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# NONCOMMUTATIVE LINE BUNDLES ASSOCIATED TO TWISTED MULTIPULLBACK QUANTUM ODD SPHERES

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Abstract. We construct a noncommutative deformation of odd-dimensional spheres that preserves the natural partition of the $(2N+1)$-dimensional sphere into $(N+1)$-many solid tori. This generalizes the case $N=1$ referred to as the Heegaard quantum sphere. Our twisted odd-dimensional quantum sphere $C^*$-algebras are given as multipullback $C^*$-algebras. We prove that they are isomorphic to the universal $C^*$-algebras generated by certain isometries, and use this result to compute the $K$-groups of our odd-dimensional quantum spheres. Furthermore, we show that the natural (diagonal) $U(1)$-actions on our twisted-quantum-sphere $C^*$-algebras are $C^*$-free, and define twisted multipullback quantum complex projective spaces through fixed-point subalgebras for these actions. In the untwisted case, we prove that the fixed-point subalgebras yield the independently defined $C^*$-algebras of the quantum complex projective spaces constructed from Toeplitz cubes. This leads to the main result stating that the noncommutative line bundles associated to multipullback quantum odd spheres, which are noncommutative line bundles over these quantum complex projective spaces, are pairwise stably *non*-isomorphic.

Contents

## 1. Introduction

Each odd-dimensional sphere decomposes into a union of solid tori, along the lines of the Heegaard splitting of the 3-sphere. Under this decomposition, the embedding of each component torus in the sphere is equivariant for the diagonal actions of the rank-one unitary group. Taking quotients by the $U(1)$-actions, we obtain a covering of the complex projective space by quotients of solid tori, which form a closed restriction of the usual affine covering.

We study a noncommutative deformation of this decomposition, using the point of view from [15] that the Toeplitz algebra can be regarded as the $C^*$-algebra of a noncommutative disc. In [4], the authors constructed a decomposition of a 3-dimensional quantum sphere along these lines by taking a pullback of two copies of the tensor product of the circle algebra and the Toeplitz algebra. Subsequently, in his Ph.D. thesis, Rudnik extended this construction to five dimensions using multipullback $C^*$-algebras. One of his main results was establishing the stable nontriviality of the *dual* tautological line bundle over the multipullback complex quantum projective plane [13, Theorem 2.4]. In this paper, we carry this idea further to all odd integers bigger than one, obtaining multipullback odd-dimensional noncommutative sphere algebras $C(S_H^{2N+1})$ (see Definition 3.2), and establishing that the winding numbers of associated noncommutative line bundles are $K_0$-invariants (see Theorem 1.1(4)). We thereby obtain a noncommutative complex-projective-space algebra as the fixed-point algebra of $C(S_H^{2N+1})$ for its natural $U(1)$ action. We prove that this fixed-point algebra is isomorphic to the noncommutative complex-projective-space algebra $C(\mathbb{P}^N(\mathcal{T}))$ introduced in [10] to study lattice-theoretical properties of sets of $C^*$-ideals.

Our main result (Theorem 1.1(4)) says that the noncommutative line bundles associated to the aforementioned $U(1)$-action are pairwise stably non-isomorphic. In particular, we prove that the tautological line-bundle module over any one of these algebras $C(\mathbb{P}^N(\mathcal{T}))$ is not stably free. To state it, we recall some background. Given a $C^*$-algebra $A$, we write $C(U(1), A)$ for the $C^*$-algebra of norm-continuous functions from $U(1)$ to $A$. Each action $\alpha$ of $U(1)$ on $A$ determines a homomorphism

$$\delta : A \longrightarrow C(U(1), A) \quad \text{by} \quad \delta(a)(\lambda) := \alpha_\lambda(a), \qquad a \in A,\ \lambda \in U(1). \tag{1.1}$$

An action $\alpha$ is *$C^*$-free* if and only if

$$\overline{\operatorname{span}}\{a\,\delta(b) \mid a, b \in A\} = C(U(1), A), \tag{1.2}$$

where $\overline{\operatorname{span}}$ stands for the closed linear span. The general definition of freeness of a quantum-group action on a $C^*$-algebra is due to Ellwood [7], and the special case of any compact Hausdorff topological group acting on a unital $C^*$-algebra looks exactly as above.

Next, recall that for each integer $m$ (winding number) and a $U(1)$-action $\alpha$, the spectral subspace $A_m$ is

$$A_m := \{a \in A \mid \alpha_\lambda(a) = \lambda^m a \text{ for all } \lambda \in U(1)\}. \tag{1.3}$$

The subspace $A_0$ is the fixed-point subalgebra $A^\alpha$ of $A$, and since $A_m A_n \subseteq A_{m+n}$ for all $m, n$, the spectral subspaces are always $A^\alpha$-bimodules. When $\alpha$ is $C^*$-free, they are finitely-generated projective (left) $A^\alpha$-modules [6, Theorem 1.2] encoding associated noncommutative line bundles.

The key results of this paper can be summarized as follows:

**Theorem 1.1.** *Fix an integer $N \geq 1$.*

(1) *The odd Heegaard quantum sphere $C^*$-algebra $C(S_H^{2N+1})$ is universal for isometries $s_0, \ldots, s_N$ that pairwise commute and $^*$-commute and satisfy the sphere equation*
$$\textstyle\prod_{i=0}^{N}(1 - s_i s_i^*) = 0.$$
(2) *The formulae $\alpha_\lambda(s_i) = \lambda s_i$, $i \in \{0, \ldots, N\}$, determine a $C^*$-free action $\alpha$ of $U(1)$ on $C(S_H^{2N+1})$.*
(3) *The fixed-point algebra $C(S_H^{2N+1})^\alpha$ is isomorphic to the quantum projective space $C^*$-algebra $C(\mathbb{P}^N(\mathcal{T}))$.*
(4) *The spectral subspaces $C(S_H^{2N+1})_m$, regarded as left $C(\mathbb{P}^N(\mathcal{T}))$-modules, are pairwise stably nonisomorphic. In particular, the section module $C(S_H^{2N+1})_{-1}$ of the tautological line bundle is not stably free.*

The paper is organized as follows. We begin by covering the background necessary for the new construction we wish to describe. This involves briefings on multipullbacks, Heegaard splittings of odd spheres, gauge actions of $U(1)$, coverings of complex projective spaces and twisted higher rank graph $C^*$-algebras. We then move on to the basic quantum-space philosophy behind our construction. We use the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0 \tag{1.4}$$

to regard the Toeplitz algebra $\mathcal{T}$ as the algebra of a two-dimensional quantum ball, or disc. The one-dimensional quantum sphere then corresponds to the quotient $\mathcal{T}/\mathcal{K}$. From this perspective, $\mathcal{T}^{\otimes N}$ can be regarded as the algebra of a Cartesian product of $N$ two-dimensional balls, and therefore as a copy of a $2N$-dimensional (non-round) quantum ball. The quotient $\mathcal{T}^{\otimes N+1}/\mathcal{K}^{\otimes N+1}$ is then viewed as the algebra of the boundary of the quantum ball—that is, a quantum sphere of dimension $2N+1$. In the same spirit, $\mathcal{T}^{\otimes N} \otimes C(\mathbb{T})$ is regarded as the algebra of the Cartesian product of a $2N$-ball and a circle, which is to say a $(2N+1)$-dimensional noncommutative solid torus.

Classically, the Heegaard splitting shows how to obtain a $(2N+1)$-dimensional sphere by gluing $N+1$ copies of the $N$-dimensional solid torus. By taking an appropriate multipullback of $N+1$ copies of the noncommutative solid torus algebra $\mathcal{T}^{\otimes N} \otimes C(\mathbb{T})$, we therefore obtain the algebra $C(S_H^{2N+1})$ of continuous functions on an odd quantum sphere. The new construction we present in this paper is to use an antisymmetric matrix $\theta$ to add a twisting to this formulation. Specifically, we use the matrix to define twisted quantum balls $\mathcal{T}_\theta^{N+1}$, and then apply the multipullback construction to define the twisted Heegaard quantum sphere. These algebras each carry a $C^*$-free $U(1)$-action whose fixed-point subalgebra defines the twisted multipullback quantum complex projective space.

Our analysis of $C(S_{H,\theta}^{2N+1})$ is facilitated by realising it as a universal algebra. Specifically, Theorem 3.3 shows that $C(S_{H,\theta}^{2N+1})$ is isomorphic to a twisted $(N+1)$-graph $C^*$-algebra. This identification allows us to show in Appendix A that the $K$-theory of $C(S_{H,\theta}^{2N+1})$ coincides with that of the algebra of continuous functions on the classical $(2N+1)$-sphere. Furthermore, for $\theta = 0$ this identification also allows us to define $U(1)$-equivariant $C^*$-homomorphisms from $C(S_H^{2N+1})$ to $C(S_H^{2N-1})$. Combining this fact with $C^*$-freeness of the $U(1)$-action of the previous paragraph we may use the Chern-Galois theory of [3] to derive the main result of this paper, Theorem 1.1(4). During this analysis we prove that the fixed-point algebra for the $U(1)$-action on $C(S_H^{2N+1})$ is isomorphic to $C(\mathbb{P}^N(\mathcal{T}))$.

Finally in Appendix B, we use the same methods to give a technically simpler proof of the fact that the noncommutative line bundles associated to Vaksman-Soibelman spheres [20] are pairwise stably nonisomorphic. This result was earlier obtained in [5, Proposition 4] using the index pairing.

## 2. Background

### 2.1. **Multipushouts, multipullbacks and the cocycle condition.**

In what follows, we will construct algebras of functions on quantum spaces as multipullbacks of $C^*$-algebras. To make sure that this construction dually corresponds to the presentation of a quantum space as a "union of closed subspaces" (no self gluings of closed subspaces or their partial multipushouts; see [14] for an in-depth discussion of these issues), we assume the cocycle condition (see Definition 2.1). First we need some auxiliary definitions.

Let $(\pi^i_j : A_i \to A_{ij})_{i,j\in J, i\neq j}$ be a finite family of surjective $C^*$-algebra homomorphisms. For all distinct $i,j,k \in J$, we define $A^i_{jk} := A_i/(\ker\pi^i_j + \ker\pi^i_k)$ and denote by $[\cdot]^i_{jk} : A_i \to A^i_{jk}$ the canonical surjections. Next, we introduce the family of maps

$$\pi^{ij}_k : A^i_{jk} \longrightarrow A_{ij}/\pi^i_j(\ker\pi^i_k), \qquad [b_i]^i_{jk} \longmapsto \pi^i_j(b_i) + \pi^i_j(\ker\pi^i_k), \tag{2.1}$$

for all distinct $i,j,k \in J$. Note that these maps $\pi^{ij}_k$ are isomorphisms when all the $\pi^i_j$'s are surjective $C^*$-algebra homomorphisms, as assumed herein.

**Definition 2.1** (Proposition 9 in [4])**.** We say that a finite family $(\pi^i_j : A_i \to A_{ij})_{i,j\in J,i\neq j}$ of $C^*$-algebra surjections satisfies the *cocycle condition* if and only if, for all distinct $i,j,k \in J$,

(1) $\pi^i_j(\ker\pi^i_k) = \pi^j_i(\ker\pi^j_k)$, and
(2) the isomorphisms $\phi^{ij}_k := (\pi^{ij}_k)^{-1} \circ \pi^{ji}_k : A^j_{ik} \to A^i_{jk}$ satisfy $\phi^{ik}_j = \phi^{ij}_k \circ \phi^{jk}_i$.

