle \quad \text{[HR, Chapter 9]} \\
&= -n. \qquad (19)
\end{aligned}$$

The number $n$ appears basically because the multiplicity provided by the core has given us $n$ copies of the Dirac operator at each point.

Now one can add the handle to the $n$-point 2-torus to get the 2-graph $\Lambda_n$. The core becomes $\mathcal{K}^n$ and an argument entirely analogous to the above shows again that the number $n$ is an invariant. □

**Remarks 7.3.** It is clear how to generalise the above example to an $n$-point $k$-torus with a "handle". A very similar argument to that of Lemma 7.1 shows that the resulting $k$-graph $\Lambda_n^k$ satisfies $C^*(\Lambda_n^k) \cong \mathcal{K} \otimes C(\mathbf{T}^k)$ independent of $n$, but that the core $F_n^k$ is always isomorphic to $\oplus_{i=1}^n \mathcal{K}$. We can therefore see that $n$ is an invariant for the pair $(C^*(\Lambda_n^k), F_n^k)$ in each case.

We now discuss the relationship between the $KK$-index pairing with values in $K_0(F)$ and the semfinite index theorem.

**Theorem 7.4.** *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the gauge spectral triple (relative to $(\mathcal{N}, \tilde{\tau})$ of a locally convex, locally finite $k$-graph without sinks, and $(X, \mathcal{D})$ the corresponding Kasparov module with class in $KK^k(A, F)$. Let $x \in K_k(A)$ be a $K$-theory class. Then*

$$\tilde{\tau}_*([x \times (X, \mathcal{D})]) = Ch_{(\mathcal{A},\mathcal{H},\mathcal{D})}(Ch(x)).$$

*Proof.* Let us first consider the even case, where we have the $K_0(F)$-valued index of $p\mathcal{D}_+p$ on $X$, where $p$ is a projection in $A$. The projections defining $\ker(p\mathcal{D}_+p)$ and $\operatorname{coker}(p\mathcal{D}_+p)$ are compact endomorphisms of the module $X$, and moreover map $X_c$ to itself. This last assertion follows because $\mathcal{D}$ maps $X_c$ to itself, and $p$ may be chosen to lie in $A_c$ which preserves $X_c$. The reason we can do this is that $K_0(A) = \lim K_0(\phi_n A \phi_n)$ where $\phi_n$ is any local approximate unit for $A_c$, [R1]. Hence the kernel and cokernel projections are actually endomorphisms preserving $X_c$.

Now such endomorphisms extend to act on the Hilbert space in a unique way. Since $\mathcal{H} = \overline{X_c}$ with respect to the norm coming from the inner product, we see that the Hilbert space kernel and cokernel projections are given by the extension of the $C^*$-module projections. So we have, by Lemma 6.13,

$$\tilde{\tau} - \operatorname{Index}(p\mathcal{D}_+p) = \tilde{\tau}(Q_{\ker(p\mathcal{D}_+p)} - Q_{\operatorname{coker}(p\mathcal{D}_+p)}) = \tilde{\tau}_*([\operatorname{Index}(p\mathcal{D}_+p)]) = \tilde{\tau}_*([p] \times [(X, \mathcal{D})]).$$

The argument for the odd pairing is now exactly the same, except that we consider the kernel and cokernel projections of $PuP$ where $P$ is the non-negative spectral projection of $\mathcal{D}$ and $u$ is unitary. The upshot is that

$$\tilde{\tau} - \operatorname{Index}(PuP) = \tilde{\tau}_*([\operatorname{Index}(PuP)]) = \tilde{\tau}_*([u] \times [(X, \mathcal{D})]).$$

Now we wish to relate the $\tilde{\tau}$ index to the pairing of Chern characters. However, by [CPRS4], this is precisely the main theorems of [CPRS2, CPRS3] in the odd and even cases respectively, and so we are done. □

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