EQUIVALENT GROUPOIDS HAVE MORITA EQUIVALENT STEINBERG ALGEBRAS

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Abstract. Let G and H be Hausdorff ample groupoids and let R be a commutative unital ring. We show that if G and H are equivalent in the sense of Muhly-Renault-Williams, then the associated Steinberg algebras of locally constant R-valued functions with compact support are Morita equivalent. We deduce that collapsing a "collapsible subgraph" of a directed graph in the sense of Crisp and Gow does not change the Morita-equivalence class of the associated Leavitt path R-algebra, and therefore a number of graphical constructions which yield Morita equivalent C^*-algebras also yield Morita equivalent Leavitt path algebras.

1. Introduction

Two groupoids G and H are equivalent if they act freely and properly on the left and right (respectively) of a space Z in such a way that the quotient of Z by the action of G is homeomorphic to the unit space of H and vice versa. It was shown in [14] that if second-countable, locally compact, Hausdorff groupoids G and H are equivalent, then the associated full C^*-algebras are Morita equivalent. This result descends to reduced C^*-algebras, and also persists for groupoids which are locally Hausdorff (see [20]). The proof of this statement in [20] proceeds by constructing a linking groupoid L from copies of G, H, Z and the opposite space Z^{op} so that the groupoid C^*-algebra of L is a linking algebra for a $C^*(G)$–$C^*(H)$-imprimitivity bimodule.

Given a Hausdorff ample groupoid G and a commutative unital ring R, we consider the convolution R-algebra $A_R(G)$ of locally constant functions with compact support from G to R. We call $A_R(G)$ the *Steinberg algebra* associated to G. These algebras were introduced in [22] as a model for discrete inverse semigroup algebras. In the situation where $R = \mathbb{C}$, $A_{\mathbb{C}}(R)$ is a dense subalgebra of $C^*(G)$. Complex Steinberg algebras also include complex Kumjian-Pask algebras [2] and hence complex Leavitt path algebras. Uniqueness theorems and simplicity criteria for complex Steinberg algebras are established in [4] and [6]. These results indicate that the groupoid approach is a good unifying framework for understanding the striking similarities between the theory of graph C^*-algebras and the theory of Leavitt path algebras, which have attracted a lot of attention in recent years.

In this paper we present further evidence for this viewpoint. First we show that all Leavitt path R-algebras can be realised as Steinberg algebras (see example 3.2). Next we show that if G and H are Hausdorff ample groupoids, and if Z is a G–H equivalence,

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then the linking-groupoid construction of [20] yields another Hausdorff ample groupoid L. We then show that the Steinberg algebra $A_R(L)$ is, in the appropriate sense, a linking algebra for a surjective Morita context between $A_R(G)$ and $A_R(H)$, and hence that these two algebras are Morita equivalent.

We conclude by applying our result to the "collapsible subgraph" construction of Crisp and Gow [7]. They identify a specific type of subgraph T of a countable directed graph E and a collapsing process that yields a new graph F with vertices $E^0 \setminus T^0$, and show that $C^*(E)$ and $C^*(F)$ are Morita equivalent by realising one as a full corner of the other. We show that this is an instance of the Morita-equivalence theorem of [14] using the notion of an abstract transversal of the groupoid of E (see [14, Example 2.7]). We conclude that for arbitrary directed graphs E and commutative unital rings R, Crisp and Gow's collapsible subgraph construction yields Morita equivalent Leavitt path R-algebras $L_R(E)$ and $L_R(F)$.

2. Preliminaries

A groupoid is a small category in which every morphism has an inverse. Given a groupoid G, we write $r(\alpha)$ and $s(\alpha)$ for the *range* and *source* of $\alpha \in G$. We call the common image of r and s the *unit space* of G and denote it $G^{(0)}$. We identify the set of identity morphisms of G with $G^{(0)}$.

An *étale* groupoid is a groupoid G endowed with a topology so that composition and inversion are continuous, and the source map s is a local homeomorphism. In this case, r is also a local homeomorphism and there is a basis of *open bisections*; that is, a basis of sets $B \subseteq G$ such that s and r restricted to B are homeomorphisms. We say a groupoid is *ample* if it has a basis of compact open bisections. Note that a Hausdorff groupoid is ample if and only if it is locally compact, Hausdorff and étale and its unit space is totally disconnected (see [6, Lemma 2.1]). See [16] for more details on étale and ample groupoids.

