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TOPOLOGICAL REALIZATIONS AND FUNDAMENTAL GROUPS OF HIGHER-RANK GRAPHS

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Abstract We investigate topological realizations of higher-rank graphs. We show that the fundamental group of a higher-rank graph coincides with the fundamental group of its topological realization. We also show that topological realization of higher-rank graphs is a functor and that for each higher-rank graph Λ, this functor determines a category equivalence between the category of coverings of Λ and the category of coverings of its topological realization. We discuss how topological realization relates to two standard constructions for k-graphs: projective limits and crossed products by finitely generated free abelian groups.

Keywords: k-graph; fundamental group; CW-complex; functor; covering; projective limit

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1. Introduction

Higher-rank graphs are higher-dimensional analogues of directed graphs introduced by Kumjian and Pask in [**6**]. Their motivation was the study of associated C^*-algebras as common generalizations of the graph C^*-algebras of [**7**] and the higher-rank Cuntz–Krieger algebras of [**15**].

In [**6**], Kumjian and Pask described *skew products* of k-graphs by group-valued functors c. They showed that if Λ is a k-graph and $c\colon \Lambda \to G$ is a functor into an abelian group, then the C^*-algebra associated to the skew-product graph $\Lambda \times_c G$ is isomorphic to the crossed product of $C^*(\Lambda)$ by an induced action $\tilde{c}$ of the dual group $\hat{G}$.

Pask *et al*. extended this result to non-abelian groups [**12**, **13**]. Generalizing results of [**1**] for directed graphs, they showed that if $c\colon \Lambda \to G$ is a functor into any discrete group and H is any subgroup of G, then the C^*-algebra of the relative skew product $\Lambda \times_c G/H$ is isomorphic to a restricted crossed product of $C^*(\Lambda)$ by a coaction of G.

They showed how to interpret relative skew products as *coverings* of k-graphs and they showed that every covering arises this way by introducing the fundamental group of a k-graph Λ and showing that G can be taken to be $\pi_1(\Lambda)$ and H can be taken such that $H \cong \pi_1(\Lambda \times_c G/H)$. They also indicated [**12**, § 6] how one might construct a topological realization of a k-graph by gluing open cells into the interiors of commuting cubes in the category and indicated that one would expect the fundamental group of the resulting space to coincide with the fundamental group of the k-graph.

In this paper, we make this precise. We define the topological realization X_Λ of a k-graph Λ and show by example that a number of standard surfaces arise from this construction applied to 2-graphs. We then show that the assignment $\Lambda \to X_\Lambda$ preserves fundamental groups. We go on to show that each k-graph morphism $\varphi\colon \Lambda \to \Gamma$ induces a continuous map $\tilde{\varphi}\colon X_\Lambda \to X_\Gamma$ and that the pair $(\Lambda \mapsto X_\lambda, \varphi \mapsto \tilde{\varphi})$ is a functor from the category of k-graphs with k-graph morphisms to the category of topological spaces with continuous maps. The situation is particularly nice for the coverings studied in [**13**]: for each k-graph Λ, the assignment $\varphi \mapsto \tilde{\varphi}$ determines a category equivalence between the category of algebraic coverings of Λ and the category of topological coverings of X_Λ that takes a universal covering of Λ to a universal covering of X_Λ.

We finish off by describing how our construction behaves with respect to two existing constructions from the theory of k-graphs. Firstly, by analogy with our construction for discrete k-graphs, we propose a notion of topological realization for a topological k-graph in the sense of Yeend [**18**]. Given a sequence of finite-to-one coverings $p_n\colon \Lambda_n \to \Lambda_{n-1}$ of k-graphs, the projective limit $\varprojlim(\Lambda_n, p_n)$ is a topological k-graph [**14**]. We show that the topological realization $X_{\varprojlim(\Lambda_n, p_n)}$ is homeomorphic to the projective limit $\varprojlim(X_{\Lambda_n}, \tilde{p}_n)$ and, in particular, that

$$\pi_1(X_{\varprojlim(\Lambda_n, p_n)}) \cong \varprojlim(\pi_1(\Lambda_n), (p_n)^*).$$

Secondly, we consider the crossed products of k-graphs studied in [**2**] and demonstrate that if α is an action of $\mathbb{Z}^l$ on a k-graph Λ, then the topological realization $X_{\Lambda \times_\alpha \mathbb{Z}^l}$ of the crossed-product k-graph is homeomorphic to the mapping torus $M(\tilde{\alpha})$ for the induced homeomorphism $\tilde{\alpha}$ of X_Λ.

2. Background

In this paper $\mathbb{N}$ denotes the natural numbers, which we take to include 0 and regard as a monoid under addition. For $k \geqslant 1$ we regard $\mathbb{N}^k$, the set of k-tuples from $\mathbb{N}$, as a monoid under pointwise operations. When convenient, we will also regard it as a category with a single object. We denote the identity element by 0 and we write $\mathbf{1}_k$ for the element $(1, 1, \dots, 1) \in \mathbb{N}^k$. We denote the canonical generators of $\mathbb{N}^k$ by $e_1, \dots, e_k$, and for $n \in \mathbb{N}^k$ we write $n_1, \dots, n_k$ for the coordinates of n; that is, $n = (n_1, n_2, \dots, n_k) = \sum_{i=1}^k n_i e_i$. We write $|n|$ for $\sum_{i=1}^k n_i$.

For $m, n \in \mathbb{N}^k$, we write $m \leqslant n$ if $m_i \leqslant n_i$ for all i, and $m < n$ if $m \leqslant n$ and $m \neq n$; in particular, $m < n$ does not mean that $m_i < n_i$ for all i. We write $m \vee n$ for the coordinatewise maximum of m and n; we then have $m, n \leqslant m \vee n$.

As in [**6**], a *k-graph* is a countable small category Λ endowed with a functor $d\colon \Lambda \to \mathbb{N}^k$ satisfying the following factorization property: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m+n$, there exist unique elements $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$. We write Λ^n for $d^{-1}(n)$. The map $o \mapsto \mathrm{id}_o$ is a bijection between the objects of Λ and the elements of Λ^0. We use this to regard the codomain and domain maps on Λ as maps $r, s\colon \Lambda \to \Lambda^0$ and observe that μ and ν are composable if $s(\mu) = r(\nu)$. We adopt the following notational convention of [**12**] for k-graphs. Given $\lambda \in \Lambda$ and $S \subseteq \Lambda$, we write $\lambda S = \{\lambda\mu\colon \mu \in S,\ r(\mu) = s(\lambda)\}$ and $S\lambda = \{\mu\lambda\colon \mu \in S,\ s(\mu) = r(\lambda)\}$. In particular, if $v \in \Lambda^0$, then $vS = r^{-1}(v) \cap S$ and $Sv = s^{-1}(v) \cap S$.

If $m \leqslant n \leqslant l \in \mathbb{N}^k$ and $\lambda \in \Lambda^l$, then two applications of the factorization property show that there exist unique paths $\lambda' \in \Lambda^m$, $\lambda'' \in \Lambda^{n-m}$ and $\lambda''' \in \Lambda^{l-n}$ such that $\lambda = \lambda'\lambda''\lambda'''$. We define $\lambda(m,n) = \lambda''$. Since $\lambda = r(\lambda)\lambda'(\lambda''\lambda''')$, we then have $\lambda(0,m) = \lambda'$ and similarly $\lambda(n,l) = \lambda'''$.

We emphasize that while many other papers on k-graphs require that Λ be finitely aligned and/or have no sources, we make no such assumptions in this paper, though many of our key examples are in fact row-finite.

3. The topological realization of a higher-rank graph

Let Λ be a k-graph. Given $t \in \mathbb{R}^k$, we will write $\lceil t \rceil$ for the least element of $\mathbb{Z}^k$ that is coordinatewise greater than or equal to t and $\lfloor t \rfloor$ for the greatest element of $\mathbb{Z}^k$ that is coordinatewise less than or equal to t. Observe that $\lfloor t \rfloor \leqslant t \leqslant \lceil t \rceil \leqslant \lfloor t \rfloor + \mathbf{1}_k$ for all $t \in \mathbb{R}^k$.

Given $p \leqslant q \in \mathbb{N}^k$, we denote by $[p,q]$ the *closed interval* $\{t \in \mathbb{R}^k\colon p \leqslant t \leqslant q\}$ and we denote by (p,q) the *relatively open interval* $\{t \in [p,q]\colon p_i < t_i < q_i \text{ whenever } p_i < q_i\}$. Observe that (p,q) is not open in $\mathbb{R}^k$ unless $p_i < q_i$ for all i, but it is open as a subspace of $[p,q]$. The set (p,q) is never empty: in particular, if $p = q$, then $(p,q) = [p,q] = \{p\}$. In general, as a subset of $\mathbb{R}^k$, the dimension of (p,q) is $|\{i \leqslant k\colon p_i < q_i\}|$. If $p_i < q_i$, then the ith-coordinate projection of (p,q) is (p_i, q_i) and if $p_i = q_i$, then the ith-coordinate projection of (p,q) is $\{p_i\}$.

Remark 3.1. Let $m \in \mathbb{N}^k$. If $m \leqslant \mathbf{1}_k$, then for all $t \in (0,m)$, $\lfloor t \rfloor = 0$ and $\lceil t \rceil = m$.

We define a relation on the topological disjoint union $\bigsqcup_{\lambda \in \Lambda} \{\lambda\} \times [0, d(\lambda)]$ by

$$(\mu, s) \sim (\nu, t) \quad \Longleftrightarrow \quad \mu(\lfloor s \rfloor, \lceil s \rceil) = \nu(\lfloor t \rfloor, \lceil t \rceil) \quad \text{and} \quad s - \lfloor s \rfloor = t - \lfloor t \rfloor. \tag{3.1}$$

It is straightforward to see that this is an equivalence relation.

Definition 3.2. Let Λ be a k-graph. With notation as above, we define the *topological realization* X_Λ of Λ to be the quotient space

$$\Big(\bigsqcup_{\lambda \in \Lambda} \{\lambda\} \times [0, d(\lambda)] \Big) \Big/ \sim .$$

The following alternative characterization of the equivalence relation $\sim$ will simplify arguments later in the paper.

Lemma 3.3. *The relation $\sim$ is generated as an equivalence relation by the relation*

$$\{((\alpha\lambda\beta, t + d(\alpha)), (\lambda, t)) \colon d(\lambda) \leqslant \mathbf{1}_k \text{ and } t \in [0, d(\lambda)]\}. \tag{3.2}$$

Proof. The relation (3.2) is contained in $\sim$ by definition of the latter. Now suppose that $(\mu, s) \sim (\nu, t)$. We must show that $((\mu, s), (\nu, t))$ belongs to the equivalence relation generated by (3.2). Let

$$\alpha_\mu = \mu(0, \lfloor s \rfloor), \quad \lambda_\mu = \mu(\lfloor s \rfloor, \lceil s \rceil) \quad \text{and} \quad \beta_\mu = \mu(\lceil s \rceil, d(\mu))$$

and similarly for ν. By the definition of $\sim$, we have $(\lambda_\mu, s - \lfloor s \rfloor) = (\lambda_\nu, t - \lfloor t \rfloor)$. Since $((\mu, s), (\lambda_\mu, s - \lfloor s \rfloor))$ and $((\nu, t), (\lambda_\nu, t - \lfloor t \rfloor))$ belong to (3.2), $((\mu, s), (\nu, t))$ belongs to the equivalence relation generated by (3.2). □

Notation 3.4. Let $[\lambda, t]$ denote the equivalence class of an element (λ, t). If $u \in \Lambda^0$, we often write u in place of $[u, 0] \in X_\Lambda$ to simplify notation.

For each $m \leqslant \mathbf{1}_k$ and each $\lambda \in \Lambda^m$, define

$$Q_\lambda = \{[\lambda, t] \colon t \in (0, m)\} \subset X_\Lambda$$

and let $\bar{Q}_\lambda$ denote its closure in X_Λ. We call Q_λ the *open cube* associated to λ and $\bar{Q}_\lambda$ the *closed cube* associated to Λ.

