AN ELEMENTARY APPROACH TO C^*-ALGEBRAS ASSOCIATED TO TOPOLOGICAL GRAPHS

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Abstract. We develop notions of a representation of a topological graph E and of a covariant representation of a topological graph E which do not require the machinery of C^*-correspondences and Cuntz-Pimsner algebras. We show that the C^*-algebra generated by a universal representation of E coincides with the Toeplitz algebra of Katsura's topological-graph bimodule, and that the C^*-algebra generated by a universal covariant representation of E coincides with Katsura's topological graph C^*-algebra. We exhibit our results by constructing the isomorphism between the C^*-algebra of a row-finite directed graph E with no sources and the C^*-algebra of the topological graph arising from the shift map acting on infinite path space E^∞.

1. Introduction

Let E be a countable directed graph with vertex set E^0, edge set E^1 and range and source maps $r, s : E^1 \to E^0$. The Toeplitz-Cuntz-Krieger algebra $\mathcal{T}C^*(E)$ is the universal C^*-algebra generated by a family of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a family of partial isometries $\{s_e : e \in E^1\}$ such that

(1) $s_e^* s_e = p_{s(e)}$ for every $e \in E^1$, and
(2) $p_v \geq \sum_{e \in F} s_e s_e^*$ for every $v \in E^0$ and finite $F \subset r^{-1}(v)$.

The Cuntz-Krieger algebra $C^*(E)$ is universal for families as above satisfying the additional relation that $p_v = \sum_{r(e)=v} s_e s_e^*$ whenever $r^{-1}(v)$ is nonempty and finite.

These C^*-algebras have been studied very extensively over the last fifteen years, part of their appeal being precisely that they can be defined, as above, in the space of a paragraph. Combined with key structure theorems like the gauge-invariant uniqueness theorem and the Cuntz-Krieger uniqueness theorem [1], or Fowler, Laca and Raeburn's analogue of Coburn's Theorem for the Toeplitz algebra of a directed graph [4], the above elementary presentation often makes it easy to verify that a given C^*-algebra is isomorphic to $C^*(E)$ or $\mathcal{T}C^*(E)$ as appropriate.

Some years after the introduction of graph C^*-algebras, Katsura introduced topological graphs and their C^*-algebras [5]. His construction is based on that of [6], which in turn was a modification of Pimsner's construction of C^*-algebras associated to C^*-correspondences in [9]. Roughly speaking, a topological graph is like a directed graph except that E^0 and E^1 are locally compact Hausdorff spaces rather than countable discrete sets, and r and s are required to be topologically well-behaved. Building on Fowler and Raeburn's construction of a C^*-correspondence from a directed graph given in [4], Katsura associated to each topological graph E a C^*-correspondence $X(E)$, and defined the C^*-algebra of the

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topological graph E to be the C^*-algebra $\mathcal{O}_{X(E)}$ he had associated to the module $X(E)$ in [6].

The drawback of this approach is that the relations defining the C^*-algebra are quite complicated and cannot be stated without first introducing at least the rudiments of C^*-correspondences, which is quite a bit of overhead — see Section 3. Clearly it would be handy to be able to circumvent all this technical overhead when defining and dealing with the C^*-algebras of topological graphs. This paper takes the first step in this direction. Our main theorems show that if E is a topological graph, then the Toeplitz algebra $\mathcal{T}(E)$ and Katsura's topological graph algebra $\mathcal{O}(E)$ of [5] can be described as C^*-algebras universal for relatively elementary relations involving elements of $C_0(E^0)$ and $C_c(E^1)$. In particular, our presentations can be stated without any reference to C^*-correspondences. We state our results so that they apply to arbitrary topological graphs, but we also show how our crucial covariance condition simplifies when the range map $r : E^1 \to E^0$ is a local homeomorphism and $r(E^1)$ is closed. Even this special situation is interesting: it includes, for example, Cantor minimal systems, and all crossed products of abelian C^*-algebras by $\mathbb{Z}$.

To emphasise how little background is needed to present our definition of a representation of a topological graph and our covariance condition, we begin by stating all our key definitions and main theorems in Section 2. In Section 3 we recall Katsura's construction of a C^*-correspondence from a topological graph and his definitions of the associated C^*-algebras. In Section 4, we prove our main results. We finish in Section 5 by considering the topological graphs $\widehat{E}$ obtained from row-finite directed graphs E with no sources by taking $\widehat{E}^0 = \widehat{E}^1 = E^\infty$ and defining the range map to be the identity map, and the source map to be the left-shift map σ. We apply our results to provide a relatively elementary proof that $\mathcal{O}(\widehat{E})$ is canonically isomorphic to $C^*(E)$. This result could be recovered from [2, 3], but working out the details provides a good example of the efficacy of our results.

2. Main results

Definition 2.1 ([5, Definition 2.1]). A quadruple $E = (E^0, E^1, r, s)$ is called a *topological graph* if E^0, E^1 are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is a continuous map, and $s : E^1 \to E^0$ is a local homeomorphism.

We think of E^0 as a space of vertices, and we think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$. If E^0, E^1 are both countable and discrete, then E is a directed graph in the sense of [8, 10].

Given $x \in C_c(E^1)$ and $f \in C_0(E^0)$, we define $x \cdot f$ and $f \cdot x$ in $C_c(E^1)$ by

$$(x \cdot f)(e) = x(e)f(s(e)) \quad \text{and } (f \cdot x)(e) = f(r(e))x(e). \tag{2.1}$$

To define our notion of a representation of a topological graph, we need a couple of preliminary ideas. An *s-section* in a topological graph E is a subset $U \subset E^1$ such that $s|_U$ is a homeomorphism. An *r-section* is defined similarly, and a *bisection* is a set which is both an s-section and an r-section.

If $x \in C_c(E^1)$ then $\operatorname{osupp}(x)$ denotes the precompact open set $\{e \in E^1 : x(e) \neq 0\}$, and $\operatorname{supp}(x)$ is the closure of $\operatorname{osupp}(x)$. If $x \in C_c(E^1)$ and $\operatorname{osupp}(x)$ is an s-section, we define $\widehat{x} : E^0 \to \mathbb{C}$ by

$$\widehat{x}(s(e)) := |x(e)|^2 \text{ for } e \in \operatorname{osupp}(x), \text{ and } \widehat{x}(v) = 0 \text{ for } v \notin s(\operatorname{osupp}(x)). \tag{2.2}$$

Definition 2.2. Let E be a topological graph. A *Toeplitz representation of* E in a C^*-algebra B is a pair (ψ, π) where $\psi : C_c(E^1) \to B$ is a linear map, $\pi : C_0(E^0) \to B$ is a homomorphism, and

(1) $\psi(f \cdot x) = \pi(f)\psi(x)$, for all $x \in C_c(E^1)$, $f \in C_0(E^0)$;
(2) for $x \in C_c(E^1)$ such that $\mathrm{supp}(x)$ is contained in an open s-section, $\pi(\widehat{x}) = \psi(x)^*\psi(x)$; and
(3) for x, $y \in C_c(E^1)$ such that $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are contained in disjoint open s-sections, $\psi(x)^*\psi(y) = 0$.

We say that a Toeplitz representation (ψ, π) of E in B is *universal* if for any Toeplitz representation (ψ', π') of E in C, there exists a homomorphism $h : B \to C$, such that $h \circ \psi = \psi'$ and $h \circ \pi = \pi'$.

Remark 2.3. Suppose that $x \in C_c(E^1)$ and $\mathrm{supp}(x)$ is contained in an open s-section U. Since E^1 is locally compact, we may cover $\mathrm{supp}(x)$ with precompact open sets $\{V_i : i \in I\}$, and since $\mathrm{supp}(x)$ is compact, we may pass to a finite subcover $\{V_i : i \in F\}$. Then each $V_i \cap U$ is precompact and open, so $\bigcup_{i \in F}(V_i \cap U)$ is a precompact open s-section containing $\mathrm{supp}(x)$. Thus we may assume in Conditions (2) and (3) that the open s-sections containing $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are precompact.

To state our first main theorem, recall that if E is a topological graph, then $\mathcal{T}(E)$ denotes the Toeplitz algebra of Katsura's topological-graph bimodule (see Notation 3.6).

Theorem 2.4. *Let E be a topological graph. Then there is a universal Toeplitz representation (i_1, i_0) of E which generates $\mathcal{T}(E)$. Moreover the C^*-algebra generated by the image of any universal Toeplitz representation of E is isomorphic to $\mathcal{T}(E)$.*

Remark 2.5. Since the map ψ occurring in a Toeplitz representation of E is not a homomorphism, it will not usually be norm-decreasing with respect to the supremum norm. So it is not clear that one can just extend by continuity a linear map ψ_0 defined on a dense subspace of $C_c(E^1)$. We show in Proposition 4.12 how to get around this difficulty: the map ψ *is* norm-decreasing with respect to the supremum norm when applied to functions supported on s-sections.

We now describe the covariance condition for a Toeplitz representation of a topological graph. The condition is somewhat technical, but we will indicate how it simplifies under additional hypotheses in Corollary 2.15.

We first need a little notation from [5].

Definition 2.6 ([5, Definition 2.6])**.** Let E be a topological graph. We define

(1) $E^0_{\mathrm{sce}} = E^0 \setminus \overline{r(E^1)}$.
(2) $E^0_{\mathrm{fin}} = \{v \in E^0 :$ there exists a neighbourhood N of v such that $r^{-1}(N)$ is compact$\}$.
(3) $E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} \setminus \overline{E^0_{\mathrm{sce}}}$.

Remark 2.7. The set E^0_{fin} is open in E^0. To see this, fix $v \in E^0_{\mathrm{fin}}$. Then there exists a neighbourhood U of v such that $r^{-1}(U)$ is compact and there exists an open neighbourhood V of v contained in U. So $V \subset E^0_{\mathrm{fin}}$, whence E^0_{fin} is open. It follows that E^0_{sce}, E^0_{fin}, E^0_{rg} are all open in E^0. Moreover E^0_{rg} is the intersection of E^0_{fin} with the interior of $\overline{r(E^1)}$. Finally, as proved by Katsura (see [5, Lemma 1.23]), for any compact subset $K \subset E^0_{\mathrm{fin}}$, the set $r^{-1}(K)$ is compact in E^1.