Theorem 1 of [14] implies that a finite family $(\pi^i_j : A_i \to A_{ij})_{i,j\in J,i\neq j}$ of $C^*$-algebra surjections satisfies the cocycle condition if and only if, for all $K \subsetneq J$, $k \in J\setminus K$, and $(b_i)_{i\in K} \in \bigoplus_{i\in K} A_i$ such that $\pi^i_j(b_i) = \pi^j_i(b_j)$ for all distinct $i,j \in K$, there exists $b_k \in A_k$ such that also $\pi^i_k(b_i) = \pi^k_i(b_k)$ for all $i \in K$. One can easily see that dually this corresponds to the statement "a quantum space is a pushout of parts, and all partial pushouts are embedded in this quantum space". This is what we usually have in mind when constructing a space from parts.

### 2.2. **Heegaard-type splittings of odd spheres.**

In this subsection we describe how to obtain the odd-dimensional sphere $S^{2N+1}$ as a topological quotient of a disjoint union of $N+1$ copies of the solid torus $D^N \times \mathbb{T}$, where $D$ is the unit disc. By the $(2N+1)$-dimensional sphere, we mean the space

$$S^{2N+1} := \big\{(z_0,\dots,z_N) \in \mathbb{C}^{N+1} \mid |z_0|^2 + \cdots + |z_N|^2 = 1\big\}. \tag{2.2}$$

To realize this space as a gluing of solid tori, for each $0 \leq i \leq N$, define

$$V_i := \big\{(z_0, \dots, z_N) \in S^{2N+1} \mid |z_i| = \max\{|z_0|, \dots, |z_N|\}\big\}. \tag{2.3}$$

Then there exist homeomorphisms $\phi_i : V_i \to D^i \times \mathbb{T} \times D^{N-i}$ such that the element $z := (z_0, \dots, z_N) \in V_i$ satisfies

$$\phi_i(z) = \Big(\frac{z_0}{|z_i|}, \dots, \frac{z_N}{|z_i|}\Big). \tag{2.4}$$

The inverse is described by

$$\phi_i^{-1}(d) = \frac{d}{\sqrt{1 + \sum_{j \neq i} |d_j|^2}}, \tag{2.5}$$

where $d := (d_0, \dots, d_N) \in D^i \times \mathbb{T} \times D^{N-i}$.

It follows that $S^{2N+1}$ can be presented as a multipushout of closed solid tori. For each $i$, let $X_i$ denote the closed solid torus $X_i := D^i \times \mathbb{T} \times D^{N-i}$, and for $i < j$, let

$$X_{i,j} := D^i \times \mathbb{T} \times D^{j-i-1} \times \mathbb{T} \times D^{N-j}. \tag{2.6}$$

Then $S^{2N+1}$ is the multipushout of the $N+1$ closed solid tori $X_0, \dots X_N$ determined by the diagrams

$$\begin{tikzcd}
 & & S^{2N+1} & & \\
X_i \arrow[urr, dashed] & V_i \arrow[l, "\phi_i"'] \arrow[ur, hook] & & V_j \arrow[ul, hook'] \arrow[r, "\phi_j"'] & X_j \arrow[ull, dashed] \\
X_{i,j} \arrow[u, hook] & & V_i \cap V_j \arrow[ll, "\phi_{ij}"'] \arrow[rr, "\phi_{ji}"] \arrow[ul, hook] \arrow[ur, hook'] & & X_{i,j} \arrow[u, hook']
\end{tikzcd} \;. \tag{2.7}$$

(Note that $\phi_{ji} \circ \phi_{ij}^{-1} = \mathrm{id}_{X_{i,j}}$.) Thus $S^{2N+1}$ is homeomorphic to the quotient of

$$\coprod_{0 \leq i \leq N} D^i \times \mathbb{T} \times D^{N-i} \tag{2.8}$$

by the equivalence relation generated by

$$\phi_{i,j}(d) \sim \phi_{j,i}(d) \quad \text{for all } i < j \text{ and all } d \in V_{i,j}. \tag{2.9}$$

To deal with quantum spheres later on, we dualize this picture of $S^{2N+1}$ as a gluing of solid tori to the multipullback picture of the $C^*$-algebra $C(S^{2N+1})$ of continuous functions. Let $\sigma : \mathcal{T} \to C(\mathbb{T})$ be the symbol map. For $i < j$, we write

$$\begin{aligned} \pi^i_j : C(D)^{\otimes i} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes N-i} &\longrightarrow \\ & C(D)^{\otimes i} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes j-i-1} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes N-j} \end{aligned} \tag{2.10}$$

for the surjection $\mathrm{id}^{\otimes j} \otimes \sigma \otimes \mathrm{id}^{\otimes N-j}$. Then $C(S^{2N+1})$ is naturally isomorphic to

$$\big\{(f_0, \dots, f_N) \in \textstyle\bigoplus_{i=0}^N C(D)^{\otimes i} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes N-i} \,|\, \pi^i_j(f_j) = \pi^j_i(f_i) \ \text{ for all } i < j\big\}. \tag{2.11}$$

2.3. **Gauging diagonal actions and coactions.** Throughout this paper, we denote a right action of a group $G$ on a space $X$ by juxtaposition, that is $(x, g) \mapsto xg$. The general idea for converting between diagonal and rightmost actions of a group $G$ on a topological space $X$ is as follows. We regard $X \times G$ as a right $G$-space in two different ways, which we distinguish notationally as follows.

- We write $(X \times G)^R$ for the product $X \times G$ with $G$-action $(x, g) \cdot h := (x, gh)$.
- We write $X \times G$ the same space with diagonal $G$-action $(x, g)h = (xh, gh)$.

There is a $G$-equivariant homeomorphism $\kappa : (X \times G)^R \to X \times G$ determined by

$$\kappa(x, g) = (xg, g), \tag{2.12}$$

with the inverse given by

$$\kappa^{-1}(x, g) = (xg^{-1}, g). \tag{2.13}$$

In general, given any cartesian product of $G$-spaces we will regard it as a $G$-space with the diagonal action, except for those of the form $(X \times G)^R$ just described.

In what follows, the tensor product means completed tensor product, and we use the Heynemann-Sweedler notation (with the summation sign suppressed) for this completed product. Since all $C^*$-algebras that we tensor are nuclear, this completion is unique. Also, we will often identify the unit circle $\mathbb{T}$ with the unitary group $U(1)$, and use the quantum group structure on $C(U(1))$. Even though we only use the classical compact Hausdorff group $U(1)$, we are forced to use the quantum-group language of coactions, etc., to write explicit formulas, and carry out computations.

Let $H := C(G)$ be the $C^*$-algebra of continuous functions on a compact Hausdorff group $G$ acting on a unital $C^*$-algebra $A$. Then $S\colon H \to H$, given by $S(h)(g) := h(g^{-1})$, is the antipode, $\varepsilon(h) := h(e)$ (where $e$ is the neutral element of $G$) defines the counit,

$$\begin{gathered}\Delta\colon H \longrightarrow H \otimes H \cong C(G \times G),\\ \Delta(h)(g_1, g_2) := h(g_1 g_2) =: (h_{(1)} \otimes h_{(2)})(g_1, g_2) = h_{(1)}(g_1)h_{(2)}(g_2),\end{gathered} \tag{2.14}$$

is a coproduct, and

$$\delta\colon A \longrightarrow A \otimes H \cong C(G, A), \quad \delta(a)(g) := \alpha_g(a) =: (a_{(0)} \otimes a_{(1)})(g) = a_{(0)}a_{(1)}(g) \tag{2.15}$$

is a coaction.

Consider $A\otimes H$ as a $C^*$-algebra with the diagonal coaction $p\otimes h \mapsto p_{(0)}\otimes h_{(1)}\otimes p_{(1)}h_{(2)}$, and denote by $(A \otimes H)^R$ the same $C^*$-algebra but now equipped with the coaction on the rightmost factor $p \otimes h \mapsto p \otimes h_{(1)} \otimes h_{(2)}$. Then the following map is a $G$-equivariant (i.e., intertwining the coactions) isomorphism of $C^*$-algebras:

$$\widehat{\kappa} : (A \otimes H) \longrightarrow (A \otimes H)^R, \quad a \otimes h \longmapsto a_{(0)} \otimes a_{(1)}h. \tag{2.16}$$

Its inverse is explicitly given by

$$\widehat{\kappa}^{-1} : (A \otimes H)^R \longrightarrow (A \otimes H), \quad a \otimes h \longmapsto a_{(0)} \otimes S(a_{(1)})h. \tag{2.17}$$

2.4. **Affine closed coverings of complex projective spaces.** The space $S^{2N+1}$ is a $U(1)$-principal bundle. The diagonal action of $U(1)$ on $S^{2N+1}$ is given by

$$(z_0, \dots, z_N)\lambda = (z_0\lambda, \dots, z_N\lambda). \tag{2.18}$$

Since the action of $U(1)$ on $D$ by rotation leaves the boundary invariant, the above diagonal action restricts to solid tori, and makes the multipushout given by the diagrams (2.7) $U(1)$-equivariant.

Next, to obtain a multipushout presentation of $\mathbb{P}^N(\mathbb{C}) = S^{2N+1}/U(1)$, we need to gauge the diagonal actions to actions on the rightmost components. This will yield an alternative multipushout presentation of $S^{2N+1}$. Using the notation of Section 2.3, we write $\kappa : (D^N \times U(1))^R \to D^N \times U(1)$ for the gauging homeomorphism. Identifying $U(1)$ with $\mathbb{T}$, and writing $F_{i,N} : D^N \times U(1) \to D^i \times \mathbb{T} \times D^{N-i}$ for the map

$$F_{i,N}(d_0, \dots, d_{i-1}, d_i, d_{i+1} \dots, d_{N-1}, d_N) = (d_0, \dots, d_{i-1}, d_N, d_{i+1} \dots, d_{N-1}, d_i), \tag{2.19}$$

we obtain a $U(1)$-equivariant homeomorphism

$$h_i := F_{i,N} \circ \kappa : (D^N \times U(1))^R \longrightarrow D^i \times \mathbb{T} \times D^{N-i}. \tag{2.20}$$

Let $X_i^R := (D^N \times U(1))^R$ for all $i$. For $i < j < N$, let

$$\begin{gathered} X^R_{i,j} := \big(D^i \times \mathbb{T} \times D^{N-i-1} \times U(1)\big)^R, \quad X^R_{j,i} := \big(D^{j-1} \times \mathbb{T} \times D^{N-j} \times U(1)\big)^R, \\ X_{i,j} := D^i \times \mathbb{T} \times D^{j-i-1} \times \mathbb{T} \times D^{N-j} =: X_{j,i}. \end{gathered} \tag{2.21}$$

For $i \neq j$, we define $h_{ij} := h_i\big|_{X^R_{j,i}} : X^R_{j,i} \to X_{i,j} = X_{j,i}$.

We use the $h_i$ and $h_{ij}$ to transform the multipushout structure of $S^{2N+1}$ described by diagram (2.7). Explicitly, for $0 \leq i < j \leq N$ we have the following commutative diagram:

$$\begin{tikzcd}
X_i^R \arrow[r, "h_i"] & X_i & & X_j & X_j^R \arrow[l, "h_j"'] \\
X^R_{j,i} \arrow[u, hook] \arrow[drr, "h_{ij}"] & & & & X^R_{i,j} \arrow[u, hook] \arrow[dll, "h_{ji}"'] \\
& & X_{i,j} \arrow[uul, hook] \arrow[uur, hook] & &
\end{tikzcd} . \tag{2.22}$$

For $i < j$, we define $\chi_{ij} := h_{ji}^{-1} \circ h_{ij} : X^R_{j,i} \to X^R_{i,j}$. With this notation, $S^{2N+1}$ is homeomorphic to the quotient of

$$\coprod_{0 \leq i \leq N} (D^N \times U(1))^R = \coprod_{0 \leq i \leq N} X_i^R \tag{2.23}$$

by the smallest equivalence relation such that $d \sim \chi_{ij}(d)$ for all $d \in X^R_{j,i}$. The equivalence relation $\sim$ respects the $U(1)$-actions, so that we obtain a multipushout presentation of $S^{2N+1}/U(1)$ by everywhere restricting $U(1)$ to a point.