Lemma 3.5. *Let Λ be a k-graph. Then $X_\Lambda = \bigcup_{m \leqslant \mathbf{1}_k} \bigcup_{\lambda \in \Lambda^m} Q_\lambda$ and $Q_\lambda \cap Q_\mu = \emptyset$ for distinct $\lambda, \mu \in \bigcup_{m \leqslant \mathbf{1}_k} \Lambda^m$. For each $m \leqslant \mathbf{1}_k$ and each $\lambda \in \Lambda^m$, the map $[\lambda, t] \mapsto t$ is a homeomorphism of Q_λ onto $(0, m) \subset \mathbb{R}^k$, so Q_λ is homeomorphic to the open unit cube in $\mathbb{R}^{|m|}$. Furthermore, $\bar{Q}_\lambda = \{[\lambda, t] \colon 0 \leqslant t \leqslant m\}$.*

Define $X_\Lambda^0 = \bigcup_{v \in \Lambda^0} Q_v$ and recursively define

$$X_\Lambda^{r+1} = X_\Lambda^r \cup \Bigg(\bigcup_{\substack{d(\lambda) \leqslant \mathbf{1}_k, \\ |\lambda| = r+1}} Q_\lambda \Bigg).$$

Then a subset U of X_Λ is open if and only if $U \cap X_\Lambda^r$ is relatively open for each $r \leqslant k$.

Proof. Write $Y_\Lambda = \bigsqcup_{\lambda \in \Lambda} \{\lambda\} \times [0, d(\lambda)]$ and let $q \colon Y_\Lambda \to X_\Lambda$ be the quotient map.

Fix $[\mu, t] \in X_\Lambda$. Then, $\lceil t \rceil - \lfloor t \rfloor \leqslant \mathbf{1}_k$. Moreover, $0 \leqslant t - \lfloor t \rfloor \leqslant \lceil t \rceil - \lfloor t \rfloor$. Let $\lambda = \mu(\lfloor t \rfloor, \lceil t \rceil)$. Whenever $\lfloor t \rfloor_i < \lceil t \rceil_i$ we have $t_i \notin \mathbb{Z}$, and hence $\lfloor t \rfloor_i < t_i < \lceil t \rceil_i$. So

$$[\mu, t] = [\lambda, t - \lfloor t \rfloor] \in Q_\lambda,$$

whence $X_\Lambda = \bigcup_{m \leqslant \mathbf{1}_k} \bigcup_{\lambda \in \Lambda^m} Q_\lambda$.

We now show that the Q_λ are mutually disjoint. Fix λ, μ with $0 \leqslant d(\lambda)$, $d(\mu) \leqslant \mathbf{1}_k$ and suppose that $[\lambda, s] = [\mu, t] \in Q_\lambda \cap Q_\mu$. We must show that $\lambda = \mu$. By Remark 3.1, $\lceil s \rceil = d(\lambda)$, $\lceil t \rceil = d(\mu)$ and $\lfloor s \rfloor = \lfloor t \rfloor = 0$. So $s - \lfloor s \rfloor = t - \lfloor t \rfloor$ forces $s = t$, and thus

$$d(\lambda) = \lceil s \rceil = \lceil t \rceil = d(\mu).$$

The definition of $\sim$ then forces

$$\lambda = \lambda(\lfloor s \rfloor, \lceil s \rceil) = \mu(\lfloor t \rfloor, \lceil t \rceil) = \mu.$$

Fix $\lambda \in \Lambda$ with $d(\lambda) \leqslant \mathbf{1}_k$. The above argument also shows that $[\lambda, t] \mapsto t$ is a well-defined bijection from Q_λ onto $(0, m) \subset \mathbb{R}^k$. So it remains to check that the map is a homeomorphism. To see this, observe that if U is relatively open in Q_λ, then in particular, $\{(\lambda, t) \colon [\lambda, t] \in U\}$ is open in $\{\lambda\} \times (0, d(\lambda)) \subseteq \{\lambda\} \times [0, d(\lambda)]$, and hence $\{t \colon [\lambda, t] \in U\}$ is open in $(0, d(\lambda))$. So $[\lambda, t] \mapsto t$ is an open map. To see that it is continuous, fix an open subset V of $(0, d(\lambda))$. Define $W \subseteq Y_\Lambda$ by

$$W = \bigcup \{(\alpha\lambda\beta, t) \colon s(\alpha) = r(\lambda),\ r(\beta) = s(\lambda) \\ \text{and } t_i - d(\alpha)_i \in V \text{ whenever } d(\lambda)_i \neq 0\}.$$

Then W is open in Y_Λ, so $q(W)$ is open in X_Λ. By the definition of $\sim$, we have $q(W) \cap Q_\lambda = \{[\lambda, t] \colon t \in V\}$. So $[\lambda, t] \mapsto t$ is continuous as required.

To see that $\bar{Q}_\lambda = \{[\lambda, t] \colon t \in [0, d(\lambda)]\}$, we observe that

$$q^{-1}(Q_\lambda) \cap (\{\mu\} \times [0, d(\mu)]) \\ = \{(\mu, t) \colon \mu(\lfloor t \rfloor, \lceil t \rceil) = \lambda \text{ and } d(\lambda)_i \neq 0 \implies t_i \notin \mathbb{N}\}.$$

So the closure in $\{\mu\} \times [0, d(\mu)]$ of $q^{-1}(Q_\lambda) \cap (\{\mu\} \times [0, d(\mu)])$ is

$$\{(\mu, t) \colon \mu(\lfloor t \rfloor, \lceil t \rceil) = \lambda\}$$

and the image of this closure under q is precisely $\{[\lambda, t] \colon t \in [0, d(\lambda)]\}$.

It remains to check that U is open in X_Λ if and only if $U \cap X_\Lambda^r$ is relatively open for each $r \leqslant k$. Of course, if U is open, then each $U \cap X_\Lambda^r$ is relatively open. On the other hand, if each $U \cap X_\Lambda^r$ is relatively open, then in particular, $U = U \cap X_\Lambda = U \cap X_\Lambda^k$ is open. □

Corollary 3.6. *Let Λ be a k-graph. Then X_Λ is a k-dimensional CW-complex with r-skeleton X_Λ^r as defined in Lemma 3.5 for each $r \leqslant k$. In particular, the open cells in the CW-complex are the Q_λ, where $d(\lambda) \leqslant \mathbf{1}_k$, and the closed cells are the $\bar{Q}_\lambda$.*

In fact, [**10**, Appendix A] shows that each k-graph Λ gives rise to a cubical set whose r-cubes are $\bigcup_{m \leqslant \mathbf{1}_k, |m| = r} \Lambda^m$ and [**10**, Theorem B.2] shows that X_Λ is homeomorphic to the topological realization of that cubical set, providing a somewhat indirect alternative proof of the preceding corollary.

Remark 3.7. We will mainly be interested in connected k-graphs in this paper. However, it is not difficult to check that Λ is connected if and only if X_Λ is connected, and that the connected components of X_Λ are precisely the topological realizations of the connected components of Λ.

Remark 3.8. Let X be a connected CW-complex. Since every CW-complex is locally contractible (see [**4**, Proposition A.4]), X is path connected, locally connected and semi-locally simply connected. Hence, X possesses a universal cover. Let Λ be a connected k-graph; then, since X_Λ is a connected CW-complex, it possesses a universal cover as well.

3.1. Examples

In [**10**] a number of examples of 2-graphs are presented using 'planar diagrams'. Here we will show that the topological realizations of those 2-graphs are homeomorphic to the spaces whose homology they were constructed to reflect.

Recall from [**10**] that the 2-cubes of a 2-graph Λ are the morphisms of degree $(1,1)$ and that a commuting diagram (in the category Λ) that includes all 2-cubes as commuting squares is called a *planar diagram* for Λ. To see how these diagrams relate to the topological realizations of the corresponding 2-graphs, we present another description of X_Λ in terms of cubes.

Lemma 3.9. *Let Λ be a k-graph. Then X_Λ is homeomorphic to the quotient of the topological disjoint union $\bigsqcup_{d(\lambda)\leqslant \mathbf{1}_k}\{\lambda\}\times[0,d(\lambda)]$ by the equivalence relation R generated by*

$$\bigcup_{m\leqslant \mathbf{1}_k}\bigcup_{m_i=1}(\{((\lambda\alpha,t),(\lambda,t))\colon \lambda\in\Lambda^{m-e_i},\ \alpha\in s(\lambda)\Lambda^{e_i}\}\\ \cup\{((\alpha\lambda,t+d(\alpha)),(\lambda,t))\colon \lambda\in\Lambda^{m-e_i},\ \alpha\in\Lambda^{e_i}r(\lambda)\}).$$

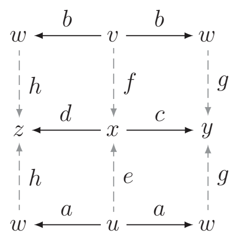
Proof. By [**10**, Theorem B.2], X_Λ is homeomorphic to the topological realization of the associated cubical set. Since in the topological realization of a cubical set every point has a representative in a non-degenerate cube, the topological realization of the cubical set of Λ is precisely the quotient described above. □

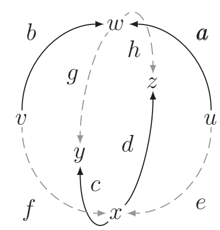
The preceding lemma implies that if E is a planar diagram for a 2-graph Λ, then the topological realization of Λ is homeomorphic to the space obtained by pasting a unit square into each commuting square in E and then identifying all instances of any given edge or vertex in an orientation-preserving way.

To describe the examples in this section, recall that the 1-skeleton, or just skeleton, of a k-graph is the directed graph E with vertices Λ^0 and edges $\bigsqcup_{i=1}^{k}\Lambda^{e_i}$ drawn using k different colours to distinguish the different degrees. There is a complete characterization of which k-coloured graphs give rise to k-graphs [**3, 5**] but for our purposes it suffices to recall the following special case of the construction of [**6**, § 6]: if E is a 2-coloured graph (with edges coloured blue and red, say) and if, for every bi-coloured path ef in E^2 there is a unique bi-coloured $f'e'$ with the same range and source but with the colours occurring in the reverse order, then there is a unique 2-graph Λ whose 1-skeleton is E.

The diagrams in the following examples are reproduced from [**10**]; edges of degree $(1,0)$ are drawn as solid black arrows and edges of degree $(0,1)$ are drawn as grey dashed arrows.

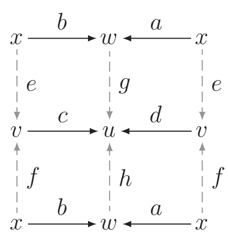
Example 3.10 (Kumjian *et al.* [10, Example 5.4]). Let Λ be the 2-graph with the planar diagram and skeleton shown below:

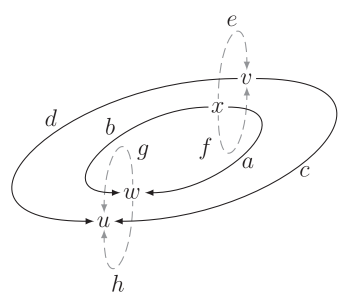




If we paste a square into each commuting square in the planar diagram on the left and then identify all instances of any given edge or vertex, the resulting space is that obtained by pasting a square onto each commuting square in the skeleton on the right, so is homeomorphic to a sphere. In particular, the fundamental group of this 2-graph is trivial.

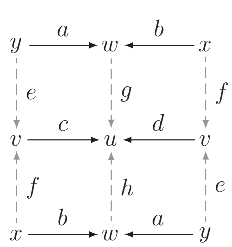
Example 3.11 (Kumjian *et al.* [10, Example 5.5]). Consider the 2-graph Σ with planar diagram on the left and skeleton on the right in the following diagram:

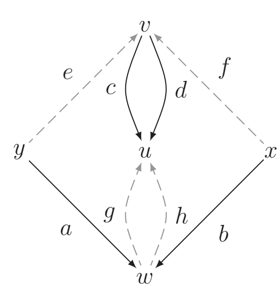




The argument of the preceding example shows that the topological realization of this 2-graph is a 2-torus. In particular, its fundamental group is $\mathbb{Z}^2$.

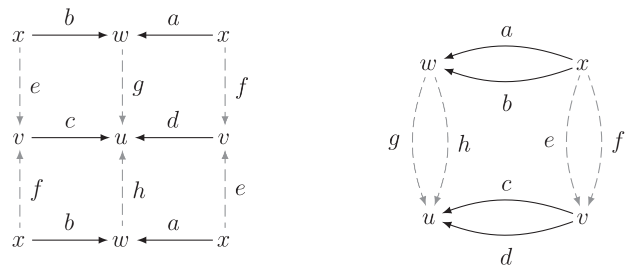
Example 3.12 (Kumjian *et al.* [10, Example 5.6]). Let Λ be the 2-graph with planar diagram on the left and skeleton on the right in the following diagram:





Arguing as in the preceding two examples, we see that the topological realization of this 2-graph is homeomorphic to the projective plane.

Example 3.13. Consider the 2-graph Λ with planar diagram on the left and skeleton on the right in the following diagram:



Arguing as above, we see that the topological realization of this 2-graph is a Klein bottle and, in particular, that its fundamental group is $\mathbb{F}_2/\langle abab^{-1}\rangle$.