Notation 2.8. Let X be a locally compact Hausdorff space and U be an open subset of X. Then the standard embedding ι_U of $C_0(U)$ as an ideal of $C_0(X)$ is given by

$$\iota_U(f)(x) := \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

We will usually suppress the map ι_U and just identify each of $C_0(U)$ and $C_c(U)$ with their images in $C_0(X)$ under ι_U. That is, we think of $C_0(U)$ as an ideal of $C_0(X)$, and we regard $C_c(U)$ as an (algebraic) ideal of $C_0(X)$.

Remark 2.9. Let X be a Hausdorff space, and let $K \subset X$ be compact. Fix a finite cover $\mathcal{N}$ of K by open subsets of X. By [7, Chapter 5.W], there exists a partition of unity $\{h_N : N \in \mathcal{N}\}$ on K subordinate to $\{N \cap K : N \in \mathcal{N}\}$, where $h_N(x) \in [0, 1]$, for all $N \in \mathcal{N}$, $x \in K$. Since X may not be normal, we cannot necessarily extend this to a partition of unity on X. Nevertheless, if $f \in C_c(X)$ with $\operatorname{supp}(f) \subset K$, then the functions $f_N : X \to \mathbb{C}$ given by

$$f_N(x) := \begin{cases} f(x)h_N(x) & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

belong to $C_c(X)$. Hence each $\operatorname{osupp}(f_N) \subset N \cap K$, and $\sum_{N\in\mathcal{N}} f_N = f$.

Definition 2.10. Let E be a topological graph, and let (ψ, π) be a Toeplitz representation of E in a C^*-algebra B. We call (ψ, π) *covariant* if there exists a collection $\mathcal{G} \subset C_c(E^0_{\mathrm{rg}})$ of nonnegative functions which generates $C_0(E^0_{\mathrm{rg}})$ as a C^*-algebra, and for each $f \in \mathcal{G}$ there exist a finite cover $\mathcal{N}_f$ of $r^{-1}(\operatorname{supp}(f))$ by open s-sections, and a collection of nonnegative continuous functions $\{f_N : N \in \mathcal{N}_f\}$ such that

(1) $\operatorname{osupp}(f_N) \subset N \cap r^{-1}(\operatorname{supp}(f))$ for all $N \in \mathcal{N}_f$;
(2) $\sum_{N\in\mathcal{N}_f} f_N = f \circ r$; and
(3) $\pi(f) = \sum_{N\in\mathcal{N}_f} \psi\big(\sqrt{f_N}\big)\psi\big(\sqrt{f_N}\big)^*$.

We say that a covariant Toeplitz representation (ψ, π) of E in B is *universal* if for any covariant Toeplitz representation (ψ', π') of E in C, there exists a homomorphism $h : B \to C$, such that $h \circ \psi = \psi'$ and $h \circ \pi = \pi'$.

Remark 2.11. Definition 2.10 is formulated so as to make it easy to check that a given pair (ψ, π) is covariant. However, when *using* covariance of a pair (ψ, π) it is helpful to know that (3) of Definition 2.10 holds for every $f \in C_c(E^0_{\mathrm{rg}})$, every $\mathcal{N}_f$, and every $\{f_N : N \in \mathcal{N}_f\}$ satisfying (1) and (2) of Definition 2.10. We prove this in Proposition 4.8.

Remark 2.12. In Definition 2.10, we implicitly used Remark 2.9 because $r^{-1}(\operatorname{supp}(f))$ is compact by Remark 2.7. Observe that the covariance condition for a Toeplitz representation of E only involves functions in $C_0(E^0_{\mathrm{rg}}) \subset C_0(E^0)$.

To state our second main theorem, recall that if E is a topological graph, then $\mathcal{O}(E)$ denotes Katsura's topological graph C^*-algebra (see Notation 3.6).

Theorem 2.13. *Let E be a topological graph. Then there is a universal covariant Toeplitz representation (j_1, j_0) of E which generates $\mathcal{O}(E)$. Moreover the C^*-algebra generated by the image of any universal covariant Toeplitz representation of E is isomorphic $\mathcal{O}(E)$.*

Although Definition 2.10 looks complicated, the hypotheses are relatively easy to check in specific instances. To give some intuition, we indicate how the definition simplifies if the range map $r : E^1 \to E^0$ is a local homeomorphism and $r(E^1)$ is closed. This situation still includes many interesting examples.

Definition 2.14. Let E be a topological graph. A pair $(\mathcal{U}, V)$ is called a *local r-fibration* if V is a subset of E^0, and $\mathcal{U}$ is a finite collection of mutually disjoint bisections such that $r^{-1}(V) = \bigcup_{U\in\mathcal{U}} U$, and $r(U) = V$ for each $U \in \mathcal{U}$. A local r-fibration is *precompact* if each $U \in \mathcal{U}$ is precompact and V is precompact. A local r-fibration is *open* if each $U \in \mathcal{U}$ is open and V is open.

Suppose that $(\mathcal{U}, V)$ is an open local r-fibration. Suppose that $U \in \mathcal{U}$ and that $f \in C_c(V)$. We write $r_U^* f \in C_c(U)$ for the function

$$r_U^* f : e \mapsto \begin{cases} f(r(e)) & \text{if } e \in U \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.15. *Let E be a topological graph. Suppose that r is a local homeomorphism and $r(E^1)$ is closed. Let (ψ, π) be a Toeplitz representation of E. Then (ψ, π) is covariant if and only if there exists a collection $\mathcal{G} \subset C_c(E^0_{rg})$ of nonnegative functions which generates $C_0(E^0_{rg})$ as a C^*-algebra, and for each $f \in \mathcal{G}$ there exists an open local r-fibration $(\mathcal{U}, V)$ such that* $\operatorname{supp}(f) \subset V$, *and*

$$\pi(f) = \sum_{U\in\mathcal{U}} \psi\big(\sqrt{r_U^* f}\big)\psi\big(\sqrt{r_U^* f}\big)^*. \tag{2.4}$$

If (ψ, π) is covariant, then Equation 2.4 *holds for every $f \in C_c(E^0_{rg})$ and open local r-fibration $(\mathcal{U}, V)$ with* $\operatorname{supp}(f) \subset V$.

Remark 2.16. Let E be a topological graph. Fix $f \in C_c(E^0_{\mathrm{rg}})$ and suppose that $(\mathcal{U}, V)$ is an open local r-fibration such that $\operatorname{supp}(f) \subset V$. By Remark 2.7, $r^{-1}(\operatorname{supp}(f))$ is compact. Since $\mathcal{U}$ is an open cover of $r^{-1}(\operatorname{supp}(f))$ by disjoint open bisections, we apply Remark 2.9 to obtain functions $\{f_U : U \in \mathcal{U}\}$ such that $\operatorname{osupp}(f_U) \subset U \cap r^{-1}(\operatorname{supp}(f))$, and $\sum_{U\in\mathcal{U}} f_U = f \circ r$, for all $U \in \mathcal{U}$. We then have $r_U^* f = f_U$, for all $U \in \mathcal{U}$.

3. C^*-correspondences, C^*-algebras and Katsura's construction

We recall some background on Hilbert C^*-modules. For more detail see [11].

Definition 3.1. Let A be a C^*-algebra. A *right Hilbert A-module* is a right A-module X equipped with a map $\langle\cdot,\cdot\rangle_A : X \times X \to A$ such that for $x, y \in X$, $a \in A$,

1. $\langle x, x\rangle_A \geq 0$ with equality only if $x = 0$;
2. $\langle x, y \cdot a\rangle_A = \langle x, y\rangle_A a$;
3. $\langle x, y\rangle_A = \langle y, x\rangle_A^*$; and
4. X is complete in the norm[1] defined by $\|x\|_A^2 = \|\langle x, x\rangle_A\|$.

Recall from, for example, [10, Page 72], that a right Hilbert A-module X is a C^*-*correspondence* over A if there is a left action of A on X such that

$$\langle a^* \cdot y, x\rangle_A = \langle y, a \cdot x\rangle_A \quad \text{for all } a \in A,\ x,\ y \in X. \tag{3.1}$$

[1]It is not supposed to be immediately obvious that this defines a norm, but it is true (see [11]).

An operator $T : X \to X$ is *adjointable* if there exists $T^* : X \to X$ such that $\langle T^*y, x\rangle_A = \langle y, Tx\rangle_A$, for all x, $y \in X$. The adjoint T^* is unique, and T is automatically bounded and linear. The set $\mathcal{L}(X)$ of adjointable operators on X is a C^*-algebra. Equation (3.1) implies that there is a homomorphism $\phi : A \to \mathcal{L}(X)$ such that $\phi(a)x = a \cdot x$ for all $a \in A$ and $x \in X$.

Definition 3.2. Let A be a C^*-algebra and let X be a right Hilbert A-module. Fix x, $y \in X$, we define $\Theta_{x,y} : X \to X$ by $\Theta_{x,y}(z) = x \cdot \langle y, z\rangle_A$, for all $z \in X$. We define $\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in X\}$.

A calculation shows that $T\Theta_{x,y} = \Theta_{Tx,y}$, and $\Theta_{x,y}^* = \Theta_{y,x}$, for all x, $y \in X$, $T \in \mathcal{L}(X)$. Hence $\mathcal{K}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Toeplitz representations and Toeplitz algebras of C^*-correspondences were introduced and studied in [9]. Cuntz-Pimsner algebras were also introduced in [9], but the definition was later modified by Katsura in [6, Definition 3.5] so as to include graph algebras as a special case [10, Example 8.13]. We use Katsura's definition in this paper.