The complex projective space $\mathbb{P}^N(\mathbb{C})$ is defined as

$$\mathbb{P}^N(\mathbb{C}) := (\mathbb{C}^{N+1} \setminus \{0\})/\mathbb{C}^\times, \tag{2.24}$$

where the group $\mathbb{C}^\times$ of nonzero complex numbers acts by scalar multiplication. We denote by $[x_0 : \ldots : x_N]$ the class of $(x_0, \ldots, x_N)$ in $\mathbb{P}^N(\mathbb{C})$.

There is a well-known affine open covering of $\mathbb{P}^N(\mathbb{C})$ consisting of the sets

$$U_i := \{[x_0 : \ldots : x_N] \in \mathbb{P}^N(\mathbb{C}) \mid x_i \neq 0\}, \quad 0 \leq i \leq N. \tag{2.25}$$

For each $i$, there is a homeomorphism $U_i \to \mathbb{C}^N$ given by

$$[x_0 : \ldots : x_N] \longmapsto \Big(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i}\Big). \tag{2.26}$$

Here instead we consider the affine closed covering of $\mathbb{P}^N(\mathbb{C})$ consisting of the sets

$$V_i := \big\{[x_0 : \ldots : x_N] \in \mathbb{P}^N(\mathbb{C}) \bigm| |x_i| = \max\{|x_0|, \ldots, |x_N|\}\big\}, \quad 0 \leq i \leq N. \tag{2.27}$$

For each $i$, there is a homeomorphism $\psi_i : V_i \to D^N$ defined by

$$\psi_i\big([x_0 : \ldots : x_N]\big) := \Big(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i}\Big). \tag{2.28}$$

Its inverse is given by

$$\psi_i^{-1}(d_1, \ldots, d_N) = [d_1 : \ldots : d_i : 1 : d_{i+1} : \ldots : d_N]. \tag{2.29}$$

Now we can use the homeomorphisms $\psi_i$ to transport the multipushout presentation of $\mathbb{P}^N(\mathbb{C})$ associated to the covering $\{V_i\}_i$ to a presentation as a multipushout of products of discs. For distinct $i, j \in \{0, \ldots, N\}$, we define $\psi_{ij} := \psi_i|_{V_i \cap V_j}$.

Thus, for $i < j$, we obtain the commutative diagram

$$\begin{tikzcd}
 & & \mathbb{P}^N(\mathbb{C}) & & \\
D^N \arrow[urr, dashed] & V_i \arrow[l, "\psi_i"'] \arrow[ur, hook] & & V_j \arrow[r, "\psi_j"] \arrow[ul, hook'] & D^N \arrow[ull, dashed] \\
D^{j-1} \times \mathbb{T} \times D^{N-j} \arrow[u, hook] & & V_i \cap V_j \arrow[ll, "\psi_{ij}"'] \arrow[rr, "\psi_{ji}"] \arrow[ul, hook'] \arrow[ur, hook] & & D^i \times \mathbb{T} \times D^{N-i-1}. \arrow[u, hook']
\end{tikzcd} \tag{2.30}$$

For $i < j$, the gluing maps $\Upsilon_{ij} := \psi_{ji} \circ \psi_{ij}^{-1} : D^{j-1} \times \mathbb{T} \times D^{N-j} \to D^i \times \mathbb{T} \times D^{N-i-1}$, satisfy

$$\begin{aligned} &\Upsilon_{ij}(d_1, \ldots, d_{j-1}, \lambda, d_{j+1}, \ldots, d_N) \\ &\qquad = (\lambda^{-1}d_1, \ldots, \lambda^{-1}d_i, \lambda^{-1}, \lambda^{-1}d_{i+1}, \ldots, \lambda^{-1}d_{j-1}, \lambda^{-1}d_{j+1}, \ldots, \lambda^{-1}d_N). \end{aligned} \tag{2.31}$$

Since, for all $i < j$, homeomorphism $\Upsilon_{ij}$ coincides with $\chi_{ij}$ with the rightmost component deleted, the multipushout structure of $S^{2N+1}/U(1)$ agrees with the above multipushout presentation.

Dualizing this construction, and writing $\sigma_i$ for the homomorphism of $C(D)^{\otimes N}$ onto $C(D)^{\otimes i} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes N-i-1}$ given by the restriction of functions, we see that the $C^*$-algebra $C(\mathbb{P}^N(\mathbb{C}))$ is isomorphic to the multipullback $C^*$-algebra

$$\Big\{a \in \textstyle\bigoplus_{i=0}^N C(D)^{\otimes N} \mid \Upsilon_{ij}^*(\sigma_i(a_j)) = \sigma_j(a_i) \text{ for all } 0 \leq i < j \leq N\Big\}. \tag{2.32}$$

2.5. **Twisted higher-rank graph $C^*$-algebras.** We briefly recall the background we need about $k$-graphs and twisted relative Cuntz–Krieger algebras. For details, see [16, 17].

Recall from [16] that for a positive integer $k$, a *$k$-graph* is a countable small category $\Lambda$ with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the *factorisation property*: if $d(\eta) = m + n$, then there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\eta = \mu\nu$. We write $\Lambda^n := d^{-1}(n)$. It follows from the factorisation property that $\Lambda^0$ is precisely the set of identity morphisms (or objects), so the domain and codomain maps in the category $\Lambda$ determine maps $s, r : \Lambda \to \Lambda^0$ such that $r(\mu)\mu = \mu = \mu s(\mu)$ for all $\mu$.

We use the following notational convention: if $E$ is a subset of $\Lambda$ and $\alpha \in \Lambda$, then we write $\alpha E := \{\alpha\beta \mid \beta \in E,\ s(\alpha) = r(\beta)\}$. The set $\mathrm{MCE}(\mu, \nu)$ of *minimal common extensions* of $\mu, \nu \in \Lambda$ is defined as $\mathrm{MCE}(\mu, \nu) := \mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}$, where $d(\mu) \vee d(\nu)$ is the coordinatewise maximum of $d(\mu), d(\nu) \in \mathbb{N}^k$. Given $\mu \in \Lambda$ and $E \subseteq r(\mu)\Lambda$, we define $\mathrm{Ext}(\mu; E) := \bigcup_{\nu\in E}\{\alpha \mid \mu\alpha \in \mathrm{MCE}(\mu, \nu)\}$.

For $v \in \Lambda^0$ we say that $F \subseteq v\Lambda \setminus \{v\}$ is *exhaustive* if and only if $\mathrm{Ext}(\lambda; F) \neq \emptyset$ for every $\lambda \in v\Lambda$. We write $r(F) := v$. A collection $\mathcal{E}$ of finite exhaustive sets is *satiated* if and only if

(S1) if $F \in \mathcal{E}$ and $\mu \in r(F)\Lambda \setminus \{r(F)\}$, then $F \cup \{\mu\} \in \mathcal{E}$;
(S2) if $F \in \mathcal{E}$ and $\mu \in r(F)\Lambda \setminus F\Lambda$, then $\mathrm{Ext}(\mu; F) \in \mathcal{E}$;
(S3) if $F \in \mathcal{E}$ and $\mu$ and $\mu\nu$ both belong to $F$ with $\mu\nu \neq \mu$, then $F \setminus \{\mu\nu\} \in \mathcal{E}$;
(S4) if $F \in \mathcal{E}$, $\mu \in F$ and $G \in \mathcal{E}$ with $r(G) = s(\mu)$, then $F \setminus \{\mu\} \cup \mu G \in \mathcal{E}$.

The *satiation* $\overline{\mathcal{E}}$ of a collection $\mathcal{E}$ of finite exhaustive sets is the smallest satiated collection of finite exhaustive sets that contains $\mathcal{E}$.

A *2-cocycle* on a $k$-graph $\Lambda$ is a map $c : \{(\mu, \nu) \in \Lambda \times \Lambda \mid s(\mu) = r(\nu)\} \to \mathbb{T}$ such that

$$\begin{aligned} &c(r(\mu), \mu)) = 1 = c(\mu, s(\mu)) \text{ for all } \mu, \text{ and} \\ &c(\mu, \nu)c(\mu\nu, \eta) = c(\nu, \eta)c(\mu, \nu\eta) \text{ for all composable } \mu, \nu, \eta \in \Lambda. \end{aligned} \tag{2.33}$$

Given a $k$-graph $\Lambda$, a collection $\mathcal{E}$ of finite exhaustive sets, and a 2-cocycle $c$, the *twisted relative Cuntz–Krieger algebra* $C^*(\Lambda, c; \mathcal{E})$ is the universal $C^*$-algebra generated by elements $\{s^c_{\mathcal{E}}(\mu) \mid \mu \in \Lambda\}$ satisfying

(TCK1) $\{s^c_{\mathcal{E}}(v) \mid v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(TCK2) $s^c_{\mathcal{E}}(\mu)s^c_{\mathcal{E}}(\nu) = c(\mu, \nu)s^c_{\mathcal{E}}(\mu\nu)$ for all $\mu, \nu \in \Lambda$ with $s(\mu) = r(\nu)$;
(TCK3) $s^c_{\mathcal{E}}(\mu)^* s^c_{\mathcal{E}}(\mu) = s^c_{\mathcal{E}}(s(\mu))$ for all $\mu \in \Lambda$;
(TCK4) $s^c_{\mathcal{E}}(\mu)s^c_{\mathcal{E}}(\mu)^* s^c_{\mathcal{E}}(\nu)s^c_{\mathcal{E}}(\nu)^* = \sum_{\eta\in\mathrm{MCE}(\mu,\nu)} s^c_{\mathcal{E}}(\eta)s^c_{\mathcal{E}}(\eta)^*$ for all $\mu, \nu \in \Lambda$; and
(CK) $\prod_{\mu\in F}\left(s^c_{\mathcal{E}}(r(F)) - s^c_{\mathcal{E}}(\mu)s^c_{\mathcal{E}}(\mu)^*\right) = 0$ for all $F \in \mathcal{E}$.

Its universal property ensures that $C^*(\Lambda, c; \mathcal{E})$ carries a $U(1)^k$-action $\gamma^c_{\mathcal{E}}$ such that

$$(\gamma^c_{\mathcal{E}})_z(s^c_{\mathcal{E}}(\mu)) = z^{d(\mu)}s^c_{\mathcal{E}}(\mu) \text{ for all } z = (z_1, \ldots, z_k) \in U(1)^k \text{ and } \mu \in \Lambda, \tag{2.34}$$

where $z^{d(\mu)} := \prod_{i=1}^k z_i^{d(\mu)_i}$.

If $\{t^c_{\mathcal{E}}(\mu) \mid \mu \in \Lambda\}$ is a twisted relative Cuntz–Krieger family in a $C^*$-algebra $B$, then, by the universal property of $C^*(\Lambda, c; \mathcal{E})$, there is a map $\pi : C^*(\Lambda, c; \mathcal{E}) \to B$ characterized by $\pi(s^c_{\mathcal{E}}(\mu)) = t^c_{\mathcal{E}}(\mu)$. The gauge-invariant uniqueness theorem [17, Theorem 3.16] reads as follows. Suppose that the projection $\pi(s^c_{\mathcal{E}}(v))$ is nonzero for every vertex $v \in \Lambda^0$, that the projection $\pi\big(\prod_{\mu\in E} s^c_{\mathcal{E}}(r(E)) - s^c_{\mathcal{E}}(\mu)s^c_{\mathcal{E}}(\mu)^*\big)$ is nonzero for every finite exhaustive set $E \notin \overline{\mathcal{E}}$, and that $B$ carries a $U(1)^k$-action $\beta$ such that $\beta_z(t^c_{\mathcal{E}}(\mu)) = z^{d(\mu)}t^c_{\mathcal{E}}(\mu)$. Then $\pi$ is injective.