We use the notational convention that if A, B are subsets of a groupoid G, then

$$AB := \{\alpha\beta : \alpha \in A, \beta \in B, s(\alpha) = r(\beta)\}.$$

If $A = \{\alpha\}$, then we write αB for $\{\alpha\}B$. The *orbit* of a unit $x \in G^{(0)}$ is the set

$$[x] := s(xG) = r(Gx) \subseteq G^{(0)}.$$

An (algebraic) isomorphism $\Phi : G \to H$ of groupoids is a bijection from G to H that carries units to units, preserves the range and source maps and satisfies $\Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta)$ whenever α and β are composable in G. Uniqueness of inverses implies that $\Phi(\alpha^{-1}) = \Phi(\alpha)^{-1}$. If G and H are topological groupoids then an isomorphism $\Phi : G \to H$ is an algebraic isomorphism that is also a homeomorphism.

The next example demonstrates how groupoids are useful in the study of graph algebras.

Example 2.1. Let $E = (E^0, E^1, r_E, s_E)$ be an arbitrary directed graph.[1] We denote the infinite-path space by E^∞ and the finite-path space by E^*. We use the convention that a path x is a sequence of edges x_i in which each $s_E(x_i) = r_E(x_{i+1})$ and we write $|x|$ for the length of x. A *source* in E is a vertex v such that $r_E^{-1}(v) = \emptyset$, and an infinite receiver is a vertex v such that $r_E^{-1}(v)$ is infinite.

[1] To avoid confusion, we adopt the convention that an unadorned r or s will always denote the range or source map in a groupoid, and the range and source maps associated to a graph E will always be decorated with a subscript E.

The following construction of a groupoid G_E from a graph E can be found in [15]. This generalises the construction in [13]. Unlike [13] and [15], we do not require our graphs to be countable. More general versions are described in [9, 12, 18, 26].

Define

$$X := E^\infty \cup \{\mu \in E^* \mid s_E(\mu) \text{ is a source}\} \cup \{\mu \in E^* \mid s_E(\mu) \text{ is an infinite receiver}\}.$$

Let

$$G_E := \{(\alpha x, |\alpha| - |\beta|, \beta x) \mid \alpha, \beta \in E^*, x \in X, s_E(\alpha) = s_E(\beta) = r_E(x)\}$$

We view each $(x, k, y) \in G_E$ as a morphism with range x and source y. The formulas

$$(x, k, y)(y, l, z) := (x, k + l, z) \quad \text{and} \quad (x, k, y)^{-1} := (y, -k, x)$$

define composition and inverse maps on G_E making it a groupoid with

$$G_E^{(0)} = \{(x, 0, x) : x \in X\} \text{ which we identify with } X.$$

Next, we describe a topology on G. For $\mu \in E^*$, the cylinder set $Z(\mu) \subseteq X$ is the set

$$Z(\mu) := \{\mu x \mid x \in X, s_E(\mu) = r_E(x)\}.$$

For $\mu \in E^*$ and a finite $F \subseteq r_E^{-1}(s_E(\mu))$, define

$$Z(\mu \setminus F) := Z(\mu) \cap \Big(\bigcup_{\alpha \in F} Z(\mu\alpha)\Big)^c.$$

The sets $Z(\mu \setminus F)$ are a basis of compact open sets for a locally compact, Hausdorff topology on $X = G_E^{(0)}$ (see [24, Theorem 2.1]).

For $\mu, \nu \in E^*$ with $s_E(\mu) = s_E(\nu)$, and for a finite $F \subseteq E^*$ such that $s_E(\mu) = r_E(\alpha)$ for all $\alpha \in F$, we define

$$Z(\mu, \nu) := \{(\mu x, |\mu| - |\nu|, \nu x) : x \in X, s_E(\mu) = r_E(x)\},$$

and then

$$Z((\mu, \nu) \setminus F) := Z(\mu, \nu) \cap \Big(\bigcup_{\alpha \in F} Z(\mu\alpha, \nu\alpha)\Big)^c.$$

The $Z((\mu, \nu) \setminus F)$ form a basis of compact open sets for a locally compact Hausdorff topology on G_E under which it is étale. Hence, G_E is a Hausdorff ample groupoid. We will come back to this example in Example 3.2 and again in Section 6.