4. The fundamental group of a higher-rank graph

In proving that the algebraic and topological fundamental groups of a k-graph are isomorphic, on the algebraic side we need to pass from the fundamental groupoid to the fundamental group. Here we show how to do this.

First of all, we quote the following theorems from topology (see, for example, [**11**]). Let X be a connected k-dimensional CW-complex.

(1) The inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \cong \pi_1(X)$ [**11**, Theorem VII.4.1].

(2) Let $\iota\colon X^1 \hookrightarrow X^2$ denote the inclusion map. Denote by $Q = ((0,0),(1,1))$ the open unit square in $\mathbb{R}^2$, let $\bar{Q}$ be the closed unit square and let $\partial Q = \bar{Q} \setminus Q$. Each 2-cell attached to form X^2 is determined by a 'characteristic map' $f_i\colon \bar{Q} \to X^2$, namely, a continuous map taking Q homeomorphically onto an open set in $X^2 \setminus X^1$ such that $f_i(\partial Q) \subset X^1 \setminus X^2$. Let φ denote a generator of $\pi_1(\partial Q)$. Then $\iota_*\colon \pi_1(X^1) \to \pi_1(X^2)$ is a surjective homomorphism whose kernel is the normal subgroup generated by the images $f_{i*}(\varphi)$ under the characteristic maps [**11**, Theorem VII.2.1].

(3) $\pi_1(X^1) \cong \mathbb{F}_n$, where n is the cardinality of the set of edges in E^1 remaining after a maximal tree has been removed [**11**, Theorem VI.5.2].

We next give some background from [**16**]. Let $\mathcal{C}$ be a small category.

A *congruence relation* on $\mathcal{C}$ is an equivalence relation R on $\mathcal{C}$ such that

$$\text{if } (\alpha,\beta) \in R, \text{ then } s(\alpha) = s(\beta) \text{ and } r(\alpha) = r(\beta) \tag{4.1}$$

and

$$\text{if } (\alpha,\beta),(\lambda,\mu) \in R \text{ and } s(\alpha) = r(\lambda), \text{ then } (\alpha\lambda, \beta\mu) \in R. \tag{4.2}$$

In this case the quotient $\mathcal{C}/R$ is a category and the quotient map $Q\colon \mathcal{C} \to \mathcal{C}/R$ is a functor.

If $S \subset \mathcal{C} \times \mathcal{C}$ satisfies (4.1), then there is a smallest congruence relation on $\mathcal{C}$ containing S, which we say is *generated* by S.

We are primarily interested in the case in which $\mathcal{C}$ is a groupoid, which we will typically denote by $\mathcal{G}$. Also, we will make the standing assumption that $\mathcal{G}$ is connected in the sense that $v\mathcal{G}u \neq \emptyset$ for all units v, u of $\mathcal{G}$.

A subgroupoid $\mathcal{N}$ of $\mathcal{G}$ is *normal* if

$$\mathcal{N}^0 = \mathcal{G}^0 \tag{4.3}$$

and

$$\beta\alpha\beta^{-1} \in \mathcal{N} \text{ for all } \alpha \in \mathcal{N}(u),\ \beta \in \mathcal{G}u, \text{ and } u \in \mathcal{G}^0. \tag{4.4}$$

The following is our main technical tool, allowing us to pass from the fundamental groupoid to the fundamental group. Recall that for a unit u of a groupoid $\mathcal{G}$, we write $\mathcal{G}(u)$ for the isotropy group $u\mathcal{G}u$ at u.

Proposition 4.1. *Let $S \subset \mathcal{G} \times \mathcal{G}$ satisfy (4.1), let R be the congruence relation on $\mathcal{G}$ generated by S, let $\mathcal{H} = \mathcal{G}/R$ be the quotient groupoid and let $Q\colon \mathcal{G} \to \mathcal{H}$ be the quotient map. Fix $u \in \mathcal{G}^0$ and for each $v \in \mathcal{G}^0$ choose $\kappa_v \in v\mathcal{G}u$ with $\kappa_u = u$. Let K be the normal subgroup of $\mathcal{G}(u)$ generated by*

$$\{\kappa_{r(\alpha)}^{-1}\alpha\beta^{-1}\kappa_{r(\alpha)}\colon (\alpha,\beta) \in S\}. \tag{4.5}$$

Then $\mathcal{H}(u) = \mathcal{G}(u)/K$.

Proof. Let $\mathcal{N} = \ker Q$ so that $\mathcal{H} = \mathcal{G}/\mathcal{N}$. Set

$$T = \{\alpha\beta^{-1}\colon (\alpha,\beta) \in S\}.$$

Then $\mathcal{N}$ is the normal subgroupoid of $\mathcal{G}$ generated by T and K is the normal subgroup of $\mathcal{G}(u)$ generated by

$$\bigcup_{v\in\mathcal{G}^0} \kappa_v^{-1}(T \cap \mathcal{G}(v))\kappa_v. \tag{4.6}$$

It suffices to show that $\mathcal{N}(u) = K$.

Since $T \subset \bigcup_{v\in\mathcal{G}^0} \mathcal{G}(v)$, the group $\mathcal{N}(u)$ coincides with the subgroup of $\mathcal{G}(u)$ generated by

$$\bigcup_{\substack{v\in\mathcal{G}^0\\ \beta\in v\mathcal{G}u}} \beta^{-1}(T \cap \mathcal{G}(v))\beta.$$

We need to know that $\mathcal{N}(u)$ is generated as a normal subgroup by the smaller set (4.6). Since K is normal, it is easy to see that for each $v \in \mathcal{G}^0$ and $\beta \in v\mathcal{G}u$ we have

$$\beta^{-1}(T \cap \mathcal{G}(v))\beta \subset \kappa_v^{-1}(T \cap \mathcal{G}(v))\kappa_v,$$

and the result follows. □

Our next goal is to show how to pass from the fundamental groupoid to the fundamental group, which we do in Corollary 4.5. Let Λ be a connected k-graph and let E be its 1-skeleton. Let $\mathcal{G}(\Lambda)$ and $\mathcal{G}(E)$ denote the fundamental groupoids of Λ and E, respectively. We will find it convenient to package the commuting squares of Λ as 'commutativity conditions'* in E: each commuting square is of the form $\lambda = ef = gh$ with $d(e) = d(h) = e_i$, $e(f) = d(g) = e_j$ and $i \neq j$. We regard the edge-paths ef and gh as elements of $\mathcal{G}(E)$, we associate with λ the pair $(ef, gh) \in \mathcal{G}(E) \times \mathcal{G}(E)$ and we let S denote the set of all such pairs (and we abuse terminology by referring to these pairs as commuting squares also).

As discussed in the paragraph following [**12**, Definition 5.6], it follows from [**12**, Theorem 5.5] that

$$\mathcal{G}(\Lambda) \cong \mathcal{G}(E)/R,$$

where R is the congruence relation on $\mathcal{G}(E)$ generated by S. In the following corollary, we identify the fundamental group; this corollary can be regarded as making precise the discussion in [**12**, § 6].

Corollary 4.2. *Let $S \subset \mathcal{G}(E) \times \mathcal{G}(E)$ be the commuting squares of a connected k-graph Λ and let R be the congruence relation on $\mathcal{G}(E)$ generated by S. Then for any vertex $u \in \Lambda^0$ we have*

$$\pi_1(\Lambda, u) \cong \pi_1(E, u)/K,$$

where K is the normal subgroup of $\pi(E, u)$ generated by the set

$$\{\kappa_{r(\alpha)}^{-1} \alpha \beta^{-1} \kappa_{r(\alpha)} \colon (\alpha, \beta) \in S\}.$$

Proof. This follows immediately from Proposition 4.1. □

We now proceed towards our main result on the fundamental groups, namely, $\pi_1(\Lambda, u) \cong \pi_1(X_\Lambda, u)$.

First, some notation. For each $n \geqslant 1$ and each commuting n-cube $\lambda \in \Lambda$, let f_λ be the associated map attaching an n-cell to X^{n-1} in the formation of X^n. Let Q^n be the open unit cube in $\mathbb{R}^n$. Recall that

$$Q_\lambda = f_\lambda(Q^n) \quad \text{and} \quad \bar{Q}_\lambda = f_\lambda(\bar{Q^n}).$$

Moreover, if $e \in E^1$, then the homotopy class $[f_e]$ may be regarded as an element of $\mathcal{G}(X^1)$, the fundamental groupoid of the 1-skeleton.

Lemma 4.3. *Define $\theta \colon E^1 \to \mathcal{G}(X^1)$ by*

$$\theta(e) = [f_e].$$

Then θ extends to a groupoid homomorphism of $\mathcal{G}(E)$ into $\mathcal{G}(X^1)$ and for each $u \in \Lambda^0$, θ restricts to an isomorphism $\theta \colon \pi_1(E, u) \to \pi_1(X^1, u)$.

* This follows the terminology of [**16**]

Proof. This is routine, although the result in this form does not appear to be readily available in the literature. The first part follows from the techniques in the proof of [**17**, Theorem 3.7.3] and then the second part follows from the observations that since Λ is connected, so are the 1-skeleton E and the one-dimensional CW-complex X^1, and both fundamental groups $\pi_1(E,u)$ and $\pi_1(X^1,u)$ are free, with the same number of generators (see, for example, [**17**, Corollary 3.7.5] and [**11**, Theorem 6.5.2]). □

For the following lemma, recall from the beginning of the section that φ denotes the boundary of the unit square in $\mathbb{R}^2$.

Lemma 4.4. *Let K be as in Corollary 4.2 and let L be the normal subgroup of $\pi_1(X^1,u)$ generated by $\{f_\lambda(\varphi)\colon \lambda$ is a commuting square in $\Lambda\}$. Then $\theta(K)=L$.*

Proof. Let $\lambda = ef = gh$, where $d(e)=d(h)=e_i$ and $d(f)=d(g)=e_j$ with $i\neq j$. It then follows from the definitions that

$$\theta(\kappa_{r(e)}^{-1} efh^{-1}g^{-1}\kappa_{r(e)}) \text{ is equal either to } f_\lambda(\varphi) \text{ or to } f_\lambda(\varphi^{-1}),$$

and the lemma follows. □

Corollary 4.5. *The isomorphism $\pi_1(E,u)\cong\pi_1(X^1,u)$ of fundamental groups of 1-skeletons induces an isomorphism $\pi_1(\Lambda,u)\cong\pi_1(X_\Lambda,u)$.*

Proof. This follows from Lemmas 4.3 and 4.4 because

$$\pi_1(\Lambda,u)\cong\pi_1(E,u)/K \quad\text{and}\quad \pi_1(X_\Lambda,u)\cong\pi_1(X^1,u)/L.$$

□

5. Functoriality

We prove here that quasi-morphisms of k-graphs induce continuous maps of topological realizations. In particular, topological realization is a functor from the category of higher-rank graphs and quasi-morphisms to that of topological spaces and continuous maps. For quasi-morphisms that carry edges to edges (for example, k-graph morphisms) the induced map of topological realizations is injective if and only if the original quasi-morphism is injective, and it is surjective if and only if the original quasi-morphism is surjective.

Recall from [**10**] that if $\pi\colon \mathbb{N}^k\to\mathbb{N}^l$ is a homomorphism, and if Λ is a k-graph and Γ an l-graph, then a π-quasi-morphism from Λ to Γ is a functor $\phi\colon\Lambda\to\Gamma$ such that $d(\phi(\lambda))=\pi(d(\lambda))$ for all $\lambda\in\Lambda$.

Proposition 5.1. *Let Λ be a k-graph and Γ an l-graph. Fix a homomorphism $\pi\colon\mathbb{N}^k\to\mathbb{N}^l$. Extend this to a homomorphism $\tilde\pi\colon\mathbb{R}^k\to\mathbb{R}^l$ by $\tilde\pi(t)=\sum_{i=1}^k t_i\pi(e_i)$. Suppose that $\varphi\colon\Lambda\to\Gamma$ is a π-quasi-morphism. There is then a continuous map $\tilde\varphi\colon X_\Lambda\to X_\Gamma$ defined by*

$$\tilde\varphi([\lambda,t])=[\varphi(\lambda),\tilde\pi(t)].$$

Moreover, if $\pi'\colon\mathbb{N}^l\to\mathbb{N}^h$ is another homomorphism, Σ is an h-graph and $\varphi'\colon\Gamma\to\Sigma$ is a π'-quasi-morphism, then $\varphi'\circ\varphi$ is a $\pi'\circ\pi$-quasi-morphism and $\tilde\varphi'\circ\tilde\varphi=(\varphi'\circ\varphi)^\sim$.