## 3. Twisted multipullback quantum odd spheres

Recall that we regard the Toeplitz algebra $\mathcal{T}$ as the quantum-disc $C^*$-algebra. By analogy with the Heegaard splitting of $S^{2N+1}$ in the preceding section, we define the algebra $C(S^{2N+1}_H)$ of continuous functions on the Heegaard quantum sphere as a multipullback of the $C^*$-algebras $\mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-i}$ with respect to the maps

$$\pi^i_j\colon \mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-i} \longrightarrow \mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes j-i-1} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-j}, \quad i<j, \tag{3.1}$$

that are given by applying the symbol map

$$\sigma : \mathcal{T} \longrightarrow C(\mathbb{T}), \qquad s \longmapsto u, \tag{3.2}$$

in the $j$th (counting from zero) tensor factor (cf. (2.10)). Here $s$ is the isometry generating $\mathcal{T}$ and $u$ is the unitary generating $C(\mathbb{T})$.

In Section 3.4, we will realize $C(S^{2N+1}_H)$ as the special case where $\theta = 0$ of a multipullback of twisted tensor products of the same sort. We begin by defining the twisted Toeplitz algebras $\mathcal{T}^{N+1}_\theta$, which we view as twisted-quantum-ball $C^*$-algebras.

**Definition 3.1.** Fix $N > 0$, and suppose that $\theta = (\theta_{i\,j})^N_{i,j=0} \in M_{N+1}(\mathbb{R})$ is antisymmetric in the sense that $\theta_{i\,j} = -\theta_{j\,i}$. We define the twisted Toeplitz algebra $\mathcal{T}^{N+1}_\theta$ to be the universal $C^*$-algebra generated by isometries $\{w_0, \dots, w_N\}$ such that

$$w_j w_k = e^{2\pi i\theta_{jk}} w_k w_j \quad \text{and} \quad w_j^* w_k = e^{-2\pi i\theta_{jk}} w_k w_j^* \quad \text{for all } j \neq k.$$

With this in hand, we are ready to present our definition of the twisted Heegaard quantum sphere $S^{2N+1}_{H,\theta}$, which we view as the boundary of a twisted quantum ball. Thus we generalize the 3-dimensional case $S^3_{H,\theta}$ introduced and analyzed in [2].

**Definition 3.2.** Resume the notation of Definition 3.1. For $0 \le i \le N$, let $I_i$ denote the ideal of $\mathcal{T}^{N+1}_\theta$ generated by $1 - w_i w_i^*$, and for $i \neq j$, let $I_{ij} := I_i + I_j$. Let $B_i := \mathcal{T}^{N+1}_\theta / I_i$ and $B_{ij} := \mathcal{T}^{N+1}_\theta / I_{ij}$. Also, let

$$\sigma_i : \mathcal{T}^{N+1}_\theta \longrightarrow B_i \quad \text{and} \quad \pi^i_j : B_i \longrightarrow B_{ij}$$

be the natural quotient maps. We define the *twisted Heegaard quantum sphere* $C^*$-algebra as the multipullback of the algebras $B_i$ over the homomorphisms $\pi^i_j$, that is

$$C(S^{2N+1}_{H,\theta}) := \left\{ (b_0, \dots, b_N) \in \bigoplus_{i=0}^N B_i \;\middle|\; \pi^i_j(b_i) = \pi^j_i(b_j) \text{ for all } 0 \le i < j \le N \right\}.$$

The universal property of $\mathcal{T}^{N+1}_\theta$ ensures that it carries a $U(1)^{N+1}$-action satisfying $(\lambda_0, \dots, \lambda_N) \cdot w_j = \lambda_j w_j$. We call this the *gauge action* on $\mathcal{T}^{N+1}_\theta$. This action descends to each $B_i$ and each $B_{ij}$, and hence induces a $U(1)^{N+1}$-action on $C(S^{2N+1}_{H,\theta})$, also called the gauge action. Restricting to the diagonal in $U(1)^{N+1}$ gives a $U(1)$-action $\alpha$ on $C(S^{2N+1}_{H,\theta})$ such that

$$\alpha_\lambda(b_0, \ldots, b_N) = (\lambda \cdot b_0, \ldots, \lambda \cdot b_N). \tag{3.3}$$

3.1. **Universal presentation.** We prove that this twisted Heegaard quantum sphere $C^*$-algebra can be described in terms of a universal property.

**Theorem 3.3.** *Consider an integer $N \geq 1$ and a skew-symmetric matrix $\theta \in M_{N+1}(\mathbb{R})$. Let $A_\theta(N+1)$ be the universal $C^*$-algebra generated by isometries $s_0, \ldots, s_N$ satisfying*

$$s_i s_j = e^{2\pi i \theta_{i\,j}} s_j s_i \quad \text{and} \quad s_i s_j^* = e^{-2\pi i \theta_{i\,j}} s_j^* s_i, \tag{3.4}$$

*and the sphere equation*

$$\prod_{i=0}^{N} (1 - s_i s_i^*) = 0. \tag{3.5}$$

*Then there is a $U(1)$-action on $A_\theta(N+1)$ such that $\lambda \cdot s_i = \lambda s_i$ for all $i$, and there is a $U(1)$-equivariant isomorphism $\phi_\theta : A_\theta(N+1) \to C(S^{2N+1}_{H,\theta})$ such that*

$$\phi_\theta(s_i) = (\sigma_0(w_i), \ldots, \sigma_N(w_i)) \quad \text{for all } i.$$

*Furthermore, the maps $\pi^i_j : B_i \to B_{ij}$ satisfy the cocycle condition of Defintion 2.1.*

The existence of the $U(1)$-action on $A_\theta(N+1)$ and of the homomorphism $\phi_\theta$ follows from the universal property of $A_\theta(N+1)$. We use the technology of higher-rank graph $C^*$-algebras to see that $\phi_\theta$ is injective. For surjectivity, and to see that the cocycle condition is satisfied, we will need the following technical lemma.

**Lemma 3.4.** *Let $A$ be a $C^*$-algebra and suppose that $I_0, \ldots, I_n$ are ideals of $A$. Suppose that $a_0, \ldots, a_n \in A$ satisfy $a_i + (I_i + I_j) = a_j + (I_i + I_j)$ for all $i, j$. Then there exists $a \in A$ such that $a + I_i = a_i + I_i$ for all $i$.*

*Proof.* We proceed by induction on $n$. The base case $n = 0$ is trivial. Suppose as an inductive hypothesis that there exists $a' \in A$ such that $a' + I_i = a_i + I_i$ for all $i < n$. Then $a' + (I_i + I_n) = a_n + (I_i + I_n)$ for all $i < n$, whence

$$a' - a_n \in \textstyle\bigcap_{i<n} (I_i + I_n). \tag{3.6}$$

Since the ideals of the $C^*$-algebra $A$ form a distributive lattice with meet given by intersection and join given by sum, we have

$$\bigcap_{i<n} (I_i + I_n) = \Big( \bigcap_{i<n} I_i \Big) + \sum_{\emptyset \neq F \subseteq \{0,\ldots,n-1\}} \Big( I_n \cap \bigcap_{i \notin F} I_i \Big) \subseteq \Big( \bigcap_{i<n} I_i \Big) + I_n. \tag{3.7}$$

Combining this with (3.6), we obtain $a' - a_n = b' - b_n$, where $b' \in \bigcap_{i=0}^{n-1} I_i$ and $b_n \in I_n$. Put $a := a' - b'$. Since $b' \in I_i$ for all $i \leq n-1$, we have $a + I_i = a' + I_i = a_i + I_i$ for $i \leq n-1$. Furthermore, we have $a = a' - b' = a_n - b_n$, and since $b_n \in I_n$, we deduce that $a + I_n = a_n + I_n$ too. □

3.2. **Twisted quantum odd spheres as $k$-graph algebras.** To establish Theorem 3.3, we will appeal to Whitehead's machinery [22] (see also [17]) of twisted relative Cuntz–Krieger algebras of higher-rank graphs as summarized in Section 2.5. Fortunately, the only higher-rank graphs we need to consider are the following elementary examples.

Let $\Lambda$ denote a copy of the monoid $\mathbb{N}^{N+1}$ under addition. This becomes an $(N+1)$-graph under the degree map $d : \Lambda \to \mathbb{N}^{N+1}$ given by the identity map on $\mathbb{N}^{N+1}$. We write $e_0, \ldots, e_N$ for the canonical generators of $\mathbb{N}^{N+1}$. Since we are viewing $\Lambda$ as a category, we write $\mu\nu$ for the composition of elements $\mu, \nu$. This is really just $\mu + \nu$ when the two are regarded as elements of $\mathbb{N}^{N+1}$. The unique vertex of $\Lambda$ is $0 \in \mathbb{N}^{N+1}$. For $\mu = (\mu_0, \ldots, \mu_N) \in \Lambda$, we write $|\mu| := \sum_{i=0}^{N} \mu_i$.

We will realize $A_\theta(N+1)$ as a twisted relative Cuntz–Krieger algebra for $\Lambda$. We need to describe an appropriate 2-cocycle $c$ and collection $\mathcal{E}$ of finite exhaustive sets.

Fix a skew-symmetric matrix $\theta \in M_{N+1}(\mathbb{R})$, and let $A(\theta)$ be the lower-triangular matrix with $A(\theta)_{i\,j} = \theta_{i\,j}$ for $i > j$ and $A(\theta)_{i\,j} = 0$ for $i \leq j$. Since $\Lambda$ has one vertex, every pair in $\Lambda$ is composable, and the function $c : \Lambda \times \Lambda \to \mathbb{T}$ given by

$$c(\mu, \nu) := e^{\pi i (d(\mu)^T A(\theta) d(\nu))} \tag{3.8}$$

is a 2-cocycle on $\Lambda$.

Since $\mu\nu \in \mu\Lambda \cap \nu\Lambda$ for all $\mu, \nu \in \Lambda$, every finite nonempty subset of $\Lambda \setminus \{0\}$ is a finite exhaustive set. Hence $\mathcal{E} := \{\{e_0, \ldots, e_N\}\}$ is a collection of finite exhaustive sets, and we can form the $C^*$-algebra $C^*(\Lambda, c; \mathcal{E})$. This $C^*$-algebra has the same universal property as $A_\theta(N+1)$. Hence we may identify the two via an isomorphism that carries the generator $w_i \in A_\theta(N+1)$ to the generator $s^c_{\mathcal{E}}(e_i)$ of $C^*(\Lambda, c; \mathcal{E})$ for each $0 \leq i \leq N$.

*Proof of Theorem 3.3.* The relations (3.4) and (3.5) are invariant under multiplication of the $s_i$ by any fixed $\lambda \in U(1)$. Thus the universal property of $A_\theta(N+1)$ yields the desired $U(1)$-action.