3. Steinberg algebras over commutative rings with 1

Throughout this section, R denotes a commutative unital ring, Γ denotes a discrete group, G denotes a Hausdorff ample groupoid, and c denotes a continuous homomorphism from G to Γ; that is, $c : G \to \Gamma$ is a continuous groupoid *cocycle*. The Steinberg algebra $A(G)$ of G, introduced in [22] [2] is the R-algebra of locally constant R-valued functions on G with compact support, where addition is pointwise and multiplication is given by convolution

$$(f * g)(\gamma) = \sum_{\alpha\beta = \gamma} f(\alpha) g(\beta).$$

It is useful to note that

$$A_R(G) = \operatorname{span}\{1_U : U \text{ is a compact open bisection of } G\} \subseteq R^G,$$

[2] Steinberg's notation is RG, but we continue to use the notation of [4, 6].

where 1_U denotes the characteristic function on U (see [22, Proposition 4.3]). We have

$$1_U * 1_V = 1_{UV}$$

for compact open bisections U and V (see [22, Proposition 4.5(3)]).

Lemma 3.1. *Suppose that R is a commutative unital ring, G is a Hausdorff ample groupoid and $c : G \to \Gamma$ is a continuous cocycle. The subsets*

$$A_R(G)_n := \{f \in A_R(G) : \operatorname{supp}(f) \subseteq c^{-1}(n)\}$$

for $n \in \Gamma$ form a Γ-grading of $A_R(G)$.

Proof. We must show that:

1. $A_R(G) = \bigoplus_{n\in\Gamma} A_R(G)_n$ as an R-module; and
2. if $f \in A_R(G)_n$ and $g \in A_R(G)_m$ then $f * g \in A_R(G)_{n+m}$.

Fix a compact open bisection $U \subseteq G$. For (1), it suffices to show that the indicator function 1_U belongs to $\bigoplus_{n\in\Gamma} A_R(G)_n$. For $n \in \Gamma$, let $V_n := U \cap c^{-1}(n)$. Since the $c^{-1}(n)$ are disjoint clopen sets and U is compact open, the V_n are disjoint compact open subsets of U. Further, since U is compact, only finitely many V_n are nonempty, and then $1_U = \sum_{V_n \neq \emptyset} 1_{V_n} \in \bigoplus_{n\in\Gamma} A_R(G)_n$.

For (2), suppose that $f \in A_R(G)_n$ and $g \in A_R(G)_m$. For $\gamma \in G$ we have $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$, and so

$$\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) \operatorname{supp}(g) \subseteq c^{-1}(n)c^{-1}(m) \subseteq c^{-1}(n + m).$$

Therefore $f * g \in A_R(G)_{n+m}$. □

Example 3.2. Every Leavitt path algebra is a Steinberg algebra. To see this, let E be an arbitrary directed graph, G_E the groupoid of Example 2.1 and R a commutative unital ring. We show that the Leavitt path algebra $L_R(E)$ is isomorphic to $A_R(G_E)$. It is routine to check that the indicator functions $q_v := 1_{Z(v)}$, $v \in E^0$ are mutually orthogonal idempotents, and that the indicator functions $t_e := 1_{Z(e,s(e))}$ and $t_{e^*} = 1_{Z(s(e),e)}$ constitute a Leavitt E-family as in [23, Definition 2.4]. So the universal property of $L_R(E)$ gives a homomorphism $\pi : L_R(E) \to A_R(G_E)$ satisfying $\pi(p_v) = q_v$, $\pi(s_e) = t_e$ and $\pi(s_{e^*}) = t_{e^*}$. An application of the graded uniqueness theorem [23, Theorem 4.8] shows that this homomorphism is injective. To see that it is surjective, observe that each $1_{Z((\mu,\nu)\setminus F)} = t_\mu t_{\nu^*} - \sum_{\alpha\in F} t_{\mu\alpha} t_{(\nu\alpha)^*}$ belongs to the range of π. Fix a compact open U. This U can be written as a union of basic open sets (because it is open), and therefore as a finite union of basic open sets (because it is compact); say $U = \bigcup_{(\mu,\nu,F)\in\mathcal{F}} Z((\mu, \nu) \setminus F)$. We claim that U can be written as a disjoint union of basic open sets. By the inclusion-exclusion principle,