Definition 3.3 ([6, Definition 2.1])**.** Let A be a C^*-algebra and let X be a C^*-correspondence over A. A *Toeplitz representation* of X in a C^*-algebra B is a pair (ψ, π), where $\psi : X \to B$ is a linear map, $\pi : A \to B$ is a homomorphism, and for any x, $y \in X$ and $a \in A$,

(1) $\psi(a \cdot x) = \pi(a)\psi(x)$; and
(2) $\psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A)$.

As in [4, Proposition 1.3], we say that Toeplitz representation (ψ, π) of X in B is *universal* if for any Toeplitz representation (ψ', π') of X in C, there exists a homomorphism $h : B \to C$, such that $h \circ \psi = \psi'$, and $h \circ \pi = \pi'$.

Remark 3.4 ([6, Page 370]). Let X be a C^*-correspondence over a C^*-algebra A and let (ψ, π) be a Toeplitz representation of X. Condition (2) of Definition 3.3 and that π is a homomorphism imply that $\psi(x \cdot a) = \psi(x)\pi(a)$, for all $x \in X$, $a \in A$, and also that ψ is bounded with $\|\psi\| \leq 1$.

Proposition 1.3 of [4] implies that there exists a C^*-algebra $\mathcal{T}_X$ generated by the image of a universal Toeplitz representation (i_X, i_A) of X. This C^*-algebra is unique up to canonical isomorphism and we call it the *Toeplitz algebra* of X. Given another Toeplitz representation (ψ, π) of X in a C^*-algebra B, we write $h_{\psi,\pi}$ for the induced homomorphism from $\mathcal{T}_X$ to B.

Recall from [9, Page 202] (see also [4, Proposition 1.6]) that if (ψ, π) is a Toeplitz representation of a C^*-correspondence X in a C^*-algebra B, then there is a unique homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ such that $\psi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$ for all x, $y \in X$.

Recall also that given a C^*-algebra A and a closed two-sided ideal J of A, we define $J^\perp := \{a \in A : ab = 0 \text{ for all } b \in J\}$. Then $J^\perp$ is also a closed two-sided ideal of A.

Definition 3.5 ([6, Definitions 3.4 and 3.5])**.** Let A be a C^*-algebra, let X be a C^*-correspondence over A, and write $\phi : A \to \mathcal{L}(X)$ for the homomorphism implementing the left action. A Toeplitz representation (ψ, π) of X is *covariant* if $\psi^{(1)}(\phi(a)) = \pi(a)$ for all $a \in \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^\perp$.

A covariant Toeplitz representation (ψ, π) of X in B is *universal* if for any covariant Toeplitz representation (ψ', π') of X in C, there exists a homomorphism h from B into C, such that $h \circ \psi = \psi'$, and $h \circ \pi = \pi'$.

Recall from [10, Page 75] that there exists a C^*-algebra $\mathcal{O}_X$ generated by the image of a universal covariant Toeplitz representation (j_X, j_A) of X. This C^*-algebra is unique up to canonical isomorphism. Given another covariant Toeplitz representation (ψ, π) of X in a C^*-algebra B, we write $\psi \times \pi$ for the induced homomorphism from $\mathcal{O}_X$ to B.

Let E be a topological graph. As in [5], define right and left actions of $C_0(E^0)$ on $C_c(E^1)$ by Equation (2.1). For x_1, $x_2 \in C_c(E^1)$, define $\langle x_1, x_2\rangle_{C_0(E^0)} : E^0 \to \mathbb{C}$ by

$$\langle x_1, x_2\rangle_{C_0(E^0)}(v) = \sum_{s(e)=v} \overline{x_1(e)} x_2(e).$$

If $s^{-1}(v) = \emptyset$, then our convention is that the sum is equal to 0. If $\operatorname{osupp}(x)$ is an s-section, then the function $\widehat{x}$ of Equation (2.2) is equal to $\langle x, x\rangle_{C_0(E^0)}$. As in [5] (see also [10, Page 79]), $\langle \cdot, \cdot\rangle_{C_0(E^0)}$ defines a $C_0(E^0)$-valued inner product on $C_c(E^1)$, and the completion $X(E)$ of $C_c(E^1)$ in the norm $\|x\|^2_{C_0(E^0)} = \|\langle x, x\rangle_{C_0(E^0)}\|$ is a C^*-correspondence over $C_0(E^0)$, which we call the *graph correspondence* associated to E.

Notation 3.6. We denote by $\mathcal{T}(E)$ [5, Definition 2.2] the Toeplitz algebra $\mathcal{T}_{X(E)}$, and we denote by $\mathcal{O}(E)$ [5, Definition 2.10] the C^*-algebra $\mathcal{O}_{X(E)}$.

4. Proofs of the main results

To prove Theorem 2.4, we must show that $\psi(x)^*\psi(y) = \pi(\langle x, y\rangle_{C_0(E^0)})$ for all $x, y \in C_c(E^1)$. We establish this formula for x, y supported on s-sections in Lemma 4.1, and then extend it to arbitrary $x, y \in C_c(E^1)$ in Proposition 4.3.

Let U, V be complex vector spaces. Then any sesquilinear form $\varphi : V \times V \to U$ which is conjugate linear in the first variable satisfies the polarisation identity

$$\varphi(v_1, v_2) = \frac{1}{4}\sum_{n=0}^{3}(-i)^n \varphi(v_1 + i^n v_2, v_1 + i^n v_2).$$

To prove this, just expand the sum.

Lemma 4.1. *Let E be a topological graph and let (ψ, π) be a Toeplitz representation of E. Fix x_1, $x_2 \in C_c(E^1)$. Suppose that $\operatorname{supp}(x_1) \cup \operatorname{supp}(x_2)$ is contained in an open s-section. Then $\pi(\langle x_1, x_2\rangle_{C_0(E^0)}) = \psi(x_1)^*\psi(x_2)$.*

Proof. The polarisation identity for $\langle \cdot, \cdot\rangle_{C_0(E^0)}$, together with Definition 2.2(2), gives

$$\begin{aligned}\pi(\langle x_1, x_2\rangle_{C_0(E^0)}) &= \frac{1}{4}\sum_{n=0}^{3}(-i)^n \pi(\langle x_1 + i^n x_2, x_1 + i^n x_2\rangle_{C_0(E^0)}) \\ &= \frac{1}{4}\sum_{n=0}^{3}(-i)^n \psi(x_1 + i^n x_2)^*\psi(x_1 + i^n x_2) = \psi(x_1)^*\psi(x_2).\end{aligned}$$ □

In the following and throughout the rest of the paper we write U^c for the complement $X \setminus U$ of a subset U of a set X.

Remark 4.2. Let X be locally compact Hausdorff space. Fix a compact subset $K \subset X$ and an open neighbourhood U of K. Since U^c is closed and disjoint from K, there is a function $f \in C(X, [0, 1])$ which is identically 1 on K and identically 0 on U^c (see for example [13, Theorem 37.A]). So $V := f^{-1}\big((1/2, 3/2)\big)$ is open and satisfies $K \subset V \subset \overline{V} \subset U$.

Proposition 4.3. *Let E be a topological graph and let (ψ, π) be a Toeplitz representation of E. Fix open s-sections U_1, $U_2 \subset E^1$, and x_1, $x_2 \in C_c(E^1)$ with* $\operatorname{supp}(x_i) \subset U_i$, *for* $i = 1, 2$. *Then* $\pi(\langle x_1, x_2\rangle_{C_0(E^0)}) = \psi(x_1)^*\psi(x_2)$.

Proof. Let $K_i = \operatorname{supp}(x_i)$, for $i = 1$, 2, and $K = K_1 \cup K_2$. Since E^1 is locally compact and Hausdorff, Remark 4.2 implies that there exist open sets $U_i' \subset E^1$ such that $K_i \subset U_i' \subset \overline{U_i'} \subset U_i$, for $i = 1, 2$.

For each $e \in K$ and $W \in \{U_i, U_i', \overline{U_i'}^c : i = 1, 2\}$ such that $e \in W$, fix an open neighbourhood $V(e, W)$ of e such that $\overline{V(e, W)} \subset W$. Define

$$N_e = \Big(\bigcap_{e \in U_i} V(e, U_i)\Big) \bigcap \Big(\bigcap_{e \in K_i} V(e, U_i')\Big) \bigcap \Big(\bigcap_{e \in U_i^c} V(e, \overline{U_i'}^c)\Big).$$

Then for $i = 1$, 2,

(1) if $e \in U_i$, then $\overline{N_e} \subset U_i$;
(2) if $e \in K_i$, then $\overline{N_e} \subset U_i'$; and
(3) if $e \notin U_i$, then $\overline{N_e} \cap \overline{U_i'} = \emptyset$.

Since K is compact, there is a finite subset $F \subset K$, such that $\{N_e : e \in F\}$ covers K.

Fix e, $f \in F$. Suppose that $\overline{N_e} \cap \overline{N_f} \neq \emptyset$. We claim that

$$\text{either } \overline{N_e} \cup \overline{N_f} \subset U_1, \text{ or } \overline{N_e} \cup \overline{N_f} \subset U_2. \tag{4.1}$$

When $e = f$, this is trivial. Suppose $e \neq f$. Assume without loss of generality that $e \in K_1$. Then (2) forces $\overline{N_e} \subset U_1'$, so $\overline{N_e} \cap \overline{N_f} \neq \emptyset$ forces $\overline{N_f} \cap \overline{U_1'} \neq \emptyset$. (3) then forces $f \in U_1$, so (1) forces $\overline{N_f} \subset U_1$ and hence $\overline{N_e} \cup \overline{N_f} \subset U_1$ as required.