The universal property of $\mathcal{T}_\theta^{N+1}$ yields a homomorphism

$$\psi_\theta : \mathcal{T}_\theta^{N+1} \longrightarrow C(S^{2N+1}_{H,\theta}) \quad \text{given by} \quad \psi_\theta(a) = (\sigma_0(a), \sigma_1(a), \ldots, \sigma_N(a)). \tag{3.9}$$

Applying Lemma 3.4 to $A = \mathcal{T}_\theta^{N+1}$ and the ideals $I_i = \ker(\sigma_i)$ shows that

$$C(S^{2N+1}_{H,\theta}) = \{(\sigma_0(a), \sigma_1(a), \ldots, \sigma_N(a)) \mid a \in \mathcal{T}_\theta^{N+1}\}, \tag{3.10}$$

so that $\psi_\theta$ is surjective. Since $\prod_{j=0}^{N}(1 - w_j w_j^*)$ belongs to $\ker \sigma_i$ for each $i$, it belongs to $\ker \psi_\theta$, and so $\psi_\theta$ descends to a surjective homomorphism $\phi_\theta : A_\theta(N+1) \to C(S^{2N+1}_{H,\theta})$, which satisfies

$$\phi_\theta(s_i) = (\sigma_0(w_i), \ldots, \sigma_N(w_i)) =: \mathbf{s}_i \quad \text{for all } i. \tag{3.11}$$

Identifying $A_\theta(N+1)$ with the $(N+1)$-graph algebra $C^*(\Lambda, c; \mathcal{E})$ as above, reduces our task to checking that the homomorphism $\rho : C^*(\Lambda, c; \mathcal{E}) \to C(S^{N+1}_{H,\theta})$ satisfying $\rho(s^c_{\mathcal{E}}(e_i)) = \phi_\theta(s_i)$ is injective. For this, we aim to apply the gauge-invariant uniqueness theorem [17, Theorem 3.15] for $C^*(\Lambda, c; \mathcal{E})$.

The homomorphism $\rho$ is equivariant for the gauge actions on $C(S^{2N+1}_{H,\theta})$ and $C^*(\Lambda, c; \mathcal{E})$. Since $s^c_{\mathcal{E}}(0)$ is the identity element of $C^*(\Lambda, c; \mathcal{E})$, we have $\rho(s^c_{\mathcal{E}}(0)) = (1, 1, \dots, 1) \neq 0$. Hence, by [17, Theorem 3.15], it suffices to show that for each finite exhaustive set $F$ in the complement of $\overline{\mathcal{E}}$ (see Section 2.5), we have $\rho\big(\prod_{\mu \in F}(s^c_{\mathcal{E}}(0) - s^c_{\mathcal{E}}(\mu)s^c_{\mathcal{E}}(\mu)^*)\big) \neq 0$.

We first have to identify $\overline{\mathcal{E}}$. Recall $\Lambda = \mathbb{N}^{N+1}$, then the set

$$\mathcal{E}' := \{F \subset \Lambda \setminus \{0\} \mid \text{ there exists } i > 0 \text{ such that } |p| > i \text{ implies } p \geq q \text{ for some } q \in F\}$$

is satiated and contains $\mathcal{E}$. An induction then shows that $\mathcal{E}' \subseteq \overline{\mathcal{E}}$. Hence the two are equal.

Now suppose that $F$ is a finite exhaustive set in the complement of $\overline{\mathcal{E}}$. Then there is a sequence $(p^i)$ in $\Lambda$ with $|p^i| \to \infty$ such that $p^i \not\geq q$ for all $q \in F$ and all $i \in \mathbb{N}$. By passing to a subsequence, we may assume that $p^i_j \to \infty$ for some $j \leq N$. Since $p^i \not\geq q$ for all $q \in F$ and all $i$, it follows that $q \in F$ implies $q_l > 0$ for some $l \neq j$. Hence there exists $l \neq j$ such that $q \geq e_l$, which forces

$$s^c_{\mathcal{E}}(q)s^c_{\mathcal{E}}(q)^* = s^c_{\mathcal{E}}(e_l)s^c_{\mathcal{E}}(q - e_l)s^c_{\mathcal{E}}(q - e_l)^* s^c_{E}e(e_l)^* \leq s^c_{\mathcal{E}}(e_l)s^c_{\mathcal{E}}(e_l)^*. \tag{3.12}$$

Thus

$$\rho\big(1 - s^c_{\mathcal{E}}(q)s^c_{\mathcal{E}}(q)^*\big) \geq \rho\big(1 - s^c_{\mathcal{E}}(e_l)s^c_{\mathcal{E}}(e_l)^*\big) = 1 - \mathbf{s}_l\mathbf{s}_l^*. \tag{3.13}$$

Applying this reasoning to each $q \in F$, we obtain

$$\rho\big(\prod_{q \in F}(1 - s^c_{\mathcal{E}}(q)s^c_{\mathcal{E}}(q)^*)\big) \geq \prod_{l \neq j}(1 - \mathbf{s}_l\mathbf{s}_l^*). \tag{3.14}$$

By (3.11), the $\mathbf{s}_l$ lie in the subalgebra $C(S^{2N+1}_{H,\theta})$ of $\bigoplus_{i=0}^N B_i$ where $B_i = \mathcal{T}_\theta^{N+1}/I_i$. Hence the $j$th coordinate of $\prod_{l \neq j}(1 - \mathbf{s}_l\mathbf{s}_l^*)$ is equal to

$$\Big(\prod_{l \neq j}(1 - \mathbf{s}_l\mathbf{s}_l^*)\Big)_j = \sigma_j\Big(\prod_{l \neq j}(1 - w_l w_l^*)\Big), \tag{3.15}$$

so that it suffices to show that the right-hand side of (3.15) is nonzero. For this, observe that $\sigma_j(\mathcal{T}_\theta^{N+1})$ is universal for the same relations as the twisted relative Cuntz–Krieger algebra $C^*(\Lambda, c; \{e_j\})$. Thus there is an isomorphism $\sigma_j(\mathcal{T}_\theta^{N+1}) \to C^*(\Lambda, c; \{e_j\})$ that carries $\sigma_j(w_l)$ to $s^c_{\{e_j\}}(e_l)$ for each $l$. One checks that the satiation $\overline{\{e_j\}}$ of $\{e_j\}$ does not contain the set $\{e_l \mid l \neq j\}$. Hence Proposition 3.9 of [17] implies that

$$\prod_{l \neq j}(1 - s^c_{\{e_j\}}(e_l)s^c_{\{e_j\}}(e_l)^*) \neq 0, \tag{3.16}$$

giving $\sigma_j\Big(\prod_{l \neq j}(1 - w_l w_l^*)\Big) \neq 0$ as required. This completes the proof that $\phi_\theta$ is an isomorphism.

Finally, observe that since each $B_i = \mathcal{T}_\theta^{N+1}/I_i$ and $B_{ij} = \mathcal{T}_\theta^{N+1}/(I_i + I_j)$ by definition, the homomorphisms $\pi^i_j$ are distributive in the sense of [14, Definition 2]. Lemma 3.4 shows in particular that given distinct $i, j, k$ and elements $b_i \in B_i$ and $b_j \in B_j$ such that $\pi^i_j(b_i) = \pi^j_i(b_j)$, there exists $b_k \in B_k$ such that $\pi^k_i(b_k) = \pi^i_k(b_i)$ and $\pi^k_j(b_k) = \pi^j_k(b_j)$. Hence Theorem 1 of [14] implies that the $\pi^i_j$ satisfy the cocycle condition of Definition 2.1. $\square$

3.3. **Twisted multipullback quantum complex projective spaces.** Our twisted multipullback quantum odd sphere $C^*$-algebras (see Definition 3.2), yield a natural construction of a family of $\theta$-twisted complex projective space $C^*$-algebras as fixed-point algebras. Using the $U(1)$-action $\alpha$ on $C(S^{2N+1}_{H,\theta})$ from equation (3.3), we define

$$C(\mathbb{P}^N_\theta(\mathcal{T})) := C(S^{2N+1}_{H,\theta})^\alpha. \tag{3.17}$$

Further study of this twisted quantum projective space algebra is beyond the scope of the current paper.

3.4. **The untwisted case.** Recall that $s$ denotes the isometry generating the Toeplitz algebra $\mathcal{T}$. Observe that the universal properties of the maximal tensor product and of the untwisted algebra $\mathcal{T}_0^{N+1}$ show that the map

$$\mathcal{T}_0^{N+1} \ni w_j \longmapsto 1^{\otimes j} \otimes s \otimes 1^{\otimes N-j} \in \mathcal{T}^{\otimes N+1} \tag{3.18}$$

is an isomorphism.

To see where Definition 3.2 comes from, and how it relates to noncommutative solid tori, recall that $\sigma$ denotes the symbol map from $\mathcal{T}$ to $C(\mathbb{T})$. When $\theta = 0$, we denote $C(S^{2N+1}_{H,\theta})$ by $C(S^{2N+1}_H)$. We have $\mathcal{T}_0^{N+1} = \mathcal{T}^{\otimes N+1}$, each $I_i$ is precisely the kernel of

$$\mathrm{id}^{\otimes i} \otimes \sigma \otimes \mathrm{id}^{\otimes N-i} : \mathcal{T}^{\otimes N+1} \longrightarrow B_i := \mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-i}, \tag{3.19}$$

and so each $B_i$ is the noncommutative solid torus algebra $\mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-i}$. The algebras $B_i$ and $B_{ij}$ and the maps $\pi^i_j$ of Definition 3.2 are then given by

$$\begin{gathered} B_{ij} := \mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes j-i-1} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-j}, \quad i<j, \quad i,j \in \{0,1,\dots,N\}, \\ B_{ij} := B_{ji}, \quad j<i, \quad i,j \in \{0,1,\dots,N\}, \quad \text{and} \\ \pi^i_j := \mathrm{id}^j \otimes \sigma \otimes \mathrm{id}^{N-j} : B_i \to B_{ij}, \quad i \neq j, \quad i,j \in \{0,1,\dots,N\}. \end{gathered} \tag{3.20}$$

So our definition of $C(S^{2N+1}_H)$ as the multipullback along the $\pi^i_j$ is a natural noncommutative dual to the Heegaard-type splitting of $S^{2N+1}$ described in Section 2.2

## 4. Strong connections on twisted multipullback quantum odd spheres

Since our paper is focused only on free $U(1)$-actions on unital $C^*$-algebras, we avoid the general coalgebraic formalism of strong connections of [3], and formulate the concept of a strong connection from [8] solely for $U(1)$-actions on unital $C^*$-algebras.

Let $A$ be a unital $C^*$-algebra carrying a $U(1)$-action. For $m \in \mathbb{Z}$, recall that $A_m$ denotes the spectral subspace $\{a \in A \mid \lambda \cdot a = \lambda^m a \text{ for all } \lambda \in U(1)\}$. We write $\mathbb{C}[u,u^*]$ for the $^*$-algebra of Laurent polynomials. Let $\ell$ be a unital linear map

$$\ell : \mathbb{C}[u,u^*] \longrightarrow \Big(\bigoplus\nolimits_{m\in\mathbb{Z}} A_m\Big) \underset{\mathrm{alg}}{\otimes} \Big(\bigoplus\nolimits_{m\in\mathbb{Z}} A_m\Big) \subseteq A \underset{\mathrm{alg}}{\otimes} A, \tag{4.1}$$

where $\bigoplus_{m\in\mathbb{Z}} A_m$ denotes the algebraic direct sum of the spectral subspaces. We say that $\ell$ is a *strong connection* for the $U(1)$-action on $A$ if, writing $m_A : A \otimes_{\mathrm{alg}} A \to A$ for the multiplication map, we have

$$(m_A \circ \ell)(h) = h(1)1_A \quad \text{for all } h \in \mathbb{C}[u,u^*], \tag{4.2}$$

and

$$\ell(u^n) \in A_{-n} \otimes A_n \quad \text{for all } n \in \mathbb{Z}. \tag{4.3}$$

Note that by [19] the existence of a strong connection is equivalent to strong grading:

$$A_m A_n = A_{m+n} \quad \text{for all } m, n \in \mathbb{Z}. \tag{4.4}$$

Moreover, by the main theorem of [1], the existence of a strong connection is equivalent to $C^*$-freeness.