$$U = \bigsqcup_{\emptyset\neq\mathcal{G}\subseteq\mathcal{F}} \Big(\Big(\bigcap_{(\mu,\nu,P)\in\mathcal{G}} Z((\mu, \nu) \setminus P)\Big) \setminus \Big(\bigcup_{(\eta,\zeta,Q)\in\mathcal{F}\setminus\mathcal{G}} Z((\eta, \zeta) \setminus Q)\Big)\Big)$$

For any $\mu, \nu, \alpha, \beta \in E^*$ with $s(\mu) = s(\nu)$ and $s(\alpha) = s(\beta)$, we have

$$Z(\mu, \nu) \cap Z(\alpha, \beta) = \begin{cases} Z(\alpha, \beta) & \text{if } \alpha = \mu\tau \text{ and } \beta = \nu\tau \\ Z(\mu, \nu) & \text{if } \mu = \alpha\tau \text{ and } \nu = \beta\tau \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$Z(\mu,\nu)\setminus Z(\alpha,\beta)=\begin{cases}Z((\mu,\nu)\setminus\{\tau\}) & \text{if } \alpha=\mu\tau \text{ and } \beta=\nu\tau\\ \emptyset & \text{otherwise.}\end{cases}$$

Using this, de Morgan's laws and distributivity of intersection and union, it is routine to check that every set of the form $\bigcap_{(\mu,\nu,P)\in\mathcal{G}} Z((\mu,\nu)\setminus P)\setminus(\bigcup_{(\eta,\zeta,Q)\in\mathcal{H}} Z((\eta,\zeta),Q))$ with $\mathcal{G},\mathcal{H}$ finite and $\mathcal{G}$ nonempty can be written as a finite disjoint union of basic open sets. Hence U can be written as a finite disjoint union of basic open sets as claimed. Thus 1_U is a finite sum of indicator functions of basic open sets, and therefore belongs to the range of π. That is, π is an isomorphism of $L_R(E)$ onto $A_R(G_E)$ as required.

Remark 3.3. If Λ is a row-finite k-graph with no sources and G_Λ is the associated groupoid (see for example [12] and [9]), the [6, Proposition 4.3] shows that $A_{\mathbb{C}}(G_\Lambda)$ is isomorphic to the Kumjian-Pask algebra $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ as defined in [2]. An argument similar to that of the preceding example generalises this to the Kumjian-Pask R-algebras associated to a locally convex row-finite k-graphs (possibly with sources) as in [5]. That is $\mathrm{KP}_R(\Lambda)\cong A_R(G_\Lambda)$.

4. Groupoid equivalence

In this section, we assume throughout that G is a locally compact Hausdorff groupoid and X is a locally compact Hausdorff space. We say G *acts on the left* of X if there is a map r_X from X onto $G^{(0)}$ and a map $(\gamma,x)\mapsto\gamma\cdot x$ from

$$G*X:=\{(\gamma,x)\in G\times X : s(\gamma)=r_X(x)\} \text{ to } X$$

such that

(1) if $(\eta,x)\in G*X$ and (γ,η) is a composable pair in G, then $(\gamma\eta,x),(\gamma,\eta\cdot x)\in G*X$ and $\gamma\cdot(\eta\cdot x)=(\gamma\eta)\cdot x$;
(2) $r_X(x)\cdot x=x$ for all $x\in X$.

We will call X a *continuous left G-space* if r_X is an open map and both r_X and $(\gamma,x)\mapsto\gamma\cdot x$ are continuous.

The action of G on X is *free* if $\gamma\cdot x=x$ implies $\gamma=r_X(x)$. It is *proper* if the map from $G*X\to X\times X$ given by $(\gamma,x)\to(\gamma\cdot x,x)$ is a proper map in the sense that inverse images of compact sets are compact.