Since $\{N_e : e \in F\}$ is a finite open cover of K, Remark 2.9 implies that there are finite collections of functions $\{x_{1,e} : e \in F\}$, $\{x_{2,e} : e \in F\} \subset C_c(E^1)$ such that $\operatorname{osupp}(x_{i,e}) \subset N_e \cap K$ for all $e \in F$, $i = 1$, 2, and $\sum_{e \in F} x_{1,e} = x_1$, $\sum_{e \in F} x_{2,e} = x_2$. Linearity of ψ gives $\psi(x_1)^*\psi(x_2) = \sum_{e,f \in F} \psi(x_{1,e})^*\psi(x_{2,f})$. Fix e, $f \in F$ such that $\overline{N_e} \cap \overline{N_f} = \emptyset$. By Remark 4.2 there are disjoint open s-sections O_e, $O_f \subset E^1$, such that $\overline{N_e} \subset O_e$, and $\overline{N_f} \subset O_f$. Thus condition (3) of Definition 2.2 gives $\psi(x_{1,e})^*\psi(x_{2,f}) = 0$ since $\operatorname{supp}(x_{i,e}) \subset \overline{N_e}$ for all $e \in F$, $i = 1, 2$. It follows that $\psi(x_1)^*\psi(x_2) = \sum_{\overline{N_e} \cap \overline{N_f} \neq \emptyset} \psi(x_{1,e})^*\psi(x_{2,f})$. Whenever $\overline{N_e} \cap \overline{N_f} \neq \emptyset$, Equation (4.1) implies that $\overline{N_e} \cup \overline{N_f}$ is contained in an open s-section, so Lemma 4.1 gives $\psi(x_1)^*\psi(x_2) = \sum_{\overline{N_e} \cap \overline{N_f} \neq \emptyset} \pi(\langle x_{1,e}, x_{2,f}\rangle_{C_0(E^0)}) = \pi(\langle x_1, x_2\rangle_{C_0(E^0)})$. □

Proposition 4.4. *Let E be a topological graph and let (ψ, π) be a Toeplitz representation of E. Then ψ is a bounded linear operator on $(C_c(E^1), \|\cdot\|_{C_0(E^0)})$. Let $\widetilde{\psi}$ be the unique extension of ψ to $X(E)$. Then the pair $(\widetilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$.*

Proof. Fix x_1, $x_2 \in C_c(E^1)$. Let $K = \operatorname{supp}(x_1) \cup \operatorname{supp}(x_2)$. For each $e \in K$, there exists an open s-section N_e containing e. Remark 4.2 yields an open neighbourhood N_e' of e such that $\overline{N_e'} \subset N_e$. Since K is compact, there is a finite subset $F \subset K$, such that $\{N_e' : e \in F\}$ covers K. By Remark 2.9, there exist $\{x_{1,e} : e \in F\}$, $\{x_{2,e} : e \in F\} \subset C_c(E^1)$ such that $\operatorname{osupp}(x_{i,e}) \subset N_e' \cap K$, for all $e \in F$, $i = 1$, 2, and $\sum_{e \in F} x_{1,e} = x_1$, $\sum_{e \in F} x_{2,e} = x_2$. Proposition 4.3 implies that

$$\pi(\langle x_1, x_2\rangle_{C_0(E^0)}) = \sum_{e,f \in F} \pi(\langle x_{1,e}, x_{2,f}\rangle_{C_0(E^0)}) = \sum_{e,f \in F} \psi(x_{1,e})^*\psi(x_{2,f}) = \psi(x_1)^*\psi(x_2).$$

Then $\|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\| = \|\pi(\langle x,x\rangle_{C_0(E^0)})\| \le \|\langle x,x\rangle_{C_0(E^0)}\| = \|x\|^2_{C_0(E^0)}$, for all $x \in C_c(E^1)$, so ψ is bounded, and hence extends uniquely to a bounded linear map $\widetilde{\psi}$ on $X(E)$. By continuity, $(\widetilde{\psi},\pi)$ is a Toeplitz representation of $X(E)$. □

Remark 4.5. Given a Toeplitz representation (ψ,π) of $X(E)$, the pair $(\psi|_{C_c(E^1)},\pi)$ is a Toeplitz representation of E. So Proposition 4.4 implies that $(\psi,\pi) \mapsto (\psi|_{C_c(E^1)},\pi)$ is a bijection between Toeplitz representations of $X(E)$ and Toeplitz representations of E, with inverse described by Proposition 4.4.

Proof of Theorem 2.4. Let (i_X, i_A) be the universal Toeplitz representation of $X(E)$ in $\mathcal{T}(E)$. Then $(i_1, i_0) := (i_X|_{C_c(E^1)}, i_A)$ is a Toeplitz representation of E. Fix another Toeplitz representation (ψ,π) of E in a C^*-algebra B. By Proposition 4.4, ψ extends to $\widetilde{\psi} : X(E) \to B$ such that $(\widetilde{\psi},\pi)$ is a Toeplitz representation of $X(E)$. By the universal property of (i_X, i_A), there exists a homomorphism $h_{\widetilde{\psi},\pi} : \mathcal{T}(E) \to B$, such that $h_{\widetilde{\psi},\pi} \circ i_X = \widetilde{\psi}$, and $h_{\widetilde{\psi},\pi} \circ i_A = \pi$. In particular $h_{\widetilde{\psi},\pi} \circ i_1 = \psi$. Hence (i_1, i_0) is a universal Toeplitz representation of E which generates $\mathcal{T}(E)$. The second statement follows easily. □

Our next task is to prove Theorem 2.13. We first need some background results.

Remark 4.6. Let E be a topological graph. Fix $v \in E^0_{\rm fin}$. There exists a neighbourhood U of v such that $r^{-1}(U)$ is compact, and there exists an open neighbourhood V of v such that $V \subset U$. By Remark 4.2, there exists an open neighbourhood W of v such that $\overline{W} \subset V$. Since $r^{-1}(\overline{W})$ is closed and is contained in $r^{-1}(U)$, it is compact. Hence $v \in E^0_{\rm fin}$ if and only if there exists an open neighbourhood N of v such that $r^{-1}(\overline{N})$ is compact.

Let E be a topological graph. Recall from Definition 3.5 that Katsura's covariance condition for Toeplitz representations of $X(E)$ involves the ideal $\phi^{-1}(\mathcal{K}(X(E))) \cap (\ker\phi)^\perp$. Katsura computed this ideal in [5]. We quote his result and give a simple proof.

Observe that $\ker\phi = \{f \in C_0(E^0) : f(\overline{r(E^1)}) \equiv 0\}$. Hence $(\ker\phi)^\perp = \{f \in C_0(E^0) : f(\overline{E^0_{\rm sce}}) \equiv 0\}$.

Proposition 4.7 ([5, Proposition 1.24]). *Let E be a topological graph. Then*

$$\phi^{-1}(\mathcal{K}(X(E))) = C_0(E^0_{\rm fin}).$$

Moreover $\phi^{-1}(\mathcal{K}(X(E))) \cap (\ker\phi)^\perp = C_0(E^0_{\rm rg})$.

Proof. The final statement follows from the previous one and definition of $E^0_{\rm rg}$. So we just need to show that $\phi^{-1}(\mathcal{K}(X(E))) = C_0(E^0_{\rm fin})$.

Fix $f \in C_0(E^0) \setminus C_0(E^0_{\rm fin})$. We must show that $\phi(f) \notin \mathcal{K}(X(E))$. Fix $v_0 \in (E^0_{\rm fin})^c$, such that $f(v_0) \neq 0$. Let $m = |f(v_0)|$ and let $N_0 = \{v \in E^0 : |f(v)| > m/2\}$, so N_0 is an open neighbourhood of v_0. By Remark 4.6, $r^{-1}(\overline{N_0})$ is not compact. Fix $x_1,\dots,x_n$ and $y_1,\dots,y_n$ in $C_c(E^1)$. Let $K = \bigcup_{i=1}^n \operatorname{supp}(x_i) \cup \operatorname{supp}(y_i)$. Then K is compact, so $r^{-1}(\overline{N_0})$ is not contained in K. So there exists $e_0 \in r^{-1}(\overline{N_0}) \setminus K$. By Remark 4.2 there exists

$x_0 \in C_c(E^1)$ such that $x_0(e_0) = 1$. Hence

$$\begin{aligned}\Big\|\phi(f) - \sum_{i=1}^n \Theta_{x_i,y_i}\Big\| &\geq \Big\|\phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i,y_i}(x_0)\Big\|_{C_0(E^0)} \\ &\geq \Big|\Big\langle \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i,y_i}(x_0), \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i,y_i}(x_0)\Big\rangle_{C_0(E^0)}(s(e_0))\Big|^{1/2} \\ &\geq m/2.\end{aligned}$$

Thus $\|\phi(f) - a\| \geq m/2$ for all $a \in \operatorname{span}\{\Theta_{x,y} : x, y \in C_c(E^1)\}$. Since $\overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in C_c(E^1)\} = \mathcal{K}(X(E))$, it follows that $\phi(f) \notin \mathcal{K}(X(E))$.