Proof. We first show that $\tilde{\varphi}$ is well defined. Suppose that $(\lambda, s) \sim (\mu, t)$. We must show that $(\varphi(\lambda), \tilde{\pi}(s)) \sim (\varphi(\mu), \tilde{\pi}(t))$. We have $\tilde{\pi}(t) = \tilde{\pi}(\lfloor t \rfloor) + \tilde{\pi}(t - \lfloor t \rfloor)$. Since $\tilde{\pi}(\lfloor t \rfloor) = \pi(\lfloor t \rfloor) \in \mathbb{N}^l$, we have $\lfloor \tilde{\pi}(\lfloor t \rfloor) + x \rfloor = \pi(\lfloor t \rfloor) + \lfloor \tilde{\pi}(x) \rfloor$ for all $x \in \mathbb{R}^k$. Hence,

$$\begin{aligned}\tilde{\pi}(s) - \lfloor \tilde{\pi}(s) \rfloor &= (\tilde{\pi}(\lfloor s \rfloor) + \tilde{\pi}(s - \lfloor s \rfloor)) - \lfloor \tilde{\pi}(\lfloor s \rfloor) + \tilde{\pi}(s - \lfloor s \rfloor) \rfloor \\ &= \tilde{\pi}(\lfloor s \rfloor) + \tilde{\pi}(s - \lfloor s \rfloor) - \tilde{\pi}(\lfloor s \rfloor) - \lfloor \tilde{\pi}(s - \lfloor s \rfloor) \rfloor \\ &= \tilde{\pi}(s - \lfloor s \rfloor) - \lfloor \tilde{\pi}(s - \lfloor s \rfloor) \rfloor.\end{aligned}$$

Likewise, $\tilde{\pi}(t) - \lfloor \tilde{\pi}(t) \rfloor = \tilde{\pi}(t - \lfloor t \rfloor) - \lfloor \tilde{\pi}(t - \lfloor t \rfloor) \rfloor$. Since $(\lambda, s) \sim (\mu, t)$, we have $s - \lfloor s \rfloor = t - \lfloor t \rfloor$, and hence

$$\tilde{\pi}(s) - \lfloor \tilde{\pi}(s) \rfloor = \tilde{\pi}(t) - \lfloor \tilde{\pi}(t) \rfloor. \tag{5.1}$$

So to show that $(\varphi(\lambda), \tilde{\pi}(s)) \sim (\varphi(\mu), \tilde{\pi}(t))$, it remains to show that

$$\varphi(\lambda)(\lfloor \tilde{\pi}(s) \rfloor, \lceil \tilde{\pi}(s) \rceil) = \varphi(\mu)(\lfloor \tilde{\pi}(t) \rfloor, \lceil \tilde{\pi}(t) \rceil).$$

Since $\lfloor s \rfloor \leqslant s \leqslant \lceil s \rceil$, we have $\pi(\lfloor s \rfloor) \leqslant \tilde{\pi}(s) \leqslant \pi(\lceil s \rceil)$ and similarly for t. Since $\pi(\lfloor s \rfloor), \pi(\lceil s \rceil) \in \mathbb{N}^l$, it follows from the definition of the floor and ceiling functions that

$$\pi(\lfloor s \rfloor) \leqslant \lfloor \tilde{\pi}(s) \rfloor \leqslant \tilde{\pi}(s) \leqslant \lceil \tilde{\pi}(s) \rceil \leqslant \pi(\lceil s \rceil).$$

Moreover, since $(\lambda, s) \sim (\mu, t)$ and since φ is a π-quasi-morphism, we have

$$\varphi(\lambda)(\pi(\lfloor s \rfloor), \pi(\lceil s \rceil)) = \varphi(\lambda(\lfloor s \rfloor, \lceil s \rceil)) = \varphi(\mu(\lfloor t \rfloor, \lceil t \rceil)) = \varphi(\mu)(\pi(\lfloor t \rfloor), \pi(\lceil t \rceil)).$$

Moreover, (5.1) forces $\lceil \tilde{\pi}(s) \rceil - \lfloor \tilde{\pi}(s) \rfloor = \lceil \tilde{\pi}(t) \rceil - \lfloor \tilde{\pi}(t) \rfloor$. Since $(\lambda, s) \sim (\mu, t)$, we have $s - \lfloor s \rfloor = t - \lfloor t \rfloor$, and hence

$$\lfloor \tilde{\pi}(s) \rfloor - \pi(\lfloor s \rfloor) = \lfloor \tilde{\pi}(t) \rfloor - \pi(\lfloor t \rfloor).$$

So,

$$\begin{aligned}&\varphi(\lambda)(\lfloor \tilde{\pi}(s) \rfloor, \lceil \tilde{\pi}(s) \rceil) \\ &\quad = (\varphi(\lambda)(\tilde{\pi}(\lfloor s \rfloor), \tilde{\pi}(\lceil s \rceil)))(\lfloor \tilde{\pi}(s) \rfloor - \tilde{\pi}(\lfloor s \rfloor), \lfloor \tilde{\pi}(s) \rfloor - \tilde{\pi}(\lfloor s \rfloor) + (\lceil \tilde{\pi}(s) \rceil - \lfloor \tilde{\pi}(s) \rfloor)) \\ &\quad = (\varphi(\mu)(\tilde{\pi}(\lfloor t \rfloor), \tilde{\pi}(\lceil t \rceil)))(\lfloor \tilde{\pi}(t) \rfloor - \tilde{\pi}(\lfloor t \rfloor), \lfloor \tilde{\pi}(t) \rfloor - \tilde{\pi}(\lfloor t \rfloor) + (\lceil \tilde{\pi}(t) \rceil - \lfloor \tilde{\pi}(t) \rfloor)) \\ &\quad = \varphi(\mu)(\lfloor \tilde{\pi}(t) \rfloor, \lceil \tilde{\pi}(t) \rceil).\end{aligned}$$

Hence, $\tilde{\varphi}$ is well defined. To see that it is continuous, fix an open subset U of X_Γ. By the definition of the quotient topology on X_Λ, to show that $\tilde{\varphi}^{-1}(U)$ is open, we must show that for each $\lambda \in \Lambda$, the set $\{s \in [0, d(\lambda)] \colon \tilde{\varphi}([\lambda, s]) \in U\}$ is open in $[0, d(\lambda)]$. Since $\tilde{\varphi}([\lambda, s]) = [\varphi(\lambda), \tilde{\pi}(s)]$, we have

$$\{s \in [0, d(\lambda)] \colon \tilde{\varphi}([\lambda, s]) \in U\} = \tilde{\pi}^{-1}(\{t \in [0, \pi(d(\lambda))] \colon [\varphi(\lambda), t] \in U\}).$$

Since U is open in X_Γ, the set $\{t \in [0, \pi(d(\lambda))] \colon [\varphi(\lambda), t] \in U\}$ is open in $[0, \pi(d(\lambda))]$. So continuity of $\tilde{\pi}$ implies that $\{s \in [0, d(\lambda)] \colon \tilde{\varphi}([\lambda, s]) \in U\}$ is open also. That $\varphi' \circ \varphi$ is a $\pi' \circ \pi$-quasi-morphism is routine. For $[\lambda, t] \in X_\Lambda$, we have

$$\tilde{\varphi}' \circ \tilde{\varphi}([\lambda, t]) = \tilde{\varphi}'([\varphi(\lambda), \tilde{\pi}(t)]) = [\varphi' \circ \varphi(\lambda), \pi' \circ \pi(t)] = (\varphi' \circ \varphi)^{\sim}([\lambda, t]),$$

which establishes the final assertion and completes the proof. □

Remark 5.2. Suppose that $k = l$ and π is the identity map so that $\varphi\colon \Lambda \to \Gamma$ is a morphism of k-graphs. Using the decompositions

$$X_\Lambda = \bigsqcup_{0 \leqslant d(\lambda) \leqslant \mathbf{1}_k} Q_\lambda \quad \text{and} \quad X_\Gamma = \bigsqcup_{0 \leqslant d(\gamma) \leqslant \mathbf{1}_l} Q_\gamma$$

of Lemma 3.5, we see that $\tilde{\varphi}$ is determined by $\tilde{\varphi}([\lambda, t]) = [\varphi(\lambda), t]$ whenever $d(\lambda) \leqslant \mathbf{1}_k$ and $t \in (0, d(\lambda))$.

Proposition 5.3. *In addition to the hypotheses of Proposition 5.1, suppose that π is rectilinear in the sense that each $\pi(e_i)$ has the form $n_i e_{j_i}$ for some $n_i \in \mathbb{N}$ and $j_i \leqslant l$, and suppose also that φ is weakly surjective in the sense that for each $\gamma \in \Gamma$ there exists $\lambda \in \Lambda$ and $p, q \in \mathbb{N}^k$ with $p \leqslant q \leqslant \pi(d(\lambda))$ such that $\gamma = \varphi(\lambda)(p, q)$. Then $\tilde{\varphi}$ is surjective. If $k = l$ and π is the identity so that φ is a k-graph morphism, and if φ is injective, then $\tilde{\varphi}$ is also injective.*

Proof. Suppose that π is rectilinear and φ is weakly surjective. Fix $[\gamma, t] \in X_\Gamma$. Fix $\lambda \in \Lambda$ and $p \in \mathbb{N}^l$ such that $\varphi(\lambda)(p, p+d(\gamma)) = \gamma$. Then, $[\gamma, t] = [\varphi(\lambda), p+t]$. By hypothesis on π we have $\pi(d(\lambda)) = \sum_{i=1}^k d(\lambda)_i n_i e_{j_i}$. For $h \leqslant l$ define $\alpha_h = (n+t)_h / \pi(d(\lambda))_h$. Let $s = \sum_{i=1}^k \alpha_{j_i} d(\lambda)_i e_i$. Then

$$\tilde{\pi}(s) = \sum_{i=1}^k \frac{(n+t)_{j_i}}{\pi(d(\lambda))_{j_i}} d(\lambda)_i \pi(e_i).$$

Thus, for $h \leqslant l$, we have

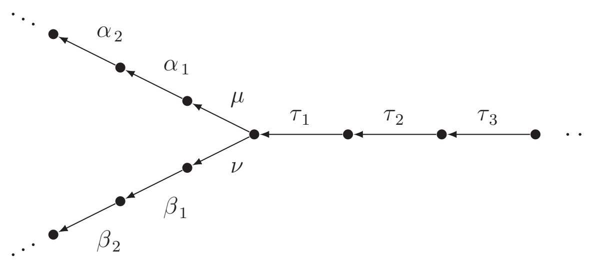
$$\pi(s)_h = \sum_{j_i = h} \frac{(n+t)_h}{\pi(d(\lambda))_h} \pi(d(\lambda)_i e_i) = \frac{(n+t)_h}{\pi(d(\lambda))_h} \pi(d(\lambda))_h = (n+t)_h.$$

Since each $\alpha_h \in [0, 1]$, we have $s \in [0, d(\lambda)]$ and we have $\tilde{\varphi}([\lambda, s]) = [\varphi(\lambda), n+t] = [\gamma, t]$, as required.

Now suppose that φ is an injective k-graph morphism. Suppose that $\tilde{\varphi}([\lambda, s]) = \tilde{\varphi}([\mu, t])$. Then $[\varphi(\lambda), \tilde{\pi}(s)] = [\varphi(\mu), \tilde{\pi}(t)]$. In particular, $\varphi(\lambda)(\lfloor s \rfloor, \lceil s \rceil) = \varphi(\mu)(\lfloor t \rfloor, \lceil t \rceil)$. Since φ is injective, it follows that $\lambda(\lfloor s \rfloor, \lceil s \rceil) = \mu(\lfloor t \rfloor, \lceil t \rceil)$. Moreover, the equality $[\varphi(\lambda), \tilde{\pi}(s)] = [\varphi(\mu), \tilde{\pi}(t)]$ implies that $s - \lfloor s \rfloor = t - \lfloor t \rfloor$. Hence, $[\lambda, s] = [\mu, t]$. $\square$

In the preceding proposition, the hypothesis used to establish that $\tilde{\varphi}$ is injective could be weakened to require just that π maps generators to generators and φ is injective. However, these two hypotheses imply that Λ consists entirely of paths of dimension at most l and that π is a generator-to-generator injection on degrees of such paths. So, locally φ is just a relabelling of an injective k-graph morphism. The next example shows that it does not suffice to ask merely that each of π and φ be injective.

Example 5.4. Let E be the 1-graph



Define $\pi\colon \mathbb{N} \to \mathbb{N}$ by $\pi(n) = 2n$. Define $\varphi\colon E^* \to E^*$ by

$$\varphi(\mu) = \mu\tau_1, \qquad \varphi(\nu) = \nu\tau_1, \qquad \varphi(\tau_i) = \tau_{2i}\tau_{2i+1},$$
$$\varphi(\alpha_i) = \alpha_{2i}\alpha_{2i-1} \quad \text{and} \quad \varphi(\beta_i) = \beta_{2i}\beta_{2i-1}.$$

Then π is injective and φ is an injective π-quasi-morphism. However, for $t \in [1/2, 1]$, we have $\tilde{\varphi}(\mu, t) = [\tau_1, 2t-1] = \tilde{\varphi}(\nu, t)$. So $\tilde{\varphi}$ is not injective.