We define right actions similarly, writing s_X for the map from X onto $G^{(0)}$, and

$$X*G:=\{(x,\gamma)\in X\times G : s_X(x)=r(\gamma)\}.$$

Definition 4.1. Let G and H be locally compact Hausdorff groupoids. A (G,H)-*equivalence* is a locally compact Hausdorff space Z such that

(1) Z is a free and proper left G-space;
(2) Z is a free and proper right H-space;
(3) the actions of G and H on Z commute;
(4) r_Z induces a homeomorphism of Z/H onto $G^{(0)}$;
(5) s_Z induces a homeomorphism of $G\setminus Z$ onto $H^{(0)}$.

Suppose that Z is a (G,H)-equivalence, and that $y,z,y',z'\in Z$ satisfy $s_Z(y)=r_Z(z)$ and $s_Z(z')=r_Z(y')$. We write ${}_G[y,z]\in G$ and $[y',z']_H\in H$ for the unique elements such that

$$ {}_G[y,z]\cdot z=y \text{ and } y'\cdot[y',z']_H=z'. \tag{4.1}$$

Let

$$Z^{\mathrm{op}} := \{\overline{z} : z \in Z\}$$

denote a homeomorphic copy of Z. For $z \in Z$, define $r_{Z^{\mathrm{op}}}(\overline{z}) = s_Z(z) \in H^{(0)}$ and $s_{Z^{\mathrm{op}}}(\overline{z}) = r_Z(z) \in G^{(0)}$, and for $\eta \in H$ with $s(\eta) = r_{Z^{\mathrm{op}}}(\overline{z})$ and $\gamma \in G$ with $r(\gamma) = s_{Z^{\mathrm{op}}}(\overline{z})$ define

$$\eta \cdot \overline{z} := \overline{z \cdot \eta^{-1}} \quad \text{and} \quad \overline{z} \cdot \gamma := \overline{\gamma^{-1} \cdot z}.$$

With this structure, Z^{op} is an (H, G)-equivalence. See [11, 14, 20] for more information on groupoid actions and equivalences.

Remark 4.2. Note that if S and T are strongly Morita equivalent inverse semigroups as in [21, Definition 2.1], then their respective universal groupoids are equivalent [21, Theorem 4.7].

The linking groupoid. Now suppose that G and H are Hausdorff ample groupoids and let Z be a (G, H)-equivalence. We show that $A_R(G)$ and $A_R(H)$ are Morita equivalent by embedding them as complementary corners of the Steinberg algebra of a *linking groupoid* L defined below. In the remainder of this section, we verify that the linking groupoid in this situation is also a Hausdorff ample groupoid and then show how $A_R(G)$ and $A_R(H)$ embed into $A_R(L)$.

If Z is a (G, H)-equivalence, the *linking groupoid of* Z is defined in [20, Lemma 2.1] as

$$L := G \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup H,$$

with $r, s : L \to L^{(0)} := G^{(0)} \sqcup H^{(0)}$ inherited from the range and source maps on each of G, H, Z and Z^{op}. We write r and s (no subscripts) to denote the range and source maps in L. Multiplication $(k, l) \mapsto kl$ in L is given by

- multiplication in G and H when (k, l) is a composable pair in G or H;
- $kl = k \cdot l$ when $(k, l) \in Z * H \sqcup G * Z \sqcup H * Z^{\mathrm{op}} \sqcup Z^{\mathrm{op}} * G$; and
- $kl = {}_G[k, h]$ if $k \in Z$ and $l = \overline{h} \in Z^{\mathrm{op}}$, and $kl = [h, l]_H$ if $l \in Z$ and $k = \overline{h} \in Z^{\mathrm{op}}$.

The inverse map is the usual inverse map in each of G and H and is given by $z \mapsto \overline{z}$ on Z and $\overline{z} \mapsto z$ in Z^{op}. Both G and H are clopen in L by construction.