Now fix a nonnegative function $f \in C_c(E^0_{\mathrm{fin}})$. We must show that $\phi(f) \in \mathcal{K}(X(E))$. Let $K' = \operatorname{supp}(f)$. For any $e \in r^{-1}(K')$, there exists an open s-section N_e containing e. Remark 2.7 shows that $r^{-1}(K')$ is compact. Hence there exists a finite subset $F \subset r^{-1}(K')$ such that $\{N_e\}_{e\in F}$ covers $r^{-1}(K')$. Since $\operatorname{supp}(f \circ r) \subset r^{-1}(K')$, Remark 2.9 yields a finite collection of functions $\{f_e : e \in F\} \subset C_c(E^1)$ such that each $\operatorname{osupp}(f_e) \subset N_e \cap r^{-1}(K')$ and $\sum_{e\in F} f_e = f \circ r$. Since the f_e are supported on the s-sections N_e, we have $\theta_{\sqrt{f_e},\sqrt{f_e}}(x)(e') = f_e(e')x(e')$ for all $e' \in E^1$. Hence

$$\phi(f) = \sum_{e\in F} \Theta_{\sqrt{f_e},\sqrt{f_e}} \in \mathcal{K}(X(E)). \tag{4.2}$$

□

Proposition 4.8. *Let E be a topological graph and let (ψ, π) be a covariant Toeplitz representation of E. Then the Toeplitz representation $(\widetilde{\psi}, \pi)$ of $X(E)$ from Proposition 4.4 is also covariant. Moreover, for any nonnegative function $f \in C_c(E^0_{\mathrm{rg}})$, any finite cover $\mathcal{N}$ of $r^{-1}(\operatorname{supp}(f))$ by open s-sections and any collection of functions $\{f_N : N \in \mathcal{N}\} \subset C_c(E^1)$, such that $\operatorname{osupp}(f_N) \subset N \cap r^{-1}(\operatorname{supp}(f))$, and $\sum_{N\in\mathcal{N}} f_N = f \circ r$, we have $\pi(f) = \sum_{N\in\mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*$.*

Proof. Let $\mathcal{G}$, $\{\mathcal{N}_f : f \in \mathcal{G}\}$, and $\{f_N : N \in \mathcal{N}_f\}$ be as in Definition 2.10. By Corollary 4.7, to prove $(\widetilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$, must show that $\widetilde{\psi}^{(1)} \circ \phi(f) = \pi(f)$, for all $f \in \mathcal{G}$. Fix $f \in \mathcal{G}$, since $\phi(f) = \sum_{N\in\mathcal{N}_f} \Theta_{\sqrt{f_N},\sqrt{f_N}}$, we have $\widetilde{\psi}^{(1)} \circ \phi(f) = \sum_{N\in\mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*$. Hence $\widetilde{\psi}^{(1)} \circ \phi(f) = \pi(f)$ by Definition 2.10. Therefore $(\widetilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$.

For the second statement observe that since $(\widetilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$,

$$\pi(f) = \widetilde{\psi}^{(1)} \circ \phi(f) = \sum_{N\in\mathcal{N}_f} \widetilde{\psi}^{(1)}(\Theta_{\sqrt{f_N},\sqrt{f_N}}) = \sum_{N\in\mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*.$$

□

Proposition 4.9. *Let E be a topological graph and let (ψ, π) be a covariant Toeplitz representation of $X(E)$. Then $(\psi|_{C_c(E^1)}, \pi)$ is a covariant Toeplitz representation of E.*

Proof. Remark 4.5 implies that $(\psi|_{C_c(E^1)}, \pi)$ is a Toeplitz representation of E. Let $\mathcal{G}$ be the set of all nonnegative functions in $C_c(E^0_{\mathrm{rg}})$. Fix $f \in \mathcal{G}$. By Equation (4.2), there exists a finite cover $\mathcal{N}$ of $r^{-1}(\operatorname{supp}(f))$ by open s-sections and a finite collection of functions $\{f_N : N \in \mathcal{N}\} \subset C_c(E^1)$, such that $\operatorname{osupp}(f_N) \subset N \cap r^{-1}(\operatorname{supp}(f))$, $\sum_{N\in\mathcal{N}} f_N = f \circ r$, and $\phi(f) = \sum_{N\in\mathcal{N}} \Theta_{\sqrt{f_N},\sqrt{f_N}}$. Since (ψ, π) is a covariant Toeplitz representation of $X(E)$, we have

$$\pi(f) = \psi^{(1)} \circ \phi(f) = \sum_{N \in \mathcal{N}} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^* = \sum_{N \in \mathcal{N}} \psi|_{C_c(E^1)}(\sqrt{f_N})\psi|_{C_c(E^1)}(\sqrt{f_N})^*.$$

Hence $(\psi|_{C_c(E^1)}, \pi)$ is covariant. □

Proof of Theorem 2.13. Propositions 4.8 and 4.9 provide a bijective map from covariant Toeplitz representations of E to covariant Toeplitz representations of $X(E)$. The result now follows from the same argument as Theorem 2.4. □

We now just need to prove Corollary 2.15. We must first show that under the hypotheses of the corollary, there are plenty of local r-fibrations (see Definition 2.14).

Lemma 4.10. *Let E be a topological graph. Suppose that r is a local homeomorphism and $r(E^1)$ is closed. Then for any $v \in E^0_{\mathrm{rg}}$, there exists a precompact open local r-fibration $(\mathcal{U}, V)$, such that $v \in V \subset \overline{V} \subset E^0_{\mathrm{rg}}$.*

Proof. Since r is a local homeomorphism, it is in particular an open map (see, for example, [5, Page 4289]). Thus $r(E^1)$ is open in E^0. Since $r(E^1)$ is also closed, the interior of $\overline{r(E^1)}$ is exactly $r(E^1)$. By Remark 2.7,

$$E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} \cap r(E^1). \tag{4.3}$$

Fix $v \in E^0_{\mathrm{rg}}$. By Remark 2.7, $r^{-1}(v)$ is a nonempty compact subset of E^1. Since r is a local homeomorphism, $r^{-1}(v)$ is a finite set. Since E^1 is locally compact Hausdorff, we can separate points in $r^{-1}(v)$ by mutually disjoint precompact open bisections $\{U_e : e \in r^{-1}(v)\}$. We can assume, by shrinking if necessary, that U_e have common range N, such that $\overline{N} \subset E^0_{\mathrm{rg}}$. We have $|r^{-1}(w)| \geq |r^{-1}(v)|$ for all $w \in N$.

We claim that there exists an open neighbourhood V of v such that $V \subset N$, and $|r^{-1}(w)| = |r^{-1}(v)|$ for all $w \in V$. Suppose for a contradiction that there exists a convergent net $(v_\alpha)_{\alpha \in \Lambda} \subset N$ with limit v satisfying $|r^{-1}(v_\alpha)| > |r^{-1}(v)|$ for all $\alpha \in \Lambda$. So for any $\alpha \in \Lambda$, there exists $e_\alpha \notin \bigcup_{e \in r^{-1}(v)} U_e$, such that $r(e_\alpha) = v_\alpha$. Since $r^{-1}(\overline{N})$ is compact, [12, Theorem IV.3] implies that there exists a convergent subnet $(e'_a)_{a \in A}$ of $(e_\alpha)_{\alpha \in \Lambda}$ with the limit e' not in $\bigcup_{e \in r^{-1}(v)} U_e$. By the continuity of r, we have $r(e') = v$, which is a contradiction.

Hence there exists an open neighbourhood V of v satisfying $V \subset N$, such that $|r^{-1}(w)| = |r^{-1}(v)|$ for all $w \in V$. So with $\mathcal{U} = \{U_e \cap r^{-1}(V) : e \in r^{-1}(v)\}$, the pair $(\mathcal{U}, V)$ is a precompact open local r-fibration. □

Lemma 4.11. *Let E be a topological graph. Suppose that r is a local homeomorphism and $r(E^1)$ is closed. Let $\mathcal{G}$ be the set of all nonnegative functions f in $C_c(E^0_{\mathrm{rg}})$ such that $\mathrm{supp}(f) \subset V$ for some open local r-fibration $(\mathcal{U}, V)$. Then $\mathcal{G}$ generates $C_0(E^0_{\mathrm{rg}})$.*

Proof. In order to prove $\mathcal{G}$ generates $C_0(E^0_{\mathrm{rg}})$, it suffices to show the linear span of $\mathcal{G}$ is $C_c(E^0_{\mathrm{rg}})$. Fix $f \in C_c(E^0_{\mathrm{rg}})$. By Lemma 4.10, for any $v \in \mathrm{supp}(f)$, there exists an open local r-fibration $(\mathcal{U}_v, V_v)$, such that $v \in V_v$. Remark 2.9 yields a finite subset $F \subset \mathrm{supp}(f)$ and a finite collection of functions $\{f_v : v \in F\} \subset C_c(E^0_{\mathrm{rg}})$, such that $\{V_v : v \in F\}$ is an open cover of $\mathrm{supp}(f)$, $\mathrm{supp}(f_v) \subset V_v$, for all $v \in F$, and $\sum_{v \in F} f_v = f$. □

Proof of Corollary 2.15. Let (ψ, π) be a covariant Toeplitz representation of E. Let $\mathcal{G}$ be the set of all nonnegative functions f in $C_c(E^0_{\mathrm{rg}})$ such that $\mathrm{supp}(f) \subset V$ for some open local r-fibration $(\mathcal{U}, V)$. Then Lemma 4.11 implies that $\mathcal{G}$ generates $C_0(E^0_{\mathrm{rg}})$. Fix $f \in \mathcal{G}$ and an open local r-fibration $(\mathcal{U}, V)$ with $\mathrm{supp}(f) \subset V$. Since $\mathcal{U}$ is a finite cover of $r^{-1}(\mathrm{supp}(f))$ by open s-sections, each $\mathrm{osupp}(r^*_U f) \subset U \cap r^{-1}(\mathrm{supp}(f))$, and $\sum_{U \in \mathcal{U}} r^*_U f = f \circ r$. By Proposition 4.8, we have $\pi(f) = \sum_{U \in \mathcal{U}} \psi(\sqrt{r^*_U f})\psi(\sqrt{r^*_U f})^*$. The second statement follows easily from the construction of $\mathcal{G}$. The converse of the first statement follows from Definition 2.10 and Remark 2.16. □

4.1. **The C^*-algebra generated by a Toeplitz representation.** In this subsection we provide some technical results which may prove useful in using our descriptions of the C^*-algebras associated to topological graphs. Proposition 4.12 is intended as an aid to constructing representations; and Proposition 4.16 provides a well-behaved collection of spanning elements for the image of any Toeplitz representation of E, and also a formula for computing products of these spanning elements.

To construct Toeplitz representations of a topological graph, one needs to build linear maps $\psi : C_c(E^1) \to B$ that are bounded in the bimodule norm $\|\cdot\|_{C_0(E^0)}$. The following technical result simplifies the task by showing that it is enough to define ψ on functions that are dense in supremum norm on $C_0(U)$ for a suitable family of open s-sections U.