The following result shows that the topological realization functor $(\Lambda \mapsto X_\Lambda, \varphi \mapsto \tilde{\varphi})$ is faithful.

Lemma 5.5. *Let $\varphi, \psi\colon \Lambda \to \Gamma$ be k-graph morphisms such that $\tilde{\varphi} = \tilde{\psi}$. Then, $\varphi = \psi$.*

Proof. The equality $\tilde{\varphi} = \tilde{\psi}$ implies that φ and ψ agree on commuting cubes and, in particular, on edges. Since Λ is generated by its edges, φ and ψ must therefore coincide by functoriality (see Proposition 5.1). □

Lemma 5.6. *Let $\varphi\colon \Lambda \to \Gamma$ be a morphism of k-graphs and let $\tilde{\varphi}\colon X_\Lambda \to X_\Gamma$ be the associated map between the topological realizations. Let $u \in \Lambda^0$ and let $v = \varphi(u)$. Then the isomorphisms $\pi_1(\Lambda, u) \cong \pi_1(X_\Lambda, u)$ and $\pi_1(\Gamma, v) \cong \pi_1(X_\Gamma, v)$ of Corollary 4.5 make the diagram*

$$\begin{array}{ccc} \pi_1(\Lambda, u) & \xrightarrow{\cong} & \pi_1(X_\Lambda, u) \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle \tilde{\varphi}_*} \\ \pi_1(\Gamma, v) & \xrightarrow[\cong]{} & \pi_1(X_\Gamma, v) \end{array}$$

commute.

Proof. Since φ is a k-graph morphism, it restricts to a morphism $\varphi^1\colon E_\Lambda \to E_\Gamma$ of 1-skeletons. Proposition 5.1 implies that φ^1 induces a homomorphism $\varphi^1_*\colon \pi_1(E_\Lambda) \to \pi_1(E_\Gamma)$. Lemma 4.3 shows that φ^1_* is compatible with the induced homomorphism $\tilde{\varphi}^1_*\colon \pi_1(X^1_\Lambda) \to \pi_1(X^1_\Gamma)$. The result then follows from Corollary 4.5. □

6. Topological realizations and coverings of higher-rank graphs

We investigate the relationship between covering maps in the algebraic and topological senses. We will assume throughout this section that all k-graphs are connected and all spaces are connected CW-complexes.

Let Λ be a k-graph. Recall from [**13**] that a *covering* of Λ is a surjective k-graph morphism $p\colon \Omega \to \Lambda$ such that for all $v \in \Omega^0$, p maps Ωv bijectively onto $\Lambda p(v)$ and maps $v\Omega$ bijectively onto $p(v)\Lambda$.

Our main purpose here is to prove the following theorem.

Theorem 6.1. *If $p\colon \Omega \to \Lambda$ is a covering of k-graphs, then $\tilde{p}\colon X_\Omega \to X_\Lambda$ is a covering map of the topological realizations.*

We know that $\tilde{p}$ is a continuous surjection. We must show that X_Λ is covered by open sets U that are *evenly covered*, i.e. $\tilde{p}^{-1}(U)$ is a disjoint union of open sets that $\tilde{p}$ maps homeomorphically onto U.

Observation 6.2. *Let $x = [\lambda, t] \in X_\lambda$ with $d(\lambda) \leqslant \mathbf{1}_k$ and $t \in Q_\lambda$.*

(1) *It follows from the covering property of p that for each $y \in p^{-1}(x)$ there is a unique ν with $d(\nu) \leqslant \mathbf{1}_k$ such that $y \in Q_\nu$ and $p(\nu) = \lambda$.*

(2) *For each $i = 1, \dots, k$, we have $0 < t_i < 1$ if $d(\lambda)_i = 1$, and $t_i = 0$ if $d(\lambda)_i = 0$.*

(3) *Suppose that $d(\mu) \leqslant \mathbf{1}_k$. Then $x \in \bar{Q}_\mu$ if and only if there exists $s \leqslant d(\mu)$ such that $[\lambda, t] = [\mu, s]$, in which case we have:*

  (a) *$s_i = t_i$ if $d(\lambda)_i = 1$;*

  (b) *$s_i \in \{0, 1\}$ if $d(\lambda)_i = 0$ and $d(\mu)_i = 1$.*

Definition 6.3. Fix $\lambda \in \Lambda$ with $d(\lambda) \leqslant \mathbf{1}_k$ and $t \in (0, d(\lambda))$. Let $x = [\lambda, t] \in Q_\lambda$. We define N_x to be the set of all $[\mu, s] \in X_\Lambda$ satisfying:

(1) $d(\mu) \leqslant \mathbf{1}_k$;

(2) $x \in \bar{Q}_\mu$;

(3) $0 < s_i < 1$ if $d(\lambda)_i = 1$;

(4) $|s_i - r_i| < 1/2$ if $[\mu, r] = [\lambda, t]$, $d(\lambda)_i = 0$ and $d(\mu)_i = 1$.

Lemma 6.4. *N_x is an open neighbourhood of x.*

Proof. Taking $\mu = \lambda$ and $s = t$ in the definition of N_x shows that $x \in N_x$. By definition of the weak topology, it suffices to show that if $d(\mu) \leqslant \mathbf{1}_k$, then the intersection $N_x \cap \bar{Q}_\mu$ is relatively open and furthermore, in order to show that $N_x \cap \bar{Q}_\mu$ is relatively open, it suffices to show that $V_\mu \subset [0, d(\mu)]$ defined by

$$V_\mu = \{s \in [0, d(\mu)]\colon [\mu, s] \in N_x \cap \bar{Q}_\mu\}$$

is open. We consider three cases.

(1) If $x \notin \bar{Q}_\mu$, then $V_\mu = \emptyset$ is open.

(2) If $\mu = \lambda$, then
$$V_\mu = \{s \in [0, d(\mu)] \colon 0 < s_i < 1 \text{ if } d(\mu)_i = 1\}.$$

(3) If $\mu \neq \lambda$ and $x \in \bar{Q}_\mu$, then $|\mu| > |\lambda|$ and x is in the boundary of the open cell Q_μ. For each $i \in \{1, \dots, k\}$ write
$$V_\mu^i = \{s_i \colon s \in V_\mu\}.$$
Then $V_\mu = \prod_{i=1}^k V_\mu^i$. So it suffices to show that each V_μ^i is relatively open in $[0, d(\mu)_i]$. Fix $i \leqslant k$. If $d(\mu)_i = 0$, then $V_\mu^i = \{0\} = [0, d(\mu)_i]$. If $d(\lambda)_i = 1$, then $V_\mu^i = (0,1)$. So we turn to the remaining case $d(\lambda)_i = 0$ and $d(\mu)_i = 1$. Then $V_\mu^i = [0, 1/2)$ or $(1/2, 1]$ *except* in the following two circumstances:

(a) $|\lambda| = 0$ and there exists $n \in \mathbb{N}^k$ such that
$$\lambda = \mu(n) = \mu(n + e_i);$$
or

(b) $|\lambda| = 1$ and there exists $n \in \mathbb{N}^k$ such that
$$\lambda = \mu(n, n + e_i) \quad \text{and} \quad \mu(n) = \mu(n + e_i).$$

In each of (a) and (b) we have $V_\mu^i = [0, 1/2) \cup (1/2, 1]$. We have shown that in every case V_μ^i is an open subset of $[0, d(\mu)_i]$.

□

Lemma 6.5. *N_x is evenly covered.*

Proof. Since the map $\tilde{p} \colon X_\Omega \to X_\Lambda$ has the form $\tilde{p}([\mu, t]) = [p(\mu), t]$, the inverse image $\tilde{p}^{-1}(N_x)$ is the disjoint union over $y \in \tilde{p}^{-1}(x)$ of the corresponding neighbourhoods N_y. We must show that:

(1) for each $y \in \tilde{p}^{-1}(x)$, the map $\tilde{p}$ restricts to a homeomorphism of N_y onto N_x;

(2) for distinct $y, z \in \tilde{p}^{-1}(x)$, we have $N_y \cap N_z = \emptyset$.

(1) Let $q_\Lambda \colon \bigsqcup_{d(\mu) \leqslant \mathbf{1}_k} \{k\} \times [0, d(\mu)] \to X_\Lambda$ be the quotient map and similarly for q_Ω. We have
$$q_\Lambda^{-1}(N_x) = \bigsqcup_{d(\mu) \leqslant \mathbf{1}_k} \{k\} \times V_\mu,$$
where V_μ is open in $[0, d(\mu)]$ for each μ, and similarly
$$q_\Omega^{-1}(N_y) = \bigsqcup_{d(\nu) \leqslant \mathbf{1}_k} \{k\} \times V_\nu^y,$$
where V_ν^y is open in $[0, d(\nu)]$ for each ν.

Define

$$p'\colon \bigsqcup_{d(\nu)\leqslant \mathbf{1}_k} \{k\}\times V^y_\nu \to \bigsqcup_{d(\mu)\leqslant \mathbf{1}_k}\{k\}\times V_\mu \quad \text{by } p'(\mu,t)=(p(\mu),t).$$

Then p' is a homeomorphism because

$$p'(\{\nu\}\times V^y_\nu)=\{p(\nu)\}\times V_{p(\nu)}.$$

Also, $(\nu,t)\sim(\omega,r)$ in $\bigsqcup_{d(\nu)\leqslant \mathbf{1}_k}\{k\}\times V^y_\nu$ if and only if $p'(\nu,t)\sim p'(\omega,r)$ in $\bigsqcup_{d(\nu)\leqslant \mathbf{1}_k}\{k\}\times V^y_{p(\nu)}$.

(2) Suppose not and take $w\in N_y\cap N_z$. Let μ and ν be the unique elements of $p^{-1}(\lambda)$ such that $y\in Q_\mu$ and $z\in Q_\nu$. Then $y=[\mu,t]$ and $z=[\nu,t]$. Let α be the unique cube in Ω such that $w\in Q_\alpha$ and let s be the unique element of $(0,d(\alpha))$ such that $w=[\alpha,s]$. By Observation 6.2 (1), we cannot have $p(\alpha)=\lambda$. Therefore, we must have $|\alpha|>|\mu|$. Since $y\in\bar{Q}_\alpha$, by Observation 6.2 (1) there exists $a\leqslant d(\alpha)$ such that $y=[\alpha,a]$, and $a_i=t_i$ except for those i for which $d(\mu)_i=0$ and $d(\alpha)_i=1$. Similarly, there exists $b\leqslant d(\alpha)$ such that $z=[\alpha,b]$, and $b_i=t_i$ except when $d(\nu)_i=0$ and $d(\alpha)_i=1$. Since $y\neq z$, there exists i such that $a_i\neq b_i$ and then we must have $d(\mu)_i=d(\nu)_i=0$ and $d(\alpha)_i=1$. Since $w\in N_y\cap N_z$, we have $|s_i-a_i|<1/2$ and $|s_i-b_i|<1/2$. But this is a contradiction since $0<s_i<1$ and a_i and b_i are distinct integers. □

Proof of Theorem 6.1. This follows from Lemma 6.4 and Lemma 6.5 because Λ is covered by the open sets N_x for $x\in X_\Lambda$. □

Lemma 6.6. *Let Λ be a k-graph and let $q\colon Y\to X_\Lambda$ be a covering map. Then there are a k-graph Ω, a covering $p\colon\Omega\to\Lambda$ and a homeomorphism $\phi\colon Y\to X_\Omega$ such that $q=\tilde{p}\circ\phi$.*

Proof. Choose $u\in\Lambda^0$ and $v\in q^{-1}(u)$. Let H' be the subgroup $q_*(\pi_1(Y,v))$ of $\pi_1(X_\Lambda,u)$ and let H be the subgroup of $\pi_1(\Lambda,u)$ corresponding to H' under the isomorphism of Corollary 4.5. By [**13**, Theorem 2.8], there are a connected k-graph Ω, a covering $p\colon\Omega\to\Lambda$ and $w\in p^{-1}(u)$ such that $H=p_*(\pi_1(\Omega,w))$. Then $\tilde{p}\colon X_\Omega\to X_\Lambda$ is a covering map and by Lemma 5.6 we have a commuting diagram

$$\begin{array}{ccc} \pi_1(\Omega,w) & \xrightarrow{\cong} & \pi_1(X_\Omega,w) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \tilde{p}_*} \\ \pi_1(\Lambda,u) & \xrightarrow[\cong]{} & \pi_1(X_\Lambda,u) \end{array}$$

Thus, $\tilde{p}_*(\pi_1(X_\Omega,w))=H'$, so by [**11**, Corollary V.6.4], the coverings (Y,q) and $(X_\Omega,\tilde{p})$ are isomorphic, that is, there is a homeomorphism $\phi\colon Y\to X_\Omega$ such that $q=\tilde{p}\circ\phi$. □

For a fixed k-graph Λ, we have a category $\mathsf{AlgCov}(\Lambda)$ of coverings of Λ and we also have a category $\mathsf{TopCov}(X_\Lambda)$ of coverings of the topological realization. Each morphism

$$\begin{array}{ccc} \Omega & \xrightarrow{\ \varphi\ } & \Gamma \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle q} & \\ & \Lambda & \end{array}$$

in $\mathsf{AlgCov}(\Lambda)$ determines a morphism (also-called a deck transformation)

$$\begin{array}{ccc} X_\Omega & \xrightarrow{\ \tilde{\varphi}\ } & X_\Gamma \\ & {\scriptstyle \tilde{p}}\searrow \qquad \swarrow{\scriptstyle \tilde{q}} & \\ & X_\Lambda & \end{array}$$

in $\mathsf{TopCov}(X_\Lambda)$.