Lemma 4.3. *Let G and H be Hausdorff ample groupoids. Suppose that Z is a (G, H)-equivalence and L is the linking groupoid of Z. Then L is a Hausdorff ample groupoid.*

Proof. Lemma 2.1 of [20] implies that L is locally compact and Hausdorff. It suffices to show that L is étale with totally disconnected unit space. We have $L^{(0)} = G^{(0)} \sqcup H^{(0)}$ which is totally disconnected because $G^{(0)}$ and $H^{(0)}$ are, so it remains to show that L is étale.[3]

We suppose that r is not a local homeomorphism, and seek a contradiction. Then there exists $z \in L$ such that r fails to be injective on every neighbourhood of z. Because G and H are étale, z is either in Z or Z^{op}. Without loss of generality, assume $z \in Z$; the case for Z^{op} is symmetric. By choosing a neighbourhood base $\{U_\alpha\}$ at z inside of Z, we can find a net $\{(x_\alpha, y_\alpha)\}$ where each $x_\alpha, y_\alpha \in U_\alpha$ such that:

(1) $x_\alpha, y_\alpha \to z$;

[3] If G and H were second-countable, then L would be as well, and then we could deduce from [17, Lemma I.2.7 and Proposition I.2.8] that L is étale by observing that $L^{(0)}$ is open in L (because each of $G^{(0)}$ and $H^{(0)}$ is open), and the Haar system on L induced from those on G and H consists of counting measures because the systems on G and H have this property.

(2) $x_\alpha \neq y_\alpha$ for all n;
(3) $r(x_\alpha) = r(y_\alpha)$ for all n.

Since G is étale, $G^{(0)}$ is open in L and so we can assume that $r(x_\alpha) \in G^{(0)}$ for all α. For each α, let $\gamma_\alpha := [x_\alpha, y_\alpha]_H$, so that $x_\alpha \cdot \gamma_\alpha = y_\alpha$ for all α. Note that $r(\gamma_\alpha) = s(x_\alpha)$. Proposition 1.15 of [25] applied to the open map $r : H \to H^{(0)}$ implies that, by passing to a subnet, we may assume that $\gamma_\alpha \to \gamma \in H$. So the continuity of the action gives

$$z \cdot \gamma = \lim x_\alpha \cdot \gamma_\alpha = \lim y_\alpha = z.$$

Since H acts freely on Z, this forces $\gamma = s(z)$. Since $H^{(0)}$ is open in H, we have $\gamma_\alpha \in H^{(0)}$ eventually. Hence $x_\alpha = y_\alpha$ eventually, contradicting (2). □

Following [20, page 108], for each $F \in A_R(L)$, define $F_{11} = F|_G$, $F_{12} = F|_Z$, $F_{21} = F|_{Z^{\mathrm{op}}}$ and $F_{22} = F|_H$. We may view each F_{ij} as an element of $A_R(L)$. We express the decomposition $F = \sum_{i,j} F_{ij}$ by writing

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

It is straightforward to check that convolution in $A_R(L)$ is given by matrix multiplication for functions written in this form. Using this notation, we see that the inclusion maps

$$f \mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \text{ and } g \mapsto \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}$$

define injective homomorphisms $A_R(G) \hookrightarrow A_R(L)$ and $A_R(H) \hookrightarrow A_R(L)$. We denote the images of these maps by $i(A_R(G))$ and $i(A_R(H))$. So

$$i(A_R(G)) \cong A_R(G) \text{ and } i(A_R(H)) \cong A_R(G). \tag{4.2}$$

5. Main result

We now have the machinery we need to show that equivalent groupoids give rise to Morita equivalent Steinberg algebras. First, we give the definition of Morita equivalent rings. Let A and B be rings, M an A–B bimodule, N a B–A bimodule, and

$$\psi : M \otimes_B N \to A \text{ and } \phi : N \otimes_A M \to B$$

bimodule homomorphisms such that

$$n' \cdot \psi(m \otimes n) = \phi(n' \otimes m) \cdot n \text{ and } m' \cdot \phi(n \otimes m) = \psi(m' \otimes n) \cdot m \tag{5.1}$$

for $n, n' \in N$ and $m, m' \in M$. Then (A, B, M, N, ψ, ϕ) is a *Morita context* between A and B; it is called *surjective* if ψ and ϕ are surjective and in this case we say A and B are *Morita equivalent.* (See [10, page 41].)