Proposition 4.12. *Let E be a topological graph, let $\mathcal{B}$ be an open base for the topology on E^1 consisting of s-sections, and let $\mathcal{F} \subset C_c(E^1)$ be a collection of nonnegative functions such that $\mathrm{osupp}(x)$ is an s-section for all $x \in \mathcal{F}$. Suppose that $\mathcal{G} \subset C_0(E^0)$ generates $C_0(E^0)$, and that for each $U \in \mathcal{B}$,*

$$\mathrm{span}\{x \in \mathcal{F} : \mathrm{osupp}(x) \subset U\} \text{ is dense in } C_0(U) \text{ under the supremum norm.} \tag{4.4}$$

Then $X_0 := \mathrm{span}\,\mathcal{F}$ is dense in $X(E)$. Let B be a C^*-algebra. Suppose that $\psi_0 : X_0 \to B$ is a linear map, that $\pi : C_0(E^0) \to B$ is a homomorphism, and that

$$\pi(\widehat{\sqrt{xy}}) = \psi_0(x)^*\psi_0(y) \quad \text{for all } x, y \in \mathcal{F} \text{ (the product in } C_c(E^1) \text{ is pointwise).} \tag{4.5}$$

Then ψ_0 extends uniquely to a bounded linear map ψ on $C_c(E^1)$. If the extension ψ satisfies

$$\psi(f \cdot x) = \pi(f)\psi_0(x) \quad \text{for all } f \in \mathcal{G} \text{ and } x \in \mathcal{F}, \tag{4.6}$$

then (ψ, π) is a Toeplitz representation of E.

Proof. Fix $x \in C_c(E^1)$. Let $K = \mathrm{supp}(x)$. For each $e \in K$, there exists an open s-section N_e containing e, such that $N_e \in \mathcal{B}$. Since K is compact, there is a finite subset $F \subset K$, such that $\{N_e : e \in F\}$ covers K. By Remark 2.9, there exists a finite collection of functions $\{x_e : e \in F\} \subset C_c(E^1)$, such that $\mathrm{osupp}(x_e) \subset N_e \cap K$ for all $e \in F$, and $\sum_{e \in F} x_e = x$. Fix $e \in F$. Since $\mathrm{osupp}(x_e) \subset N_e$, there exists a sequence $(x_{e,n}) \subset X_0 \cap C_0(N_e)$ converging to x_e in supremum norm. That $(x_{e,n})$ and x_e vanish off the s-section N_e imply that $\|x_{e,n} - x_e\|_{C_0(E^0)} = \sup_{e \in E^1} |x_{e,n} - x_e|$. Hence $\sum_{e \in F} x_{e,n} \to x$ in $\|\cdot\|_{C_0(E^0)}$ norm. Therefore X_0 is dense in $X(E)$.

Fix $x, y \in \mathcal{F}$. Since x, y are nonnegative, $\widehat{\sqrt{xy}} = \langle x, y\rangle_{C_0(E^0)}$. Hence (4.5) implies that $\pi(\langle x, y\rangle_{C_0(E^0)}) = \psi_0(x)^*\psi_0(y)$. Linearity of ψ_0 and π gives $\pi(\langle x, y\rangle_{C_0(E^0)}) = \psi_0(x)^*\psi_0(y)$, for all $x, y \in X_0$. By Remark 3.4, ψ_0 is bounded, and hence extends uniquely to a bounded linear map $\tilde{\psi}$ on $X(E)$. Then $\psi := \tilde{\psi}|_{C_c(E^1)}$ is the required map.

Equation (4.6) and continuity imply that (ψ, π) is a Toeplitz representation of E. □

Remark 4.13. To prove Proposition 4.12 we showed that Equation (4.4) implies that X_0 is dense in $X(E)$ under the $\|\cdot\|_{C_0(E^0)}$ norm, and then deduced that (ψ, π) extends to a Toeplitz representation of E. So replacing Equation (4.4) with the hypothesis that X_0 is dense in $X(E)$ would yield a formally stronger result. However, Equation (4.4) is in many instances easier to check.

Our next proposition provides a description of the C^*-algebra generated by a Toeplitz representation of E in terms of a spanning family which captures many of the key properties of the usual spanning family in the Toeplitz algebra of a directed graph. We first need some notation and two technical lemmas.

Recall that E^n denotes the space $\{\mu = \mu_1 \dots \mu_n : \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\}$ of paths of length n in E. We define $r, s : E^n \to E^0$ by $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$, and we give E^n the relative topology inherited from the product space $\prod_{i=1}^n E^1$. For $x \in C_c(E^n)$ and $f \in C_0(E^0)$ we define $f \cdot x, x \cdot f \in C_c(E^n)$ by $(f \cdot x)(\mu) = f(r(\mu))x(\mu)$ and $(x \cdot f)(\mu) = x(\mu)f(s(\mu))$.

For $x_1, \dots, x_n \in C_c(E^1)$, we define $x_1 \diamond \cdots \diamond x_n \in C_c(E^n)$ by

$$(x_1 \diamond \cdots \diamond x_n)(\mu) = \prod_{i=1}^n x_i(\mu_i) \quad \text{for } \mu = \mu_1 \dots \mu_n \in E^n.$$

We use the symbol $\diamond$ for this operation to distinguish it from the pointwise product of elements of $C_c(E^1)$ appearing in, for example, Equation (4.5).

The second assertion of the following technical result follows from the discussion preceding [9, Proposition 3.3] together with [5, Proposition 1.27] (see also [10, Proposition 9.7]). We include the result and a simple proof here for completeness.

Suppose that $x, y \in C_c(E^n)$ are supported on s-sections. Then there is a unique $H(x, y) \in C_c(E^0)$ that vanishes on $E^0 \setminus \{s(\mu) : x(\mu)y(\mu) \neq 0\}$ and satisfies

$$H(x, y)(s(\mu)) = \overline{x(\mu)}y(\mu) \text{ whenever } x(\mu)y(\mu) \neq 0.$$

Lemma 4.14. *Let E be a topological graph and suppose that $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are supported on s-sections. Let (ψ, π) be a Toeplitz representation of E. Let $x = x_1 \diamond \cdots \diamond x_n$ and $y = y_1 \diamond \cdots \diamond y_n$. Then $\pi(H(x, y)) = \psi(x_n)^* \cdots \psi(x_1)^* \psi(y_1) \cdots \psi(y_n)$. If $x = y$ then $\prod_{i=1}^n \psi(x_i) = \prod_{i=1}^n \psi(y_i)$.*

Proof. By Proposition 4.4, (ψ, π) extends to a Toeplitz representation of $X(E)$. We have $H(x, y) = \langle x_n, H(x_1 \diamond \cdots \diamond x_{n-1}, y_1 \diamond \cdots \diamond y_{n-1}) \cdot y_n \rangle$, and hence

$$\pi(H(x, y)) = \psi(x_n)^* \pi(H(x_1 \diamond \cdots \diamond x_{n-1}, y_1 \diamond \cdots \diamond y_{n-1}) \psi(y_n).$$

The first assertion now follows by induction. For the second assertion, we use the first to see that

$$\Big(\prod_{i=1}^n \psi(x_i) - \prod_{i=1}^n \psi(y_i)\Big)^* \Big(\prod_{i=1}^n \psi(x_i) - \prod_{i=1}^n \psi(y_i)\Big) = \pi(H(x, x) - H(x, y) - H(y, x) + H(y, y)),$$

which is equal to zero since $x = y$. □

Lemma 4.15. *Let E be a topological graph. Suppose that $x_1, \dots, x_n \in C_c(E^1)$ are supported on s-sections, and fix $f \in C_0(E^0)$. Then there exists $\tilde{f} \in C_c(E^0)$ such that*

$$\tilde{f}(s(\mu)) = f(r(\mu)) \quad \textit{whenever } (x_1 \diamond \cdots \diamond x_n)(\mu) \neq 0. \tag{4.7}$$

For any such $\tilde{f}$*, we have* $f \cdot (x_1 \diamond \cdots \diamond x_n) = (x_1 \diamond \cdots \diamond x_n) \cdot \tilde{f}$*, and* $\pi(f) \prod_{i=1}^n \psi(x_i) = \prod_{i=1}^n \psi(x_i)\pi(\tilde{f})$ *for any Toeplitz representation* (ψ, π) *of* E*.*

Proof. The second assertion follows immediately from the first by definition of $f \cdot (x_1 \diamond \cdots \diamond x_n)$ and $(x_1 \diamond \cdots \diamond x_n) \cdot \tilde{f}$. The final assertion then follows from Lemma 4.14. So we just need to prove the first assertion. Let $x := x_1 \diamond \cdots \diamond x_n$. Fix $f \in C_0(E^0)$. Since $K := \mathrm{supp}(x) \subseteq E^n$ is an s-section, there is a well-defined continuous function from $s(K)$ to $r(K)$ given by $s(\mu) \mapsto r(\mu)$ for $\mu \in K$. So there is a continuous function $\tilde{f}_0 \in C(s(K))$ given by $\tilde{f}_0(s(\mu)) = f(r(\mu))$ for all $\mu \in K$. Since $s(K)$ is compact, an application of the Tietze extension theorem shows that $\tilde{f}$ has an extension $\tilde{f} \in C_0(E^0)$, which satisfies (4.7) by definition. □

Proposition 4.16. *Let* E *be a topological graph and let* (ψ, π) *be a Toeplitz representation of* E*. Let* $\mathcal{B}$ *and* $\mathcal{F}$ *be as in Proposition* 4.12*, and suppose that* $\mathrm{supp}(x)$ *is an* s*-section for each* $x \in \mathcal{F}$*. Then*