4.1. **Equivariant homomorphisms and spectral subspaces.** Consider a $U(1)$-equivariant $^*$-homomorphism $f\colon A \to A'$ of unital $U(1)$-$C^*$-algebras, and suppose that the $U(1)$-action on $A$ is $C^*$-free. Then there exists a strong connection $\ell$ on $A$. It is straightforward to check that $\ell' := (f \otimes f) \circ \ell$ is a strong connection on $A'$, so that the $U(1)$-action on $A'$ is also $C^*$-free. (Here we abuse the notation by writing $f$ instead of $f$ restricted to the algebraic direct sum of spectral subspaces.) Furthermore, the $U(1)$-equivariance of $f$ guarantees that its restriction to the fixed-point subalgebra $B := A^{U(1)}$ corestricts to the fixed-point subalgebra $B' := (A')^{U(1)}$. It is important to note that $f$ turns $B'$ into a $B'$-$B$ bimodule given by the usual multiplication on the left and the formula $b' \cdot b := b' f(b)$ on the right.

Consider now $f$ corestricted to its image. Since it is a linear surjection over a field, it splits, i.e. there exists a linear map $g\colon f(A) \to A$ such that $f \circ g = \mathrm{id}$. (Now we abuse the notation by writing $f$ instead of $f$ corestricted to its image.) We have $A = g(f(A)) \oplus \ker f$. Next, let $\{a'_j\}_j$ be an extension of a basis $\{e'_i\}_i$ of $f(A)$ to a basis of $A'$. Also, let $\{e_k\}_k$ be a basis of $\ker f$. Then $\{a_l\}_l := \{g(e'_i)\}_i \cup \{e_k\}_k$ is a basis of $A$, and $f(a_l) = a'_l$ or $f(a_l) = 0$. Now, for any $n \in \mathbb{Z}$, we can write $\ell(u^n) = \sum_{l \in L} a_l \otimes r_l(u^n)$ and $\ell'(u^n) = \sum_{l \in L'} a'_l \otimes f(r_l(u^n))$. Here $L'$ and $L$ are respectively $m'$ and $m$ element sets, with $m' \leq m$, and $f(a_l) = a'_l$ for $l \leq m'$ and $f(a_l) = 0$ for $l > m'$.

It follows from the Chern-Galois theory of [3] that the existence of a strong connection guarantees that spectral subspaces are finitely generated projective as left modules over fixed-point $C^*$-algebras. (We think of them as associated noncommutative line bundles.) Moreover, given a strong connection $\ell$ and a spectral subspace $A_n$, we have an explicit formula given in [3, Theorem 3.1] for an idempotent $E^n$ representing the spectral subspace: $E^n_{kl} := r_k(u^n) a_l$ . Hence $f(E^n_{kl}) = f(r_k(u^n)) a'_l$ for $l \leq m'$ and $f(E^n_{kl}) = 0$ for $l > m'$ are the matrix coefficients of an idempotent representing $B' \otimes_B A_n$. Using the strong connection $\ell'$ and the linear basis $\{a'_l\}_l$, we conclude that the matrix coefficients of an idempotent representing $A'_n$ are also $f(r_k(u^n)) a'_l$, but with indices $k, l \in L'$.

To continue this reasoning and to take care of the range of indices, it is convenient to adopt the block-matrix notation. Let $\beta_n := (r_1(u^n), \ldots, r_m(u^n))$ and $\gamma := (a_1, \ldots, a_m)$. Much in the same way, let $\beta'_n := (f(r_1(u^n)), \ldots, f(r_{m'}(u^n)))$ and $\gamma' := (a'_1, \ldots, a'_{m'})$. Then $E^n = {\beta_n}^T \gamma \in M_m(B)$ is an idempotent matrix representing $A_n$, and $(E')^n = {\beta'_n}^T \gamma' \in M_{m'}(B')$ is an idempotent matrix representing $A'_n$. Finally, put

$$\begin{aligned} \beta''_n &:= (f(r_1(u^n)), \ldots, f(r_m(u^n)) =: (\beta'_n, \rho'_n) \qquad \text{and} \\ \gamma'' &:= (\gamma', 0, \ldots, 0) \qquad \text{(with } m - m' \text{ zeros at the end).} \end{aligned} \tag{4.5}$$

Then $(E'')^n = \beta''^T_n \gamma'' \in M_m(B')$ is an idempotent matrix representing $B' \otimes_B A_n$.

The crux of our argument is that $(E')^n$ and $(E'')^n$ represent *isomorphic* left $B'$-modules. Indeed, after extending the matrix $(E')^n$ by zeros to size $m$, we obtain a matrix conjugate[1] to $(E'')^n$:

$$\begin{pmatrix} 1 & 0 \\ -{\rho'_n}^T\gamma' & 1 \end{pmatrix}\begin{pmatrix} {\beta'_n}^T \\ {\rho'_n}^T \end{pmatrix}\begin{pmatrix} \gamma' & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ {\rho'_n}^T\gamma' & 1 \end{pmatrix} = \begin{pmatrix} {\beta'_n}^T \\ 0 \end{pmatrix}\begin{pmatrix} \gamma' & 0 \end{pmatrix}. \tag{4.6}$$

Here we used the fact that $\gamma'{\beta'_n}^T = 1$, which is condition (4.2) for the strong connection $\ell'$. Following the reasoning of the previous paragraph we have arrived at:

**Theorem 4.1.** *Let $f\colon A \to A'$ be a $U(1)$-equivariant $*$-homomorphism of unital $U(1)$-$C^*$-algebras, and let $B$ and $B'$ be the respective fixed-point $C^*$-subalgebras. Assume that the $U(1)$-action on $A$ is $C^*$-free. For each $n \in \mathbb{Z}$, let $A_n$ and $A'_n$ denote the $n^{th}$ spectral subspaces of $A$ and $A'$ respectively. Then, for any $n \in \mathbb{Z}$, there is an isomorphism of finitely generated left $B'$-modules:*

$$B' \underset{B}{\otimes} A_n \cong A'_n.$$

*In particular, the induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies*

$$(f|_B)_*\big([A_n]\big) = [A'_n] \quad \textit{for every } n \in \mathbb{Z}.$$

4.2. **A strong connection on $S^{2N+1}_{H,\theta}$.** With the help of the $\sigma_j$ of Definition 3.2, we define $\mathbf{s}_i \in C(S^{2N+1}_{H,\theta})$ by

$$\mathbf{s}_i := \big(\sigma_0(w_i), \dots, \sigma_N(w_i)\big). \tag{4.7}$$

It is trivial to verify that for all $i, j \in \{0, \dots, N\}$:

$$\mathbf{s}_i\mathbf{s}_j = e^{2\pi i\theta_{ij}}\mathbf{s}_j\mathbf{s}_i, \quad \mathbf{s}_i\mathbf{s}_j^* = e^{-2\pi i\theta_{ij}}\mathbf{s}_j^*\mathbf{s}_i, \quad \text{when } i \neq j, \tag{4.8a}$$

$$\mathbf{s}_i^*\mathbf{s}_i = 1, \quad \text{and} \tag{4.8b}$$

$$\prod_{k=0}^{N}(1 - \mathbf{s}_k\mathbf{s}_k^*) = 0. \tag{4.8c}$$

In what follows we will also need the following family of $U(1)$-fixed elements of $C(S^{2N+1}_{H,\theta})$:

$$H_N = 1, \quad H_i = \prod_{j=i+1}^{N}(1 - \mathbf{s}_j\mathbf{s}_j^*), \quad i \in \{0, \dots, N-1\}. \tag{4.9}$$

Consider the linear map

$$\ell : \mathbb{C}[u, u^*] \longrightarrow \big(\textstyle\bigoplus_{m\in\mathbb{Z}} C(S^{2N+1}_{H,\theta})_m\big) \underset{\text{alg}}{\otimes} \big(\textstyle\bigoplus_{m\in\mathbb{Z}} C(S^{2N+1}_{H,\theta})_m\big) \tag{4.10}$$

defined inductively as follows:

$$\ell(1) := 1 \otimes 1, \qquad \ell(u^n) = \mathbf{s}_0^{*n} \otimes \mathbf{s}_0^n \text{ for } n > 0, \quad \text{and} \tag{4.11}$$

$$\ell(u^{n-1}) := \sum_{0\le k\le N}\Big((\mathbf{s}_k \otimes 1)\ell(u^n)(1 \otimes \mathbf{s}_k^* H_k)\Big) \text{ for } n \le 0. \tag{4.12}$$

[1] We are grateful to Tomasz Maszczyk for pointing this out to us.

We claim that $\ell$ is a strong connection for the $U(1)$-action on $C(S^{2N+1}_{H,\theta})$. Indeed, equation (4.3) for $n \geq 0$ is trivial, and for $n < 0$ follows from an elementary induction argument. Equation (4.2) for $n \geq 0$ is trivial because $\mathbf{s}_0$ is an isometry. To check it for $n < 0$, we first use the sphere equation (4.8c) to see that

$$\sum_{k=0}^{N} \mathbf{s}_k \mathbf{s}_k^* H_k = 1. \tag{4.13}$$

The claim then follows from a straightforward induction (cf. the proof of [12, Lemma 4.2]) using the recursive formula (4.12).

## 5. Proof of the main theorem

5.1. **Proof of Theorem 1.1(1) and (2).** Statement (1) is obtained by applying Theorem 3.3 with $\theta = 0 \in M_{N+1}(\mathbb{R})$, and defining $\mathbf{s}_i$ to be the image of $s_i$ under $\phi_0$. The universal property of the $C^*$-algebra $A_0(N+1)$ shows that there is an action of $U(1)$ on $A_0(N+1)$ such that $\lambda \cdot s_i = \lambda s_i$ for all $i$, and this induces an action $\alpha$ on $C(S^{2N+1}_H)$ satisfying the desired formula. To see that this action is $C^*$-free, it suffices to show that the $C^*$-algebra is strongly graded by $\alpha$, which follows from Section 4.2 with $\theta = 0$.

5.2. **Proof of Theorem 1.1(3).** To compute the fixed-point subalgebra $C(S^{2N+1}_H)^\alpha$, we follow the general strategy of Section 2.3 of gauging diagonal coactions to coactions on the right-most components. As in the classical case (analyzed in detail in Section 2.4), to combine the gauging of coactions with the multipullback structure of $C(S^{2N+1}_H)$, we have to carefully keep track of permutations of tensorands.

5.2.1. *Notation.* For $i < j$, we denote by $(i..j) : \{0, \ldots, N\} \to \{0, \ldots, N\}$ the permutation of $\{0, \ldots, N\}$ given by

$$(i..j)(k) = \begin{cases} k+1 & \text{for } i \leq k < j, \\ i & \text{for } k = j, \\ k & \text{otherwise.} \end{cases} \tag{5.1}$$

The inverse permutation $(i..j)^{-1}$ we abbreviate to $(j..i)$.