Theorem 5.1. *Let G and H be Hausdorff ample groupoids. Suppose that Z is a (G, H)-equivalence with linking groupoid L. Let i denote the inclusion maps from $A_R(G)$ and $A_R(H)$ into $A_R(L)$. Define*

$$M := \{f \in A_R(L) \mid \operatorname{supp} f \subseteq Z\} \quad \textit{and} \quad N := \{f \in A_R(L) \mid \operatorname{supp} f \subseteq Z^{\mathrm{op}}\},$$

and let $A_R(G)$ and $A_R(H)$ act on the right and left of M and on the left and right of N by $a \cdot f = i(a) * f$ and $f \cdot a = f * i(a)$. Then there are bimodule homomorphisms

$$\psi : M \otimes_{i(A_R(H))} N \to A_R(G) \quad \textit{and} \quad \phi : N \otimes_{i(A_R(G))} M \to A_R(H)$$

determined by

$$i(\psi(f \otimes g)) = f * g \quad \text{and} \quad i(\psi(g \otimes f)) = g * f.$$

The tuple $(A_R(G), A_R(H), M, N, \psi, \phi)$ *is a surjective Morita context, and so* $A_R(G)$ *and* $A_R(H)$ *are Morita equivalent.*

Proof. That M is an $A_R(G)$–$A_R(H)$ bimodule and N is an $A_R(H)$–$A_R(G)$ bimodule is clear. The given formulas for ϕ and ψ are well-defined on the balanced tensor products because, for example,

$$f * (a \cdot g) = f * (i(a) * g) = (f * i(a)) * g = (f \cdot a) * g.$$

The maps ψ and ϕ are module homomorphisms by linearity of convolution. The formula (5.1) follows from associativity of convolution in $A_R(L)$.

To see that ψ is surjective, it suffices to fix a compact open bisection $U \subseteq G$ and show that $i(1_U)$ is in the image of ψ. For each $x \in r(U)$, choose $z_x \in Z$ such that $r(z_x) = x$. Since L is étale and Z is topologically disjoint from G, each z_x has a neighbourhood $U_x \subseteq Z$ which is a bisection of L. Since $G^{(0)}$ is locally compact, Hausdorff and totally disconnected, each x has a compact open neighbourhood W_x contained in $r(U) \cap r(U_x)$, and so by replacing each U_x with $U_x \cap r^{-1}(W_x)$, we can assume that each U_x is compact open with $r(U_x) \subseteq r(U)$. Since $r(U)$ is compact, there is a finite set $\{x_1, \dots, x_n\} \subseteq r(U)$ such that $\bigcup_i r(U_{x_i}) = r(U)$. Let $V_1 = U_{x_1}$ and iteratively define $V_i = U_{x_i} \setminus r^{-1}\big(\bigcup_{j<i} r(U_{x_j})\big)$. Then the V_i are compact open subsets of Z on which r and s are bijective, and $r(U)$ is the disjoint union of the $r(V_i)$. Therefore, writing V_i^{op} for $\{\overline{z} : z \in V_i\} \subseteq Z^{\mathrm{op}}$, we have

$$\begin{pmatrix} 1_U & 0 \\ 0 & 0 \end{pmatrix} = \sum_i \begin{pmatrix} 0 & 1_{V_i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1_{V_i^{\mathrm{op}}} & 0 \end{pmatrix}.$$

Thus $1_U = \psi(\sum_i 1_{V_i} \otimes 1_{V_i^{\mathrm{op}}})$, and so ψ is surjective. A similar argument shows that ϕ is surjective.

It follows that $(A_R(G), A_R(H), M, N, \psi, \phi)$ is a surjective Morita context, and so $A_R(G)$ and $A_R(H)$ are Morita equivalent. □

6. Applications to graph algebras

Our aim is to apply our main result to graph algebras. First we consider a useful class of examples of groupoid equivalences — those arising from *abstract transversals* of groupoids. Suppose that G is a subgroupoid[4] of H and let $Z := G^{(0)}H$. It is straightforward to check that Z is a free and proper left G-space and a free and proper right H-space where r_Z and s_Z are the range and source from H restricted to Z and the action is by multiplication in H. Because groupoid multiplication is associative, the actions of G and H commute. However, Z may not satisfy the surjectivity hypothesis of Definition 4.1 (5) required in a groupoid equivalence. The following lemma is a straightforward application of [14, Example 2.7]; we give a short proof because the construction is fundamental to our application of groupoid equivalence to graph algebras.

Lemma 6.1. *Suppose* H *is an étale groupoid and* $X \subseteq H^{(0)}$ *is a clopen subset that meets each orbit in* H*. Then* $G := XHX$ *is a clopen subgroupoid of* H*, and* $Z := XH$ *is a* (G, H)*-equivalence.*

[4]By *subgroupoid* we mean a subset that is itself a groupoid.