(1) $C^*(\psi, \pi)$ *is densely spanned by elements of the form*
$$\psi(x_1) \cdots \psi(x_n)\pi(f)\psi(y_m)^* \cdots \psi(y_1)^*$$
where $m, n \geq 0$*,* $f \in C_c(E^0)$*, the* x_i, y_j *all belong to* $\mathcal{F}$*, each* $s(\mathrm{osupp}(x_i)) \cap r(\mathrm{osupp}(x_{i+1})) \neq \emptyset$ *and each* $s(\mathrm{osupp}(y_i)) \cap r(\mathrm{osupp}(y_{i+1})) \neq \emptyset$*, and* $s(\mathrm{osupp}(x_n)) \cap s(\mathrm{osupp}(y_m)) \cap \mathrm{osupp}(f) \neq \emptyset$*.*

(2) *Let* $\psi(w_1) \ldots \psi(w_m)\pi(f)\psi(x_n)^* \ldots \psi(x_1)^*$ *and* $\psi(y_1) \ldots \psi(y_p)\pi(g)\psi(z_q)^* \ldots \psi(z_1)^*$ *be spanning elements as in* (1) *with* $p \geq n$*. Let* $x = x_1 \diamond \cdots \diamond x_n$ *and* $y = y_1 \diamond \cdots \diamond y_n$*. Fix* $k \in C_0(E^0)$ *such that* $(fH(x, y)) \cdot (y_{n+1} \diamond \cdots \diamond y_p) = (y_{n+1} \diamond \cdots \diamond y_p) \cdot k$ *as in Lemma* 4.15*. Then*

$$\begin{aligned}\big(\psi(w_1) \ldots \psi(w_m)\, \pi(f)\, \psi(x_n)^* \ldots \psi(x_1)^*\big)\big(\psi(y_1) \ldots \psi(y_p)\, \pi(g)\, \psi(z_q)^* \ldots \psi(z_1)^*\big) \\ = \psi(w_1) \ldots \psi(w_m)\psi(y_{n+1}) \ldots \psi(y_p)\, \pi(kg)\, \psi(z_q)^* \ldots \psi(z_1)^*.\end{aligned}$$

Remark 4.17. Consider the situation of Proposition 4.16(2) but with $p < n$. Let $x = x_1 \diamond \cdots \diamond x_p$ and $y = y_1 \diamond \cdots \diamond y_p$, and fix $k' \in C_0(E^0)$ such that $(H(x, y)g) \cdot (x_{p+1} \diamond \cdots \diamond x_n) = (x_{p+1} \diamond \cdots \diamond x_n) \cdot k'$. Taking adjoints in Proposition 4.16(2) gives

$$\begin{aligned}\big(\psi(w_1) \ldots \psi(w_m)\, \pi(f)\, \psi(x_n)^* \ldots \psi(x_1)^*\big)\big(\psi(y_1) \ldots \psi(y_p)\, \pi(g)\, \psi(z_q)^* \ldots \psi(z_1)^*\big) \\ = \psi(w_1) \ldots \psi(w_m)\, \pi(fk')\, \psi(x_n)^* \cdots \psi(x_{p+1})\psi(z_q)^* \ldots \psi(z_1)^*.\end{aligned}$$

Proof of Proposition 4.16. (1) Proposition 4.4 implies that $(\widetilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$, where $\widetilde{\psi}$ is the unique extension of ψ to $X(E)$. The argument of [6, Proposition 2.7] shows that $C^*(\psi, \pi)$ is densely spanned by elements of the form $\psi(x_1) \cdots \psi(x_n)\, \pi(f)\, \psi(y_m)^* \cdots \psi(y_1)^*$ where each $x_i, y_i \in C_c(E_1)$ and $f \in C_c(E^0)$. Fix x_1, $x_2 \in C_c(E^1)$ with $s(\mathrm{osupp}(x_1)) \cap r(\mathrm{osupp}(x_2)) = \emptyset$. Then

$$\begin{aligned}\|\psi(x_1)\psi(x_2)\|^2 &= \|\psi(x_2)^*\psi(x_1)^*\psi(x_1)\psi(x_2)\| = \|\psi(x_2)^*\pi(\langle x_1, x_1\rangle_{C_0(E^0)})\psi(x_2)\| \\ &= \|\psi(x_2)^*\psi(\langle x_1, x_1\rangle_{C_0(E^0)} \cdot x_2)\| = 0.\end{aligned}$$

Similarly, $\psi(x_1)\psi(x_2)^* = 0$ whenever $s(\mathrm{osupp}(x_1)) \cap s(\mathrm{osupp}(x_2)) = \emptyset$. So $C^*(\psi, \pi)$ is densely spanned by elements $\psi(x_1) \ldots \psi(x_m)\pi(f)\psi(y_m)^* \ldots \psi(y_1)^*$ where $f \in C_c(E^0)$, each $s(\mathrm{osupp}(x_i)) \cap r(\mathrm{osupp}(x_{i+1})) \neq \emptyset$, each $s(\mathrm{osupp}(y_i)) \cap r(\mathrm{osupp}(y_{i+1})) \neq \emptyset$, and $s(\mathrm{osupp}(x_n)) \cap s(\mathrm{osupp}(y_m)) \cap \mathrm{osupp}(f) \neq \emptyset$. Since Proposition 4.12 implies that $X_0 = \mathrm{span}\, \mathcal{F}$ is dense in $X(E)$ and hence in $C_c(E^1)$, the first assertion follows.

(2) Lemma 4.14 implies that

$$\pi(f)\psi(x_n)^* \dots \psi(x_1)^*\psi(y_1) \dots \psi(y_p) = \pi(fH(x,y))\psi(y_{n+1}) \dots \psi(y_p).$$

The result now follows from Lemma 4.15. □

Example 4.18. Let E be a directed graph regarded as a topological graph under the discrete topology. Let $\mathcal{G} = \{\delta_v : v \in E^0\}$ and $\mathcal{F} = \{\delta_e : e \in E^1\}$. Then $\mathcal{G}$ and $\mathcal{F}$ satisfy the hypotheses of Proposition 4.12, so we recover as an immediate consequence the isomorphism of the Toeplitz algebra $\mathcal{T}(E)$ of the graph bimodule (see [4] and [5]) with the Toeplitz algebra $\mathcal{T}C^*(E)$ of the graph E. The usual spanning family and multiplication rule for $\mathcal{T}C^*(E)$ follows from Proposition 4.16 applied to the same $\mathcal{F}$.

Remark 4.19. The multiplication formula of Proposition 4.16(2) has the drawback that the element k has no explicit formula in terms of the x_i the y_i and the function f; it is obtained by an application of the Tietze extension theorem (see Lemma 4.15). However, in practise there will frequently be a natural choice for k. Suppose, for example, that E^1 is totally disconnected. Then $\mathcal{F}$ can be taken to consist of characteristic functions of compact open s-sections. We can then take k to be the function that is identically zero off $s(\operatorname{supp}(y_{n+1} \diamond \cdots \diamond y_p))$ and satisfies $k(s(\mu)) = f(r(\mu))H(x,y)(r(\mu))$ whenever $\mu \in \operatorname{supp}(y_{n+1} \diamond \cdots \diamond y_p)$; this k is continuous because $s(\operatorname{supp}(y_{n+1} \diamond \cdots \diamond y_p))$ is clopen.

5. The topological graph arising from the shift map on the infinite path space

In this section we discuss how our results apply to the topological graph $\widehat{E}$ arising from the shift map on the infinite path space E^∞ of a row-finite directed graph E with no sources. It is known that $\mathcal{O}_{X(\widehat{E})}$ is isomorphic to $C^*(E)$ (it could be recovered from [2, 3]) but existing proofs use the universal property of $C^*(E)$ to induce a homomorphism from $C^*(E)$ to $\mathcal{O}_{X(\widehat{E})}$, invokes the gauge-invariant uniqueness theorem for $C^*(E)$ to establish injectivity, and then argues surjectivity by hand. It takes some work to show using the universal property of $\mathcal{O}_{X(\widehat{E})}$ that there is a homomorphism going in the other way.

Let E be a row-finite directed graph with no sources. We define $E^* = \bigcup_{n\geq 0} E^n$ and define $E^\infty = \{z \in \prod_{i=1}^\infty E^1 : s(z_i) = r(z_{i+1}), \text{ for all } i = 1, 2, \dots\}$. We view E^∞ as a topological space under the subspace topology coming from the ambient space $\prod_{i=1}^\infty E^1$. For any $\mu \in E^* \setminus E^0$, we define the cylinder set $Z(\mu) = \{z \in E^\infty : z_1 = \mu_1, \dots, z_{|\mu|} = \mu_{|\mu|}\}$. For $v \in E^0$ we define $Z(v) = \{z \in E^\infty : r(z_1) = v\}$. Since E has no sources, each $Z(\mu)$ is nonempty. The space E^∞ is a locally compact Hausdorff space with a base of compact open sets $\{Z(\mu) : \mu \in E^*\}$ ([8, Corollary 2.2]).

Now we construct a topological graph $\widehat{E} = (\widehat{E}^0, \widehat{E}^1, \widehat{r}, \widehat{s})$ as follows. Let $\widehat{E}^0 = \widehat{E}^1 = E^\infty$. Define $\widehat{r}$ to be the identity map, and define $\widehat{s}(z) = (z_2, z_3, \dots)$ for all $z \in \widehat{E}^1$. Since $Z(\mu)$ is a compact open $\widehat{s}$-section whenever $\mu \notin E^0$, the map $\widehat{s}$ is a local homeomorphism and hence $\widehat{E}$ is a topological graph.

For the following result recall the definition of a Cuntz-Krieger E-family from the first paragraph of the introduction.