Theorem 6.7. *With the above notation, the assignments* $(\Omega, p) \mapsto (X_\Omega, \tilde{p})$ *and* $\varphi \mapsto \tilde{\varphi}$ *give an equivalence* $\Phi\colon \mathsf{AlgCov}(\Lambda) \xrightarrow{\sim} \mathsf{TopCov}(X_\Lambda)$*. In particular, if* (Ω, p) *is a universal cover of* Λ*, then* $(X_\Omega, \tilde{p})$ *is a universal cover of* X_Λ*.*

Proof. Φ is functorial because $\Lambda \mapsto X_\Lambda$ is. We must show that Φ is

(1) faithful,

(2) full and

(3) essentially surjective.

(1) This follows from Corollary 5.5.

(2) Let $p\colon \Omega \to \Lambda$ and $q\colon \Gamma \to \Lambda$ be coverings and suppose that $\psi\colon (X_\Omega, \tilde{p}) \to (X_\Gamma, \tilde{q})$ is a morphism. Choose $v \in \Omega^0$ and let $u = p(v)$ and $w = \psi(v)$. We have $\tilde{q}(w) = \tilde{q} \circ \psi(v) = \tilde{p}(v) \in \Lambda^0$. Since q preserves degree, $\tilde{q}$ maps open n-cubes to open n-cubes and, in particular, $\tilde{q}^{-1}(\Lambda^0) = \Gamma^0$. So, $w \in \Gamma^0$. We have

$$\tilde{q}_* \circ \psi_* = \tilde{p}_*\colon \pi_1(X_\Omega, v) \to \pi_1(X_\Lambda, u),$$

so

$$\tilde{p}_*(\pi_1(X_\Omega, v)) \subset \tilde{q}_*(\pi_1(X_\Gamma, w)),$$

and hence

$$p_*(\pi_1(\Omega, v)) \subset q_*(\pi_1(\Gamma, w)).$$

Thus, by [**13**, Theorem 2.2] there is a unique morphism $\varphi\colon (\Omega, p) \to (\Gamma, q)$ taking v to w. Then both $\tilde{\varphi}$ and ψ are morphisms from $(X_\Omega, \tilde{p})$ to $(X_\Gamma, \tilde{q})$ taking v to w, and hence must coincide, by [**11**, Lemma 6.3]*.

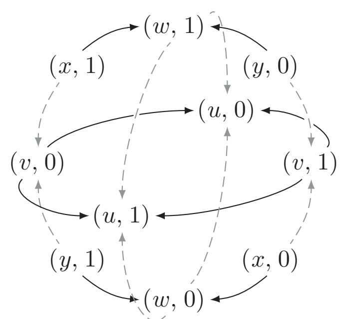
* The two quoted references do not explicitly address uniqueness of morphisms, but this follows by uniqueness of liftings.

(3) We must show that every object (Y, q) in $\mathsf{TopCov}(X_\Lambda)$ is isomorphic to one in the image of Φ. But this is exactly what Lemma 6.6 says.

The final assertion follows from the universal properties of universal covers. By [**13**, Theorem 2.7], Λ has a universal cover (Ω, p). Let $u \in \Omega^0$. Then, by [**13**, Theorem 2.7], $p_*\pi_1(\Omega, u)$ is trivial. Hence, Lemma 5.6 implies that $\tilde{p}_*\pi_1(X_\Omega, u)$ is also trivial. It follows that X_Ω is simply connected and therefore X_Ω is a universal cover of X_Λ. □

Remark 6.8. Let (Ω, p) be a covering and fix $v \in \Omega^0$. Generalizing from the context of directed graphs (see [**1**]), we call (Ω, p) *regular* if $p_*(\pi_1(\Omega, v))$ is a normal subgroup of $\pi_1(\Lambda, p(v))$. See [**13**, Corollary 2.4] for a number of equivalent conditions. The corresponding property of topological coverings is well known (see, for example, [**11**, p. 134]). Using Lemma 5.6, it is easy to verify that the covering (Ω, p) is regular if and only if the topological realization $(X_\Omega, \tilde{p})$ is.

Example 6.9 (Kumjian *et al.* [10, Example 5.8]). Arguing as in Examples 3.10–3.13, we see that the topological realization of the 2-graph with the skeleton



is a sphere. Moreover, the action α of $\mathbb{Z}/2\mathbb{Z}$ on Λ that interchanges opposite vertices induces the antipodal map on the topological realization, so the quotient is the projective plane. Indeed, as discussed in [**10**], this 2-graph is a skew product of the 2-graph of Example 3.12 by $\mathbb{Z}/2\mathbb{Z}$; the action α is then translation in $\mathbb{Z}/2\mathbb{Z}$ in the skew product, and hence the quotient 2-graph is exactly the 2-graph of Example 3.12.

Remark 6.10. Interestingly, although the 2-graphs in Examples 3.10 and 6.9 have topological realizations homeomorphic to the surface of a 3-cube, it is not hard to check that there is no 2-graph whose cell complex consists of the faces of the cube.

7. Towers of coverings and projective limits

As in [**8**], fix a sequence $(\Lambda_n)_{n=0}^\infty$ of row-finite k-graphs with no sources and a sequence $(p_n)_{n=1}^\infty$ of finite-to-one coverings $p_n\colon \Lambda_n \to \Lambda_{n-1}$. There is a unique $(k+1)$-graph $\Sigma = \Sigma(\Lambda_n, p_n)$* such that $\Sigma^0 = \bigsqcup_{n=0}^\infty \Lambda_n^0$, $\Sigma^{e_i} = \bigsqcup_{n=0}^\infty \Lambda_n^{e_i}$ for $i \leqslant k$ and $\Sigma^{e_{k+1}} =$

* In [**8**] this $(k+1)$-graph was denoted $\varprojlim(\Lambda_n, p_n)$. But in this paper we shall be discussing the construction of [**8**] in close proximity with projective limits of topological spaces, so the notation of [**8**] would be very confusing here. The notation $\Sigma(\Lambda_n, p_n)$ is reminiscent of the notation for the linking graph associated to the Ω_1-system of k-morphs obtained from the system (Λ_n, p_n) (see [**9**, Example 5.3 (iv)]).

$\bigsqcup_{n=1}^{\infty}\{f_v : v \in \Lambda_n^0\}$, and with structure maps on $\bigsqcup_{n=0}^{\infty}\Lambda_n \subseteq \Sigma$ inherited from the Λ_n, range and source on $\Sigma^{e_{k+1}}$ given by $s(f_v) = v$ and $r(f_v) = p_n(v)$ for $v \in \Lambda_n^0$, and factorization rules for edges of degree e_{k+1} determined by $f_{r(\lambda)}\lambda = p(\lambda)f_{s(\lambda)}$ (the unique path-lifting property ensures that this specifies a valid factorization property). See [**8**, Proposition 2.7 and Corollary 2.15] for details.

For $0 \leqslant m \leqslant n$, we write p_m^n for the map $p_{m+1} \circ \cdots \circ p_n : \Lambda_n \to \Lambda_m$. For each $n \geqslant 0$ and each $v \in \Lambda_n^0 \subseteq \Sigma^0$, the path $F_v := f_{p_0^n(v)} f_{p_1^n(v)} \cdots f_{p_n(v)}$ is the unique path $F_v \in \Lambda_0^0 \Sigma v$ such that $d(F_v) \in \mathbb{N}e_{k+1}$.

It is also shown in [**14**] that given a system (Λ_n, p_n) as above, the projective limit

$$\varprojlim(\Lambda_n, p_n) = \left\{(\lambda_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty}\Lambda_n : p_n(\lambda_n) = \lambda_{n-1} \text{ for all } n \geqslant 1\right\}$$

of the discrete spaces Λ_n forms a topological k-graph in the sense of Yeend [**18**] with structure maps defined coordinatewise and degree map given by $d((\lambda_n)_{n=1}^{\infty}) = d(\lambda_0)$. The thrust in [**14**] is that Yeend's topological-graph C^*-algebra of $\varprojlim(\Lambda_n, p_n)$ is isomorphic to a full corner in the k-graph algebra $C^*(\Sigma)$. Here we are interested in topological aspects of the two constructions.

We shall show that the fundamental group of Σ is identical to that of Λ_0. We will also propose a natural notion of the topological realization X_Λ of a topological k-graph Λ and we then show that $X_{\varprojlim(\Lambda_n, p_n)}$ is homeomorphic to the projective limit $\varprojlim(X_{\Lambda_n}, \tilde{p}_n)$ of the topological realizations of the Λ_n under the induced coverings arising from functoriality of the topological-realization construction. We regard this as evidence that our proposed notion of the topological realization of a topological k-graph is a reasonable one in the sense that it ensures that topological realization is continuous with respect to projective limits. We deduce that the fundamental group of $X_{\varprojlim(\Lambda_n, p_n)}$ is isomorphic to the projective limit of the fundamental groups of the X_{Λ_n}.

Lemma 7.1. *Let (Λ_n, p_n) be a system of coverings of k-graphs and let $\Sigma = \Sigma(\Lambda_n, p_n)$, as above. Suppose that $w \in \mathcal{G}(\Sigma)$ satisfies $r(w) \in \Lambda_0$. Then $w = w' F_{s(w)}$ for some $w' \in \mathcal{G}(\Lambda_0)$. Moreover, for any $v \in \Lambda_0^0$ we have $v\mathcal{G}(\Sigma)v = v\mathcal{G}(\Lambda_0)v$.*

Proof. Fix $w \in \mathcal{G}(\Sigma)$ with $r(w) \in \Lambda^0$. Write $w = \lambda_0\lambda_1^{-1}\lambda_2 \cdots \lambda_n^{(-1)^n}$ with each $\lambda_i \in \Sigma$ (we can always do this, by setting $\lambda_0 = r(w)$ if necessary). We argue by induction on n.

For the base case $n = 0$, consider $\lambda_0 \in \Sigma$ with $r(\lambda) \in \Lambda_0$. Let $p = d(\lambda)_{k+1}$ and let $m = d(\lambda) - pe_{k+1}$. By the factorization property, $\lambda = \mu\nu$ for some $\mu \in \Sigma^m$ and $\nu \in \Sigma^{pe_{k+1}}$. Since $d(\mu)_{k+1} = 0$, we have $\mu \in \bigsqcup_{n=0}^{\infty}\Lambda_n$ and, since $r(\mu) \in \Lambda_0^0$, we then have $\mu \in \Lambda_0^0$. In particular, $s(\mu) \in \Lambda_0^0$, and hence $r(\nu) \in \Lambda_0^0$. Moreover, $s(\nu) = s(w)$, so $\nu \in \Lambda_0^0\Sigma s(w)$ with $d(\nu) \in \mathbb{N}e_{k+1}$. Since $F_{s(w)}$ is the unique such path, setting $w' = \mu \in \mathcal{G}(\Lambda_0)$, we have $w = w' F_{s(w)}$ as required.

Now fix $n \geqslant 1$ and suppose that w can be written in the desired form whenever $w = \lambda_0\lambda_1^{-1}\lambda_2 \cdots \lambda_{n-1}^{(-1)^{n-1}}$ for some $\lambda_i \in \Sigma$. Fix an element $\lambda_0\lambda_1^{-1}\lambda_2 \cdots \lambda_n^{(-1)^n}$. Let $v = s(\lambda_{n-1}^{(-1)^{n-1}})$. Applying the inductive hypothesis to $\lambda_0\lambda_1^{-1}\lambda_2 \cdots \lambda_{n-1}^{(-1)^{n-1}}$ we obtain $w = zF_v\lambda_n^{(-1)^n}$ for some $z \in \mathcal{G}(\Lambda_0)$. We now consider two cases: $(-1)^n = 1$ or $(-1)^n = -1$.