Next, let $\{A_i\}_{i\in\{0,\ldots,N\}}$ be a family of $C^*$-algebras. A permutation $\tau$ of $\{0, \ldots, N\}$ induces the following $*$-homomorphism:

$$T_\tau : \bigotimes_{i=0}^{N} A_i \longrightarrow \bigotimes_{i=0}^{N} A_{\tau(i)}, \quad (a_0, \ldots, a_N) \longmapsto (a_{\tau(0)}, \ldots, a_{\tau(N)}). \tag{5.2}$$

5.2.2. *The multipullback structure of* $C(S^{2N+1}_H)^R$. We define the $C^*$-algebra $C(S^{2N+1}_H)^R$ to be the image of $C(S^{2N+1}_H)$ under $\prod_{i=0}^{N} \widehat{\kappa} \circ T_{(N..i)}$ (see (2.16)). To unravel its multipullback structure, we adopt the following notation:

$$\begin{gathered} \hat{B}_i := \mathcal{T}^{\otimes N} \otimes C(U(1)), \quad i \in \{0, \ldots, N\}, \\ \hat{B}_{ij} = \mathcal{T}^{\otimes j-1} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-j} \otimes C(U(1)), \quad i < j, \quad i,j \in \{0,1,\ldots,N\}, \\ \hat{B}_{ij} := \hat{B}_{ji}, \quad j < i, \quad i,j \in \{0,1,\ldots,N\}. \end{gathered} \tag{5.3}$$

One can view $\hat{B}_i$'s and $\hat{B}_{ij}$'s as $C(U(1))$-comodule $C^*$-algebras with both diagonal and the rightmost-component coaction. We distinguish them by adorning the algebras with the rightmost coaction by $R$-superscripts. Note that

$$B_i \xrightarrow{T_{(N..i)}} \hat{B}_i \xrightarrow{\widehat{\kappa}} \hat{B}_i^R \,. \tag{5.4}$$

Having $\hat{B}_i^R$'s as our building blocks, we can now compute morphisms $\hat{\pi}^i_j$'s that assemble them into the multipullback $C^*$-algebra $C(S_H^{2N+1})^R$. Fix any $i<j$. We determine $\hat{\pi}^i_j$ and $\hat{\pi}^j_i$ through the following commutative diagram:

$$\begin{tikzcd}
\hat{B}_i^R \arrow[d, "\widehat{\kappa}^{-1}"'] \arrow[dddrr, dashed, "\hat{\pi}^i_j"] & & & & \hat{B}_j^R \arrow[d, "\widehat{\kappa}^{-1}"] \arrow[dddll, dashed, "\hat{\pi}^j_i"'] \\
\hat{B}_i \arrow[d, "T_{(i..N)}"'] & & & & \hat{B}_j \arrow[d, "T_{(j..N)}"] \\
B_i \arrow[d, "\pi^i_j"'] & & & & B_j \arrow[d, "\pi^j_i"] \\
B_{ij} \arrow[r, "T_{(N..i)}"'] & \hat{B}_{ij} \arrow[r, "\widehat{\kappa}"'] & \hat{B}_{ij}^R & \hat{B}_{ij} \arrow[l, "\widehat{\kappa}"] & B_{ij} \,. \arrow[l, "T_{(N..i)}"]
\end{tikzcd} \tag{5.5}$$

Now one can easily verify that $C(S_H^{2N+1})^R$ can be equivalently defined as the multipullback $C^*$-algebra over the $\hat{\pi}^i_j$. Also, one immediately observes that $\hat{\pi}^i_j = \pi^i_{j-1}$ for any $i<j$.

To compute $\hat{\pi}^i_j$, let us introduce the following notation. Let $I := \{m, m+1, \ldots, n\}$ and fix $0 \le m \le k \le n \le N$ and $m \le i < j \le n \le N$. Then for $x_i \in \mathcal{T}$, we write

$$\begin{gathered}
(x_i)_{m..n} := x_m \otimes x_{m+1} \otimes \cdots \otimes x_n, \\
(x_i)^k_{m..n} := x_m \otimes x_{m+1} \otimes \cdots \otimes x_{k-1} \otimes x_{k+1} \otimes \cdots \otimes x_n, \\
\textstyle\prod_{m..n} x_i := \displaystyle\prod_{i\in I} x_i, \quad \textstyle\prod^k_{m..n} x_i := \displaystyle\prod_{i\in I\setminus\{k\}} x_i, \quad \textstyle\prod^{k,l}_{m..n} x_i := \displaystyle\prod_{i\in I\setminus\{k,l\}} x_i.
\end{gathered} \tag{5.6}$$

Let $((t_k)^j_{0..N}, h) \in \hat{B}_j^R = \mathcal{T}^{\otimes N} \otimes C(U(1))$. Substituting definitions of $\widehat{\kappa}^{-1}$ and $T$ (see (2.17) and (5.2) respectively) yields

$$\begin{aligned}
&\hat{\pi}^j_i\big(((t_k)^j_{0..N} \otimes h)\big) \\
&\quad = \big(\widehat{\kappa} \circ T_{(N..i)} \circ \pi^j_i \circ T_{(j..N)} \circ \widehat{\kappa}^{-1}\big)\big(((t_k)^j_{0..N} \otimes h)\big) \\
&\quad = \big(\widehat{\kappa} \circ T_{(N..i)} \circ \pi^j_i \circ T_{(j..N)}\big)\big(((t_{k(0)})^j_{0..N} \otimes S(\textstyle\prod^j_{0..N} t_{k(1)})h)\big) \\
&\quad = \big(\widehat{\kappa} \circ T_{(N..i)}\big)\big(((t_{k(0)})_{0..i-1} \otimes \sigma(t_{i(0)}) \otimes (t_{k(0)})_{i+1..j-1} \otimes S(\textstyle\prod^j_{0..N} t_{k(1)})h \otimes (t_{k(0)})_{j+1..N})\big) \\
&\quad = \widehat{\kappa}\big(((t_{k(0)})^i_{0..j-1} \otimes S(\textstyle\prod^j_{0..N} t_{k(1)})h \otimes (t_{k(0)})_{j+1..N} \otimes \sigma(t_{i(0)}))\big).
\end{aligned}$$

Substituting subsequently the definition of $\widehat{\kappa}$ yields:

$$\begin{aligned}
\hat{\pi}^j_i\big(((t_k)^j_{0..N} \otimes h)\big) = \big((t_{k(0)(0)})^i_{0..j-1} &\otimes S(\big(\textstyle\prod^j_{0..N} t_{k(1)}\big)_{(2)})h_{(1)} \otimes (t_{k(0)(0)})_{j+1..N} \\
&\otimes \big(\textstyle\prod^{i,j}_{0..N} t_{k(0)(1)}\big) S(\big(\textstyle\prod^j_{0..N} t_{k(1)}\big)_{(1)})h_{(2)}\sigma(t_{i(0)})\big).
\end{aligned} \tag{5.7}$$

Using the coassociativity of the coaction and the commutativity of $C(U(1))$, furthermore we obtain:

$$\begin{aligned}\hat{\pi}_i^j\big(((t_k)_{0..N}^j, h)\big) = \big((t_{k(0)})_{0..j-1}^i \otimes S\big(\textstyle\prod_{0..N}^{i,j} t_{k(3)}\big) S(t_{i(2)}) h_{(1)} \otimes (t_{k(0)})_{j+1..N}\\ \otimes \big(\textstyle\prod_{0..N}^{i,j} t_{k(1)} S(t_{k(2)})\big) S(t_{i(1)}) h_{(2)} \sigma(t_{i(0)})\big).\end{aligned} \tag{5.8}$$

Taking advantage of the antipode and counit properties in the last tensorand simplifies the above expression to:

$$\begin{aligned}&\hat{\pi}_i^j\big(((t_k)_{0..N}^j, h)\big)\\ &= \big((t_{k(0)})_{0..j-1}^i \otimes S\big(\textstyle\prod_{0..N}^{i,j} t_{k(1)}\big) S(t_{i(2)}) h_{(1)} \otimes (t_{k(0)})_{j+1..N} \otimes S(t_{i(1)}) h_{(2)} \sigma(t_{i(0)})\big).\end{aligned} \tag{5.9}$$

Finally, using in addition the $U(1)$-equivariance of the symbol map, i.e. the equality $\sigma(t_{i(0)}) \otimes t_{i(1)} = \sigma(t_i)_{(1)} \otimes \sigma(t_i)_{(2)}$, we compute:

$$\begin{aligned}&\hat{\pi}_i^j\big(((t_k)_{0..N}^j \otimes h)\big)\\ &= \big((t_{k(0)})_{0..j-1}^i \otimes S\big(\textstyle\prod_{0..N}^{i,j} t_{k(1)}\big) S(\sigma(t_i)_{(3)}) h_{(1)} \otimes (t_{k(0)})_{j+1..N} \otimes h_{(2)} \sigma(t_i)_{(1)} S(\sigma(t_i)_{(2)})\big)\\ &= \big((t_{k(0)})_{0..j-1}^i \otimes S\big(\textstyle\prod_{0..N}^{i,j} t_{k(1)}\big) S(\sigma(t_i)) h_{(1)} \otimes (t_{k(0)})_{j+1..N} \otimes h_{(2)}\big).\end{aligned}$$

5.2.3. *The multipullback structure of the fixed-point subalgebra* $C(S_H^{2N+1})^\alpha$. Define

$$\hat{\psi}_{ij} : \mathcal{T}^{\otimes^i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes^{N-i-1}} \otimes C(U(1)) \longrightarrow \mathcal{T}^{\otimes^{j-1}} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes^{N-j}} \otimes C(U(1)) \tag{5.10}$$

by

$$\begin{aligned}&\hat{\psi}_{ij}\big((t_k)_{0..i-1} \otimes h \otimes (t_k)_{i+1..N}^j \otimes h'\big)\\ &\quad := \big((t_{k(0)})_{0..j-1}^i \otimes S\big(\textstyle\prod_{0..N}^{i,j} t_{k(1)}\big) S(h) h'_{(1)} \otimes (t_{k(0)})_{j+1..N} \otimes h'_{(2)}\big).\end{aligned} \tag{5.11}$$

Then we have $\hat{\pi}_i^j = \hat{\psi}_{ij} \circ \pi_i^j$.

Since $C(S_H^{2N+1})$ and $C(S_H^{2N+1})^R$ are isomorphic as $U(1)$-$C^*$-algebras, they have isomorphic fixed-point subalgebras. Furthermore, as the multipullback structure of $C(S_H^{2N+1})^R$ is $U(1)$-equivariant, we can conclude that its fixed-point subalgebra is the multipullback $C^*$-algebra given by the maps $\hat{\pi}_j^i$ restricted and corestricted to the fixed-point subalgebras $\hat{B}_i^{\hat{\alpha}}$ and $\hat{B}_{ij}^{\hat{\alpha}}$ respectively, where $\hat{\alpha}$ always stands for the $U(1)$-action on the rightmost tensorand. It is immediate that

$$\hat{B}_i^{\hat{\alpha}} = \mathcal{T}^{\otimes N} \quad \text{and} \quad \hat{B}_{ij}^{\hat{\alpha}} = \mathcal{T}^{\otimes j-1} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-j}. \tag{5.12}$$

It is also clear that the restriction-corestriction of $\hat{\pi}_j^i$, $i < j$, becomes the appropriately restricted and corestricted $\pi_{j-1}^i$. Finally, we observe that the restriction-corestriction of $\hat{\pi}_i^j$, $i < j$, factorizes through the map $\psi_{ij}$ obtained by the restriction-corestriction of $\hat{\psi}_{ij}$ given by plugging in $h' = 1$ in (5.10) and forgetting the last tensorand:

$$\begin{aligned}&\psi_{ij} : \mathcal{T}^{\otimes^i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes^{N-i-1}} \longrightarrow \mathcal{T}^{\otimes^{j-1}} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes^{N-j}},\\ &\big((t_k)_{0..i-1} \otimes h \otimes (t_k)_{i+1..N}^j\big) \longmapsto \big((t_{k(0)})_{0..j-1}^i \otimes S\big(\textstyle\prod_{0..N}^{i,j} t_{k(1)}\big) S(h) \otimes (t_{k(0)})_{j+1..N}\big).\end{aligned} \tag{5.13}$$

As the above formula coincides with the defining formula for $C(\mathbb{P}^N(\mathcal{T}))$ (see [10, Definition 2.2]), we infer that $C(S_H^{2N+1})^\alpha \cong C(\mathbb{P}^N(\mathcal{T}))$.