Proposition 5.1. *Let E be a row-finite directed graph with no sources, and let $\widehat{E}$ be the topological graph described above.*

(1) *Let* (ψ,π) *be a covariant Toeplitz representation of* $\widehat{E}$*. For* $v\in E^0$ *define* $q_v := \pi(\chi_{Z(v)})$ *and for* $e\in E^1$ *define* $t_e := \psi(\chi_{Z(e)})$*. Then the* q_v *and the* t_e *form a Cuntz-Krieger* E*-family.*

(2) *Let* $\{q_v : v\in E^0\}$, $\{t_e : e\in E^1\}$ *be a Cuntz-Krieger* E*-family in a* C^**-algebra* B*. Then there is a unique covariant Toeplitz representation* (ψ,π) *of* $\widehat{E}$ *such that* $\psi(\chi_{Z(e\mu)}) = t_{e\mu}t_\mu^*$*, and* $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$ *for all* $e\in E^1$, $\mu\in E^*$*.*

(3) *Let* (j_1, j_0) *be the universal Toeplitz representation of* $\widehat{E}$ *in* $\mathcal{O}(E)$*, and let* $\{p_v,\ s_e : v\in E^0,\ e\in E^1\}$ *be the Cuntz-Krieger* E*-family generating* $C^*(E)$*. Then there is an isomorphism* $\mathcal{O}(\widehat{E})\cong C^*(E)$ *which carries each* $j_1(\chi_{Z(e\mu)})$ *to* $s_{e\mu}s_\mu^*$*, and each* $j_0(\chi_{Z(\mu)})$ *to* $s_\mu s_\mu^*$*.*

Proof of Proposition 5.1(1). The q_v are mutually orthogonal projections because the $\chi_{Z(v)}$ are. For $\mu\in E^*\setminus E^0$, the set $Z(\mu)$ is a compact open s-section. Thus for $e\in E^1$, relation (2) of Definition 2.2 implies that $t_e^*t_e = \pi(\widehat{\chi_{Z(e)}}) = \pi(\chi_{Z(s(e))}) = q_{s(e)}$. For $\mu\in E^*\setminus E^0$, $(\{Z(\mu)\}, Z(\mu))$ is an open local $\widehat{r}$-fibration. Since $\operatorname{supp}(\chi_{Z(\mu)}) = Z(\mu)$ and (ψ,π) is covariant, Corollary 2.15 implies that

$$\pi(\chi_{Z(\mu)}) = \psi\Big(\sqrt{r^*_{Z(\mu)}\chi_{Z(\mu)}}\Big)\psi\Big(\sqrt{r^*_{Z(\mu)}\chi_{Z(\mu)}}\Big)^* = \psi(\chi_{Z(\mu)})\psi(\chi_{Z(\mu)})^*. \tag{5.1}$$

So for $v\in E^0$, we have

$$q_v = \pi(\chi_{Z(v)}) = \sum_{r(e)=v}\pi(\chi_{Z(e)}) = \sum_{r(e)=v}\psi(\chi_{Z(e)})\psi(\chi_{Z(e)})^* = \sum_{r(e)=v} t_e t_e^*. \qquad \square$$

Proof of Proposition 5.1(2). Let $\mathcal{G} := \{\chi_{Z(\mu)} : \mu\in E^*\setminus E^0\}\subseteq C_0(\widehat{E}^0)$. Then $\operatorname{span}\mathcal{G}$ is a dense $*$-subalgebra of $C_0(\widehat{E}^0)$. We aim to define a map $\pi_0 : \operatorname{span}\mathcal{G}\to B$ by $\pi_0(\sum_{i=1}^n \alpha_i\chi_{Z(\mu_i)}) = \sum_{i=1}^n \alpha_i t_{\mu_i}t_{\mu_i}^*$. We check that π_0 is well-defined. It suffices to prove that $\sum_{i=1}^n \alpha_i\chi_{Z(\mu_i)} = 0$ implies $\sum_{i=1}^n \alpha_i t_{\mu_i}t_{\mu_i}^* = 0$, where the μ_i are distinct. Since E is row-finite and has no sources,

$$\pi_0\Big(\sum_{e\in r^{-1}(s(\mu))}\chi_{Z(\mu e)}\Big) = \sum_{e\in r^{-1}(s(\mu))} t_{\mu_e}t_{\mu_e}^* = t_\mu\Big(\sum_{e\in r^{-1}(s(\mu))} t_e t_e^*\Big)t_\mu^* = t_\mu t_\mu^* = \pi_0(\chi_{Z(\mu)}),$$

so we can assume that the μ_i have the same length. It follows that the $\chi_{Z(\mu_i)}$ are mutually orthogonal nonzero projections and hence each $\alpha_i = 0$. So π_0 is well-defined. It is obvious that π_0 is a linear map preserving the involution. [10, Corollary 1.15] implies that π_0 is a homomorphism. Now we show that π_0 is norm decreasing. Fix $\mu_1,\ldots,\mu_n$. We can assume that the μ_i are distinct and have the same length. Then

$$\begin{aligned}\big\|\pi_0(\sum_{i=1}^n\alpha_i\chi_{Z(\mu_i)})\big\|^2 &= \big\|\sum_{i=1}^n|\alpha_i|^2 t_{\mu_i}t_{\mu_i}^*\big\| \le (\max_i|\alpha_i|^2)\big\|\sum_{i=1}^n t_{\mu_i}t_{\mu_i}^*\big\| \le \max_i|\alpha_i|^2\\ &= \big\|\sum_{i=1}^n\alpha_i\chi_{Z(\mu_i)}\big\|^2,\end{aligned}$$

so π_0 is norm decreasing. Thus we obtain a unique homomorphism $\pi : C_0(\widehat{E}^0)\to C^*(E)$ by $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$ for all $\mu\in E^*$.

We next aim to define a linear map $\psi : C_c(\widehat{E}^1)\to B$ by extension of the formula $\psi(\chi_{Z(e\mu)}) = t_{e\mu}t_\mu^*$, and to show that the pair (ψ,π) is a Toeplitz representation of $\widehat{E}$. To do so, we will apply Proposition 4.12, so we need to set up the rest of the elements of the statement. Let $\mathcal{B} := \{Z(\mu) : \mu \in E^* \setminus E^0\}$, and let $\mathcal{F} := \{\chi_{Z(e\mu)} : e \in E^1, \mu \in E^*\} \subseteq C_c(E^1)$. Certainly $\mathcal{F}$ and $\mathcal{B}$ satisfy Equation 4.4. Similarly to the construction of π_0, there is a well-defined linear map $\psi_0 : \operatorname{span}\mathcal{F} \to B$ satisfying $\psi_0(\sum_{i=1}^n \alpha_i \chi_{Z(e_i\mu_i)}) = \sum_{i=1}^n \alpha_i t_{e_i\mu_i} t_{\mu_i}^*$.

Fix $x = \chi_{Z(e\mu)}$ and $y = \chi_{Z(f\nu)}$ in $\mathcal{F}$. We verify Equation 4.5. For this, observe that

$$\widehat{\sqrt{xy}} = \begin{cases} \chi_{Z(\mu)} & \text{if } e\mu = f\nu\mu' \\ \chi_{Z(\nu)} & \text{if } f\nu = e\mu\nu' \\ 0 & \text{otherwise.} \end{cases}$$

Then calculate:

$$(t_{e\mu}t_\mu^*)^* t_{f\nu}t_\nu^* = t_\mu t_{e\mu}^* t_{f\nu} t_\nu^* = \begin{cases} t_\nu t_\nu^* & \text{if } f\nu = e\mu\nu' \\ t_\mu t_\mu^* & \text{if } e\mu = f\nu\mu' \\ 0 & \text{otherwise.} \end{cases}$$

This establishes Equation (4.5). So Proposition 4.12 shows that ψ_0 extends uniquely to a linear map $\psi : C_c(\widehat{E}^1) \to B$. A similar calculation establishes Equation (4.6). Proposition 4.12 implies that (ψ, π) is a Toeplitz representation of E.

It remains to check covariance. Since the range map is a homeomorphism onto $\widehat{E}^0$ we can apply Corollary 2.15 with $\mathcal{G}$ as in the proof of part (2) and the local $\widehat{r}$-fibrations $\big\{(\{Z(\mu)\}, Z(\mu)) : \mu \in E^* \setminus E^0\big\}$ to see that (ψ, π) is covariant. □

Proof of Proposition 5.1(3). We show that $C^*(E)$ has the universal property of $\mathcal{O}(\widehat{E})$ and then invoke Theorem 2.13. Proposition 5.1(2) yields a covariant Toeplitz representation (θ_1, θ_0) of $\widehat{E}$ in $C^*(E)$ such that $\theta_1(\chi_{Z(e\mu)}) = s_{e\mu}s_\mu^*$ for all $e \in E^1$, $\mu \in E^*$, and $\theta_0(\chi_{Z(\mu)}) = s_\mu s_\mu^*$ for all $\mu \in E^*$. Fix a covariant Toeplitz representation (ψ, π) of $\widehat{E}$ in a C^*-algebra B. Then Proposition 5.1(1) gives a Cuntz-Krieger E-family $\{q_v := \pi(\chi_{Z(v)}),\ t_e := \psi(\chi_{Z(e)}) : v \in E^0,\ e \in E^1\}$ in B. So [10, Proposition 1.21] gives a homomorphism $\rho : C^*(E) \to B$ such that $\rho(p_v) = q_v$, and $\rho(s_e) = t_e$. An induction on the length of μ using Equation (5.1) show that $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$. For $e \in E^1$, and $\mu \in E^*$, we have $\rho \circ \theta_1(\chi_{Z(e\mu)}) = t_e t_\mu t_\mu^* = \psi(\chi_{Z(e)})\pi(\chi_{Z(\mu)}) = \psi(\chi_{Z(e\mu)})$. For $\mu \in E^* \setminus E^0$, we have $\rho \circ \theta_0(\chi_{Z(\mu)}) = t_\mu t_\mu^* = \pi(\chi_{Z(\mu)})$. Hence $\rho \circ \theta_1 = \psi$, and $\rho \circ \theta_0 = \pi$. Since the image of (θ_1, θ_0) generates $C^*(E)$, Theorem 2.13 implies that there is an isomorphism $\mathcal{O}(E) \cong C^*(E)$ which carries each $j_1(\chi_{Z(e\mu)})$ to $s_{e\mu}s_\mu^*$, and each $j_0(\chi_{Z(\mu)})$ to $s_\mu s_\mu^*$. □

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