First suppose that $(-1)^n = 1$. Then $\lambda_n^{(-1)^n} = \lambda_n$ with $r(\lambda_n) = v$ and we have $w = zF_v\lambda_n$. By the factorization property, we can express $F_v\lambda_n = \mu\eta$, where $d(\eta) = d(F_v) + d(\lambda_n)_{k+1}e_{k+1}$. We then have $d(\mu)_{k+1} = 0$ and, since $r(\mu) = s(w') \in \Lambda_0^0$, we then have $s(\mu) \in \Lambda_0^0$ and it follows as in the base case that $\eta = F_{s(w)}$. Hence, $w = (z\mu)F_{s(w)}$ has the desired form.

Now suppose that $(-1)^n = -1$, so $\lambda_n^{(-1)^n} = \lambda_n^{-1}$, with $s(\lambda_n) = v$. Factorize $\lambda_n = \nu\mu$, where $d(\nu) = d(\lambda_n)_{k+1}e_{k+1}$; so $w = zF_v\mu^{-1}\nu^{-1}$. Let q be the integer such that $v \in \Lambda_q^0$. By definition of the factorization rules in Σ, we have $F_{r(\mu)}\mu = p_0^q(\mu)F_{s(\mu)} = p_0^q(\mu)F_v$. Let $\mu_0 = p_0^q(\mu)$ and let $\gamma = F_{r(\mu)}\mu = \mu_0 F_v$. Then,

$$F_v\mu^{-1} = F_v\gamma^{-1}\gamma\mu^{-1} = \mu_0^{-1}F_{r(\mu)}.$$

Hence, $w = z\mu_0^{-1}F_{r(\mu)}\nu^{-1}$. Then $w' := z\mu_0^{-1}$ belongs to $\mathcal{G}(\Lambda_0)$. Since $d(\nu) = |d(\nu)|e_{k+1}$ and $s(\nu) = s(F_{r(\mu)}) = r(\mu)$, if we write m for the integer such that $r(\mu) \in \Lambda_m^0$, then $\nu = f_{p_q^m(r(\mu))} \cdots f_{p_m(r(\mu))} f_{r(\mu)}$. In particular,

$$F_{r(\mu)} = F_{p_q^m(r(\mu))}\nu = F_{r(\nu)}\nu = F_{s(w)}\nu.$$

Thus, $w = z\mu_0^{-1}F_{s(w)}\nu\nu^{-1} = (z\mu_0)F_{s(w)}$ has the required form. The first assertion of the lemma now follows by induction. For the second statement, observe that if $v \in \Lambda_0^0$, then $F_v = v$. □

Recall that a topological k-graph is a small category equipped with a second-countable locally compact Hausdorff topology and a continuous degree map $d\colon \Lambda \to \mathbb{N}^k$ satisfying the factorization property such that $r\colon \Lambda \to \Lambda^0$ is continuous, $s\colon \Lambda \to \Lambda^0$ is a local homeomorphism and composition is continuous on the space of composable pairs in Λ regarded as a subspace of $\Lambda \times \Lambda$.

Definition 7.2. Let Λ be a topological k-graph. Let $Y_\Lambda = \{(\lambda, n) \in \Lambda \times \mathbb{R}^k : 0 \leqslant n \leqslant d(\lambda)\}$ and endow Y_Λ with the relative topology induced by the product topology on $\Lambda \times \mathbb{R}^k$. Equation (3.1) determines an equivalence relation on Y_Λ just as in § 3 and we define $X_\Lambda = Y_\Lambda/\sim$ endowed with the quotient topology. We call X_Λ the *topological realization of* Λ.

Now, recall from [**14**, § 6] that if (Λ_n, p_n) is a system of coverings, then the topological projective limit

$$\varprojlim(\Lambda_n, p_n) = \left\{(\lambda_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \Lambda_n : p_n(\lambda_n) = \lambda_{n-1} \text{ for all } n\right\}$$

is a topological k-graph when endowed with pointwise operations.

Proposition 7.3. *Let (Λ_n, p_n) be a system of coverings of k-graphs. Then there is a homeomorphism*

$$\tilde{\pi}_\infty \colon X_{\varprojlim(\Lambda_n, p_n)} \to \varprojlim(X_{\Lambda_n}, (p_n)_*)$$

such that $\tilde{\pi}_\infty([(\lambda_i)_{i=0}^{\infty}, t]) = ([\lambda_i, t])_{i=0}^{\infty}$.

Proof. We first construct continuous surjections $\tilde{\pi}_n\colon X_{\varprojlim(\Lambda_n,p_n)} \to X_{\Lambda_n}$ such that $(p_n)_* \circ \tilde{\pi}_n = \tilde{\pi}_{n-1}$ for all $n \geqslant 1$. Fix $n \in \mathbb{N}$. Define

$$\pi_n^0\colon \bigcup_{m\in\mathbb{N}^k} (\varprojlim(\Lambda_n^m, p_n)) \times [0,m] \to Y_{\Lambda_n} \quad \text{by} \quad \pi_n^0((\lambda_i)_{i=1}^\infty, t) = (\lambda_n, t).$$

A basic open set in $\Lambda_n^m \times [0,m]$ has the form $U \times B(t;\varepsilon)$, where $U \subseteq \Lambda_n^m$ for some $m \in \mathbb{N}^k$ is open, $t \in [0,m]$, $\varepsilon > 0$ and the ball $B(t;\varepsilon)$ is calculated in the metric space $[0,m]$. We have $(\pi_n^0)^{-1}(U \times B(t;\varepsilon)) = Z(U,n) \times B(t;\varepsilon)$, where $Z(U,n)$ is the cylinder set $\{(\lambda_i)_{i=1}^\infty \in (\varprojlim(\Lambda_n,p_n))^m : \lambda_n \in U\}$. Since this preimage is open, π_n^0 is continuous. Now define

$$\pi_n\colon (\varprojlim(\Lambda_n,p_n))^m \times [0,m] \to X_{\Lambda_n} \quad \text{by} \quad \pi_n = q \circ \pi_n^0,$$

where $q\colon Y_{\Lambda_n} \to X_{\Lambda_n}$ is the quotient map. Then π_n is also continuous. We claim that

$$\tilde{\pi}_n([(\lambda_i)_{i=1}^\infty, t]) = \pi_n((\lambda_i)_{i=1}^\infty, t)$$

determines a well-defined map $\tilde{\pi}_n\colon X_{\varprojlim(\Lambda_n,p_n)} \to X_{\Lambda_n}$. Indeed, suppose that $[(\lambda_i)_{i=1}^\infty, t] = [(\mu_i)_{i=1}^\infty, s]$. Then $t - \lfloor t \rfloor = s - \lfloor s \rfloor$ and

$$(\lambda_i(\lfloor t \rfloor, \lceil t \rceil))_{i=1}^\infty = (\lambda_i)_{i=1}^\infty(\lfloor t \rfloor, \lceil t \rceil) = (\mu_i)_{i=1}^\infty(\lfloor s \rfloor, \lceil s \rceil) = (\mu_i(\lfloor s \rfloor, \lceil s \rceil))_{i=1}^\infty.$$

In particular, $\lambda_n(\lfloor t \rfloor, \lceil t \rceil) = \mu_i(\lfloor s \rfloor, \lceil s \rceil)$. Hence,

$$\pi_n((\lambda_i)_{i=1}^\infty, t) = [\lambda_n, t] = [\mu_n, s] = \pi_n((\mu_i)_{i=1}^\infty, s),$$

so $\tilde{\pi}_n$ is well defined as claimed. Since π_n is continuous, the definition of the quotient topology on $X_{\varprojlim(\Lambda_n,p_n)}$ ensures that $\tilde{\pi}_n$ is continuous too. Since the canonical map $P_n\colon \varprojlim(\Lambda_n,p_n) \to \Lambda_n$ is surjective for each n, the map $\tilde{\pi}_n$ is also surjective. By the definition of $(p_n)_*$, we have

$$(p_n)_* \circ \tilde{\pi}_n([(\lambda_i)_{i=1}^\infty, t]) = (p_n)_*([\lambda_n, t]) = [\lambda_{n-1}, t] = \tilde{\pi}_{n-1}([(\lambda_i)_{i=1}^\infty, t]).$$

The universal property of the projective limit $\varprojlim(X_{\Lambda_n}, (p_n)_*)$ now gives a unique continuous surjection

$$\tilde{\pi}_\infty\colon X_{\varprojlim(\Lambda_n,p_n)} \to \varprojlim(X_{\Lambda_n}, (p_n)_*)$$

defined by

$$\tilde{\pi}_\infty([(\lambda_i)_{i=0}^\infty, t]) = (\tilde{\pi}_n[(\lambda_i)_{i=0}^\infty, t])_{n=1}^\infty = [\lambda_n, t]_{n=0}^\infty.$$

To complete the proof we must show that $\tilde{\pi}_\infty$ is injective with continuous inverse. For this fix $([\lambda_n, t_n])_{n=0}^\infty \in \varprojlim(X_{\Lambda_n}, (p_n)_*)$. Then, $[p_n(\lambda_n), t_n] = [\lambda_{n-1}, t_{n-1}]$ for all n. For $n \geqslant 0$, let $\mu_n = \lambda_n(\lfloor t_n \rfloor, \lceil t_n \rceil)$ and $s_n = t_n - \lfloor t_n \rfloor$. Fix $n \geqslant 1$. By definition of the equivalence relation defining the X_{Λ_n}, we have $s_n = s_{n-1}$ and

$$p_n(\mu_n) = p_n(\lambda_n)(\lfloor t_n \rfloor, \lceil t_n \rceil) = \lambda_{n-1}(\lfloor t_{n-1} \rfloor, \lceil t_{n-1} \rceil) = \mu_{n-1}.$$

Let $s = s_0$. Then, $s_n = s_0$ for all n, and $([\lambda_n, t_n])_{n=0}^\infty = ([\mu_n, s])_{n=0}^\infty = \tilde{\pi}_\infty([(\mu_n)_{n=0}^\infty, s])$. So we may define $\theta\colon \varprojlim(X_{\Lambda_n}, (p_n)_*) \to X_{\varprojlim(\Lambda_n,p_n)}$ by $([\lambda_n, t_n])_{n=0}^\infty \mapsto [(\mu_n)_{n=0}^\infty, s]$, where the μ_n and s are obtained from the λ_n and t_n, as above. The above argument establishes that $\tilde{\pi}_\infty \circ \theta$ is the identity map on $X_{\varprojlim(\Lambda_n, p_n)}$. Hence, $\tilde{\pi}_\infty$ is surjective. On the other hand,

$$\begin{aligned}\theta \circ \tilde{\pi}_\infty([(\lambda_n)_{n=0}^\infty, t]) &= [(\lambda_n(\lfloor t \rfloor, \lceil t \rceil))_{n=0}^\infty, t - \lfloor t \rfloor] \\ &= [(\lambda_n)_{n=0}^\infty, t].\end{aligned}$$

Hence, θ is an algebraic inverse for $\tilde{\pi}_\infty$. To see that θ is continuous, it is enough, as for the other direction, to observe that if $\lambda \in \Lambda_n$ and U is open in $[0, d(\lambda)]$, then

$$\begin{aligned}\theta^{-1}(\{[(\mu_i)_{i=0}^\infty, t] \in X_{\varprojlim(\Lambda_i, p_i)} \colon \mu_n = \lambda,\ t \in U\}) \\ &= \tilde{\pi}_\infty(\{[(\mu_i)_{i=0}^\infty, t] \in X_{\varprojlim(\Lambda_i, p_i)} \colon \mu_n = \lambda,\ t \in U\}) \\ &= \{([\mu_i, t])_{i=0}^\infty \colon \mu_n = \lambda,\ t \in U\} \\ &= Z(\{[\lambda, t] \colon t \in U\}, n).\end{aligned}$$

So the preimage under θ of a sub-basic open set in the image of any connected component of $Y_{\varprojlim(\Lambda_n, p_n)}$ is the cylinder set of the image of a basic open set in some component of some Y_{Λ_n}. Continuity of θ then follows from the definition of the quotient topology. □

Corollary 7.4. *Let (Λ_n, p_n) be a system of coverings of k-graphs. Then,*

$$\pi_1(X_{\varprojlim(\Lambda_n, p_n)}) \cong \varprojlim(\pi_1(X_{\Lambda_n}), \widetilde{(p_n)}_*) \cong \varprojlim(\pi_1(\Lambda_n), (p_n)_*).$$

Proof. By [**11**, Theorem V.4.1], the covering maps $(p_n)_*$ induce injective homomorphisms $\widetilde{(p_n)}_*$ of fundamental groups. Theorems II.2.2 and II.2.3 of [**17**] imply that the covering maps $(p_n)_* \colon X_{\Lambda_{n+1}} \to X_{\Lambda_n}$ are fibrations with unique path lifting, so [**17**, Corollary VII.2.11] implies that the maps $\pi_2(X_{\Lambda_{n+1}}) \to \pi_2(X_{\Lambda_n})$ induced by the $(p_n)_*$ are isomorphisms. It therefore follows from [**4**, Proposition 4.67] that

$$\pi_1(X_{\varprojlim(\Lambda_n, p_n)}) \cong \varprojlim(\pi_1(X_{\Lambda_n}), \widetilde{(p_n)}_*).$$

That $\varprojlim(\pi_1(X_{\Lambda_n}), \widetilde{(p_n)}_*) \cong \varprojlim(\pi_1(\Lambda_n), (p_n)_*)$ follows from Lemma 5.6. □

8. Crossed products and mapping tori

Let Λ be a k-graph and let $\alpha \colon \mathbb{Z}^l \to \operatorname{Aut}(\Lambda)$ be an action by automorphisms. Recall that the crossed-product k-graph $\Lambda \times_\alpha \mathbb{Z}^l$ is equal as a set to $\Lambda \times \mathbb{N}^l$ and has operations $r(\lambda, m) = (r(\lambda), 0)$, $s(\lambda, m) = (\alpha_{-m}(s(\lambda)), 0)$ and $(\lambda, m)(\mu, n) = (\lambda\alpha_m(\mu), m + n)$.