5.3. **Proof of Theorem 1.1(4).** Let $A_0(2)$ and $A_0(N+1)$ be the universal $C^*$-algebras described in Theorem 3.3 for the zero matrix $\theta = 0$. Theorem 3.3 shows that there are $U(1)$-equivariant isomorphisms $A_0(2) \cong C(S^3_H)$ and $A_0(N+1) \cong C(S^{2N+1}_H)$. Let $s'_0 := s_0 \in A_0(2)$ and for $1 \le j \le N$ let $s'_j := s_1 \in A_0(2)$. Then $s'_0, \dots, s'_N$ satisfy the commutation relations (3.4) and the sphere equation (3.5), and so induce a $U(1)$-equivariant surjective *-homomorphism $A_0(N+1) \to A_0(2)$ carrying each generator $s_j$ of $A_0(N+1)$ to the corresponding $s'_j$. Hence we obtain a $U(1)$-equivariant surjective *-homomorphism

$$f : C(S^{2N+1}_H) \cong A_0(N+1) \longrightarrow A_0(2) \cong C(S^3_H) \tag{5.14}$$

such that $f(\mathbf{s}_0) = \mathbf{s}_0$ and $f(\mathbf{s}_j) = \mathbf{s}_1$ for every $j \ge 1$.

Combining this with the $C^*$-freeness of the diagonal $U(1)$-action on $C(S^{2N+1}_H)$, which follows from Section 4.2 for $\theta = 0$, we can apply the final statement of Theorem 4.1 to infer that the equality of $K_0$-classes $[C(S^{2N+1}_H)_m] = [C(S^{2N+1}_H)_n]$ implies the equality of $K_0$-classes $[C(S^3_H)_m] = [C(S^3_H)_n]$. Hence, by [11, Theorem 3.3], we obtain $m = n$, as needed.

## A. $K$-theory of twisted multipullback quantum odd spheres

To compute $K_*(C(S^{2N+1}_{H,\theta}))$, we first compute the $K$-theory of the untwisted quantum sphere $C(S^{2N+1}_H)$ by applying the Künneth theorem and then the six-term ideal-quotient exact sequence. We then apply results of [17] to see that the $K$-theory of $C(S^{2N+1}_{H,\theta})$ is identical to that of $C(S^{2N+1}_H)$.

Recall that $\mathcal{T}_0^{N+1}$ is canonically isomorphic to $\mathcal{T}^{\otimes N+1}$ via the map that carries the generator $w_i$ of $\mathcal{T}_0^{N+1}$ to the elementary tensor $1 \otimes \cdots \otimes 1 \otimes s \otimes 1 \otimes \cdots \otimes 1$, where the $s$ appears in the $i$th (counting from zero) tensor factor. Recall also that we have $K_0(\mathcal{T}) = \mathbb{Z}$ and $K_1(\mathcal{T}) = 0$ with the generator in $K_0$ being the class of the identity element. It then follows from the Künneth theorem (see, e.g., [21, Remarks 9.3.3]) that $K_0(\mathcal{T}_0^{N+1}) = \mathbb{Z}[1]$ and $K_1(\mathcal{T}_0^{N+1}) = 0$. For the following, given $m = (m_0, m_1, \dots, m_N) \in \mathbb{Z}^{N+1}$, we write $W_m$ for the element $\prod_{i=0}^N w_i^{m_i}$ of $\mathcal{T}_0^{N+1}$. (Note that we use the convention that $w_i^{-k} = (w_i^*)^k$ for $k \ge 0$.)

**Lemma A.1.** *For $N \ge 0$, there is an isomorphism of $\mathcal{K}(\ell^2(\mathbb{N}^{N+1}))$ onto the ideal $I$ of $\mathcal{T}_0^{N+1}$ generated by $\prod_{j=0}^N (1 - w_j w_j^*)$ that carries the matrix units $E_{pq}$ to*

$$w_p\Big(\prod_{j=0}^N (1 - w_j w_j^*)\Big) w_q^*.$$

*Proof.* Let $R := \prod_{j=0}^N (1 - w_j w_j^*)$. As the $w_i$ are commuting isometries, we see that $w_i^* R = 0 = R w_i$ for all $i$, and then we deduce that $W_p^* R = 0 = R W_p$ for all $p \in \mathbb{N}^{N+1} \setminus \{0\}$. Similarly, observe that

$$(W_p R W_q^*)(W_a R W_b^*) = (W_p R) W_q^* W_a R W_b^* = \delta_{q,a} w_p R w_b^*.$$

Since $(W_p R W_q^*)^* = W_q R W_p^*$, we see that the $W_p R W_q^*$ form a family of matrix units indexed by $\mathbb{N}^{N+1}$, and so there is a monomorphism of $\mathcal{K}(\ell^2(\mathbb{N}^{N+1})) \to I$ carrying each

$E_{pq}$ to $W_p R W_q^*$. Since $R$ is nonzero, and since $\mathcal{K}(\ell^2(\mathbb{N}^{N+1}))$ is simple, this homomorphism is injective. Surjectivity follows from

$$\prod_{j=0}^{N}(1 - w_j w_j^*) = (1 - w_0 w_0^*)R = R - w_0 R w_0^*. \tag{A.1}$$

□

The following result generalizes [2, Theorem 4.1] and [13, Theorem 3.2].

**Theorem A.2.** *Consider an integer $N \geq 1$ and a skew-symmetric matrix $\theta \in M_{N+1}(\mathbb{R})$. Then $K_0(C(S_{H,\theta}^{2N+1})) \cong \mathbb{Z} \cong K_1(C(S_{H,\theta}^{2N+1}))$, where the generator of $K_0(C(S_{H,\theta}^{2N+1}))$ is* [1].

*Proof.* We first consider the case where $\theta_{ij} = 0$ for all $i, j$. Theorem 3.3 combined with Lemma A.1 and the isomorphism $\mathcal{T}_0^{N+1} \cong \mathcal{T}^{\otimes N+1}$ given in (3.18) implies that

$$C(S_H^{2N+1}) \;\cong\; \mathcal{T}_0^{N+1}/I \;\cong\; \mathcal{T}^{\otimes N+1}/\mathcal{K}(\ell^2(\mathbb{N}^{N+1})). \tag{A.2}$$

The result will follow by examining the ideal-quotient exact sequence of $K$-groups from [21, Theorem 9.3.2].

We claim that the inclusion $\iota : \mathcal{K}(\ell^2(\mathbb{N}^{N+1})) \to \mathcal{T}_0^{N+1}$ of Lemma A.1 induces the zero map on $K$-theory. As $K_0(\mathcal{K}(\ell^2(\mathbb{N}^{N+1}))) \cong \mathbb{Z}$ is generated by $[R]$, we just have to show that $[R] = 0$ in $K_0(\mathcal{T}_0^{N+1}) \cong \mathbb{Z}$. To see this, observe that the isomorphism $\mathcal{T}_0^{N+1} \cong \mathcal{T}^{\otimes N+1}$ given by (3.18) carries $R$ to $(1 - ss^*) \otimes (1 - ss^*) \otimes \cdots \otimes (1 - ss^*)$. Since $s$ is an isometry, we have $[1 - ss^*] = [s^*s - ss^*] = 0$ in $K_0(\mathcal{T})$. As $K_1(\mathcal{T}) = 0$, the Künneth isomorphism implies that $[(1 - ss^*) \otimes (1 - ss^*) \otimes \cdots \otimes (1 - ss^*)] = 0$ in $K_0(\mathcal{T}^{\otimes N+1})$. Therefore $[R]$ is zero in $K_0(\mathcal{T}_0^{N+1})$ as claimed.

We know that $K_1(\mathcal{K}(\ell^2(\mathbb{N}^{N+1}))) = 0 = K_1(\mathcal{T}_0^{N+1})$, and so we have the following exact sequence:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] & K_0(C(S_H^{2N+1})) \arrow[d] \\ K_0(C(S_H^{2N+1})) \arrow[u] & 0 \arrow[l] & 0. \arrow[l] \end{tikzcd} \tag{A.3}$$

Now it follows that $K_0(C(S_H^{2N+1})) = \mathbb{Z}[1]$, and $K_1(C(S_H^{2N+1})) \cong \mathbb{Z}$.

For general $\theta$, identify $C(S_{H,\theta}^{2N+1})$ with $C^*(\Lambda, c; \mathcal{E})$ by Theorem 3.3 and Section 3.2. By (3.8), the cocycle $c$ on $\Lambda$ arises from exponentiation of an $\mathbb{R}$-valued cocycle, and [17, Theorem 6.1] implies that

$$K_*(C(S_{H,\theta}^{2N+1})) \;\cong\; K_*(C^*(\Lambda, c; \mathcal{E})) \;\cong\; K_*(C^*(\Lambda, 1; \mathcal{E})) \;\cong\; K_*(C(S_H^{2N+1})). \tag{A.4}$$

The isomorphisms preserve the $K_0$-class of the identity, whence the result. □

## B. Noncommutative line bundles associated to the Vaksman-Soibelman quantum spheres

The Vaksman-Soibelman quantum spheres described in [20] generalize $SU_q(2)$ from [23] and so may be viewed as a quantum three-dimensional sphere to any odd dimension $2N+1$.

For any $0 < q < 1$, the $C^*$-algebra $C(S_q^{2N+1})$ of the Vaksman-Soibelman quantum sphere is the universal $C^*$-algebra generated by $z_0, z_1, \ldots, z_N$, subject to the following relations:

$$\begin{aligned} &z_i z_j = q z_j z_i \quad \text{for } i<j, \qquad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j, \\ &z_i z_i^* = z_i^* z_i + (q^{-2}-1) \sum_{m=i+1}^{N} z_m z_m^*, \qquad \sum_{m=0}^{N} z_m z_m^* = 1. \end{aligned} \tag{B.1}$$

Note that $C(S_q^3) = C(SU_q(2))$, and the fundamental representation of $SU_q(2)$ is

$$\begin{pmatrix} z_0 & z_1 \\ -z_1^* & z_0^* \end{pmatrix}. \tag{B.2}$$

There is a natural (diagonal) $U(1)$-action on $C(S_q^{2N+1})$ given on generators by $z_i \mapsto \lambda z_i$. The map sending to zero all the generators but $z_0$ and $z_1$, and assigning to $z_0$ and $z_1$ the diagonal and the off-diagonal generator respectively, evidently defines a $U(1)$-equivariant $C^*$-algebra map:

$$C(S_q^{2N+1}) \xrightarrow{f} C(SU_q(2)), \qquad f(z_i) := 0 \text{ if } i>1 \text{ and } f(z_0) := z_0,\ f(z_1) := z_1. \tag{B.3}$$

The fixed-point subalgebra of the $U(1)$-action is, by definition, the $C^*$-algebra of the quantum complex projective space:

$$C(\mathbb{P}_q^N(\mathbb{C})) := C(S_q^{2N+1})^{U(1)}. \tag{B.4}$$

The $U(1)$-action on $C(S_q^{2N+1})$ is $C^*$-free by [18, Corollary 3]. This allows us to use the final statement of Theorem 4.1 to conclude that the induced map

$$\big(f|_{C(\mathbb{P}_q^N(\mathbb{C}))}\big)_* : K_0\big(C(\mathbb{P}_q^N(\mathbb{C}))\big) \longrightarrow K_0\big(C(\mathbb{P}_q^1(\mathbb{C}))\big) \tag{B.5}$$

satisfies

$$\big(f|_{C(\mathbb{P}_q^N(\mathbb{C}))}\big)_* \Big([C(S_q^{2N+1})_m]\Big) = [C(SU_q(2))_m] \tag{B.6}$$

for any $m \in \mathbb{Z}$. Since the index pairing computation of [9, Theorem 2.1] proves that $[C(SU_q(2))_m] = [C(SU_q(2))_k]$ implies $m = k$, we infer that:

**Theorem B.1.** *The spectral subspaces (section-modules of associated noncommutative line bundles) $C(S_q^{2N+1})_m$ are pairwise stably non-isomorphic as finitely generated projective left modules over $C(\mathbb{P}_q^N(\mathbb{C}))$, i.e., for all $N \in \mathbb{N} \setminus \{0\}$ and any $m, k \in \mathbb{Z}$, we have*

$$[C(S_q^{2N+1})_m] = [C(S_q^{2N+1})_k] \quad \Longrightarrow \quad m = k.$$

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