Now let X be a topological space and let σ be an action of $\mathbb{Z}^l$ on X by homeomorphisms. There is then an action $\sigma \times \mathrm{lt}$ of $\mathbb{Z}^l$ on $X \times \mathbb{R}^l$ given by $(\sigma \times \mathrm{lt})_m(x, t) = (\sigma_m(x), m + t)$. The *mapping torus* of σ is the orbit space

$$M(\sigma) = (X \times \mathbb{R}^l)/(\sigma \times \mathrm{lt}).$$

We denote the equivalence class of $(x,t) \in X \times \mathbb{R}^l$ in the mapping torus by $[x,t]_{M(\sigma)}$, where the subscript is to distinguish such classes from elements of topological realizations X_Λ of k-graphs Λ, or simply by $[x,t]$ if there is no possibility of confusion.

In the following lemma, we identify $\mathbb{R}^{k+l}$ with $\mathbb{R}^k \times \mathbb{R}^l$ in the standard way.

Lemma 8.1. *Let Λ be a k-graph and let α be an action of $\mathbb{Z}^l$ on Λ. Let $\tilde{\alpha}$ be the induced action of $\mathbb{Z}^l$ on X_Λ obtained from functoriality of topological realization. Then there is a homeomorphism $\varphi\colon M(\tilde{\alpha}) \cong X_{\Lambda \times_\alpha \mathbb{Z}^l}$ determined by*

$$\varphi([[\lambda,s],t]_{M(\tilde{\alpha})}) = [(\lambda, \lceil t \rceil), (s,t)] \tag{8.1}$$

whenever $t \geqslant 0$.

Proof. For any $[[\lambda,s],t]_{M(\tilde{\alpha})} \in M(\tilde{\alpha})$ and $p \in \mathbb{N}^l$ such that $p+t \geqslant 0$, we have

$$[[\lambda,s],t]_{M(\tilde{\alpha})} = [(\tilde{\alpha} \times \mathrm{lt})_p([\lambda,s],t)]_{M(\tilde{\alpha})} = [[\alpha(\lambda),s],t+p]_{M(\tilde{\alpha})},$$

so each point in X_Λ has the form $[[\lambda,s],t]_{M(\tilde{\alpha})}$, where $t > 0$. To see that φ is well defined, suppose that $[[\lambda,s],t]_{M(\tilde{\alpha})} = [[\lambda',s'],t']_{M(\tilde{\alpha})}$ with $t,t' \geqslant 0$. Then $t - \lfloor t \rfloor = t' - \lfloor t' \rfloor$ and

$$[\alpha_{\lfloor t \rfloor}(\lambda), s] = \tilde{\alpha}_{\lfloor t \rfloor}(\lambda, s) = \tilde{\alpha}_{\lfloor t' \rfloor}(\lambda', s') = [\alpha_{\lfloor t' \rfloor}(\lambda'), s'].$$

Hence, $s - \lfloor s \rfloor = s' - \lfloor s' \rfloor$ and

$$(\alpha_{\lfloor t \rfloor}(\lambda))(\lfloor s \rfloor, \lceil s \rceil) = (\alpha_{\lfloor t' \rfloor}(\lambda'))(\lfloor s' \rfloor, \lceil s' \rceil). \tag{8.2}$$

We have $(\alpha_{\lfloor t \rfloor}(\lambda), 0)((\lfloor s \rfloor, 0), (\lceil s \rceil, 0)) = (\lambda, t)((\lfloor s \rfloor, \lfloor t \rfloor), (\lceil s \rceil, \lfloor t \rfloor))$ by definition of composition in $\Lambda \times_\alpha \mathbb{Z}^l$. Substituting this and the symmetric equality for λ' into (8.2) gives

$$(\lambda, t)((\lfloor s \rfloor, \lfloor t \rfloor), (\lceil s \rceil, \lfloor t \rfloor)) = (\lambda', t')((\lfloor s' \rfloor, \lfloor t' \rfloor), (\lceil s' \rceil, \lfloor t' \rfloor)).$$

Multiplying both sides on the right by $(s(\lambda), \lceil t \rceil - \lfloor t \rfloor) = (s(\lambda'), \lceil t' \rceil - \lfloor t' \rfloor)$, we obtain

$$(\lambda, t)((\lfloor s \rfloor, \lfloor t \rfloor), (\lceil s \rceil, \lceil t \rceil)) = (\lambda', t')((\lfloor s' \rfloor, \lfloor t' \rfloor), (\lceil s' \rceil, \lceil t' \rceil)).$$

Since $s - \lfloor s \rfloor = s' - \lfloor s' \rfloor$ and $t - \lfloor t \rfloor = t' - \lfloor t' \rfloor$, we have $(s,t) - \lfloor (s,t) \rfloor = (s',t') - \lfloor (s',t') \rfloor$, whence $[(\lambda, \lceil t \rceil), (s,t)] = [(\lambda', \lceil t' \rceil), (s',t')]$, as required. In particular, given any $[[\lambda,s],t]_{M(\tilde{\alpha})}$, any two representatives of this element with positive t-value have the same image under (8.1). So there is a well-defined map $\varphi\colon M(\tilde{\alpha}) \to X_{\Lambda \times_\alpha \mathbb{Z}^l}$ satisfying (8.1). The map φ is clearly surjective. To see that it is injective, just reverse the reasoning of the preceding paragraph: if $\varphi([[\lambda,s],t]_{M(\tilde{\alpha})}) = \varphi([[\lambda',s'],t']_{M(\tilde{\alpha})})$, then

$$(s,t) - \lfloor (s,t) \rfloor = (s',t') - \lfloor (s',t') \rfloor \quad \text{and} \quad (\alpha_{\lfloor t \rfloor}(\lambda))(\lfloor s \rfloor, \lceil s \rceil) = (\alpha_{\lfloor t' \rfloor}(\lambda'))(\lfloor s' \rfloor, \lceil s' \rceil),$$

whence $[[\lambda,s],t]_{M(\tilde{\alpha})} = [[\lambda',s'],t']_{M(\tilde{\alpha})}$.

To see that φ is continuous, observe that if $d(\lambda) \leqslant \mathbf{1}_k$ and $n \leqslant \mathbf{1}_l$, then the inverse image of the closed cube $\overline{Q_{(\lambda,n)}}$ under φ is

$$\{[[\lambda,s],t]_{M(\tilde{\alpha})}\colon 0 \leqslant s \leqslant \mathbf{1}_k,\ 0 \leqslant t \leqslant \mathbf{1}_l\} = (\overline{Q_\lambda} \times [0, \lceil t \rceil])/(\tilde{\alpha} \times \mathrm{lt}), \tag{8.3}$$

which is closed.

Finally, to prove that φ is a homeomorphism, it remains to verify that the inverse $\varphi^{-1}\colon X_{\Lambda\times_\alpha\mathbb{Z}^l} \to M(\tilde{\alpha})$ is also continuous. Fix $\lambda \in \Lambda$ with $d(\lambda) \leqslant \mathbf{1}_k$ and fix $n \leqslant \mathbf{1}_l$. Then the restriction of φ^{-1} to $\overline{Q_{(\lambda,n)}}$ is a homeomorphism onto the set (8.3) and is therefore continuous. Since $X_{\Lambda\times_\alpha\mathbb{Z}^l}$ is endowed with the weak topology determined by closed cubes, this proves that φ^{-1} is continuous as required. □

Recall that a *deck transformation* of a covering map $p\colon X \to Y$ is a homeomorphism g of X such that $p \circ g = p$. The deck transformations of p form a group $D(p)$.

Proposition 8.2. *Let X be a connected CW-complex and let σ be an action of $\mathbb{Z}^l$ on X by homeomorphisms. Fix $x_0 \in X$. Let $i_X\colon X \to M(\sigma)$ denote the embedding given by $i_X(x) = [x, 0]$. Then i_X induces an injection $(i_X)_*\colon \pi_1(X, x_0) \to \pi_1(M(\sigma), [x_0, 0])$ such that $(i_X)_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(M(\sigma), [x_0, 0])$. Moreover,*

$$\pi_1(M(\sigma), [x_0, 0])/(i_X)_*(\pi_1(X, x_0)) \cong \mathbb{Z}^l.$$

Proof. Functoriality of π_1 yields a homomorphism

$$(i_X)_*\colon \pi_1(X, x_0) \to \pi_1(M(\sigma), [x_0, 0]).$$

Consider the space $X \times \mathbb{R}^l$. Let $\sigma \times \mathrm{lt}$ be the action of $\mathbb{Z}^l$ determined by $(\sigma \times \mathrm{lt})_n(x, t) = (\sigma^n(x), t+n)$. Then $M(\sigma)$ is by definition the orbit space of this action. For $(x, t) \in X \times \mathbb{R}^l$, any open neighbourhood N of the form $U \times B(t; \frac{1}{3})$ of (x, t) has the property that $(\sigma\times\mathrm{lt})_m(N) \cap (\sigma\times\mathrm{lt})_n(N) = \emptyset$ for distinct $m, n \in \mathbb{Z}^l$. So $\sigma \times \mathrm{lt}$ satisfies condition $(*)$ of [**4**, p. 72]. Hence, [**4**, Proposition 1.40] implies first that the quotient map $q\colon X \times \mathbb{R}^l \to M(\sigma)$ is a regular covering whose deck-transformation group $D(q)$ is isomorphic to $\mathbb{Z}^l$, second that $q_*(\pi_1(X \times \mathbb{R}^l, (x_0, 0)))$ is a normal subgroup of $\pi_1(M(\sigma))$, and third that the quotient is isomorphic to $D(q)$. So we just need to see that $(i_X)_*$ is an injection and that its image coincides with $q_*(\pi_1(X \times \mathbb{R}^l, (x_0, 0)))$. For this, observe that since $\pi_1(\mathbb{R}^l)$ is trivial, [**11**, Theorem II.7.1] implies that $j_X : x \mapsto (x, 0)$ from X to $X \times \mathbb{R}^l$ induces an isomorphism $(j_X)_*\colon \pi_1(X, x_0) \to \pi_1(X \times \mathbb{R}^l, (x_0, 0))$. We have $(i_X)_* = (q \circ j_X)_* = q_* \circ (j_X)_*$. Since [**11**, Theorem V.4.1] implies that q_* is injective, it follows that $(i_X)_*$ is injective with the same image as q_*, as required. □

The following corollary is an immediate consequence of Proposition 8.2, the functoriality of the fundamental group and Corollary 4.5.

Corollary 8.3. *Let Λ be a connected k-graph, let $u \in \Lambda^0$ and let α be an action of $\mathbb{Z}^l$ on Λ. Then there is an extension*

$$1 \to \pi_1(\Lambda, u) \xrightarrow{(i_\Lambda)_*} \pi_1(\Lambda \times_\alpha \mathbb{Z}^l, (u, 0)) \to \mathbb{Z}^l \to 0,$$

where $i_\Lambda\colon \Lambda \to \Lambda \times_\alpha \mathbb{Z}^l$ is the canonical embedding.

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