Proof. The set $XHX = r^{-1}(X) \cap s^{-1}(X)$ is clopen because r and s are continuous, and it is clearly a subgroupoid. Similarly, Z is a clopen subset of H, and so the open subsets of Z are the subsets of Z which are open in H. Since H is étale, r and s are open maps and so r_Z and s_Z (which are r and s restricted to Z) are also open maps. The map $r_Z : Z \to X$ is surjective by definition. To see that $s_Z : Z \to H^{(0)}$ is surjective, fix $u \in H^{(0)}$. By hypothesis, $[u] \cap X \neq \emptyset$, so there exists $\alpha \in H$ such that $r(\alpha) \in X$ and $u = s(\alpha)$. So $\alpha \in Z$ and $u = s(\alpha) \in s_Z(Z)$.

We prove that $\tilde{s} : G\backslash Z \to H^{(0)}$ is a homeomorphism; the argument that $\tilde{r}$ is a homeomorphism is similar. Clearly, $\tilde{s}$ is a surjection. If $\tilde{s}([\alpha]) = \tilde{s}([\beta])$, then $s(\alpha) = s(\beta)$, and so $\alpha\beta^{-1} \in XHX = G$ and satisfies $(\alpha\beta^{-1}) \cdot \beta = \alpha$. So $[\alpha] = [\beta]$, and $\tilde{s}$ is injective.

To see that $\tilde{s}$ is continuous, suppose $U \subseteq H^{(0)}$ is open. Then HU is open because s is continuous, and then $ZU = HU \cap Z$ is open in Z. Thus $\tilde{s}^{-1}(U) = G\backslash(ZU)$ is open by definition of the quotient topology.

Finally, if $W \subseteq G\backslash Z$ is open, then $W = G\backslash W'$ for some open $W' \subseteq Z$. Since Z is open in H, so is W' and then $\tilde{s}(W) = s(W')$ is open because s is open. $\square$

Given a graph E, Crisp and Gow identify a type of subgraph T which can be "collapsed" to yield a new graph F whose C^*-algebra is Morita equivalent to that of E [7]. We will demonstrate that G_E and G_F are equivalent groupoids. Bates and Pask's "outsplitting" move described in [3, Theorem 4.5 and Corollary 5.4] is a special case of the Crisp-Gow construction (see [7, Example iii]), as are Sørensen's moves (S) and (R) (see [19, Propositions 3.1 and 3.2]). So our result implies that applications of these moves yield Morita equivalent Leavitt path algebras regardless of the base ring.

When E is countable, our statement of the next proposition corresponds exactly to the construction of [7, Theorem 3.1] modulo the difference in edge-direction conventions. First, we need a few more graph preliminaries. Suppose E is a directed graph. For $v \in E^0$ and $S \subseteq E^0$, we write $v \geq S$ if $SE^*v \neq \emptyset$. We define the *pointed groupoid* with respect to S to be the subgroupoid of G_E consisting of groupoid elements $(\alpha x, |\alpha| - |\beta|, \beta x)$ such that $r_E(\alpha), r_E(\beta) \in S$. We define

$$E^0_{\text{sing}} := \{v \in E^0 : r_E^{-1}(v) \text{ is either empty or infinite}\}.$$

For $n \in \mathbb{N}$ we define a map $\sigma^n : \{x \in E^* \cup E^\infty : |x| \geq n\} \to E^* \cup E^\infty$ by $\sigma^n(\alpha y) = y$ for all $\alpha \in E^n$ (paths of length n) and $y \in E^* \cup E^\infty$. Notice that $G_E^{(0)}$ is invariant under σ^n. Finally, we say an acyclic path $x \in E^\infty$ is a *head* if each $r_E(x_i)$ receives only x_i and each $s_E(x_i)$ emits only x_i.

Proposition 6.2. *Let E be a directed graph with no heads and suppose that $F^0 \subseteq E^0$ satisfies $E^0_{\text{sing}} \subseteq F^0$. Suppose also that the subgraph T of E defined by $T^0 := E^0 \setminus F^0$ and*

$$T^1 := \{e \in E^1 : r_E(e), s_E(e) \in T^0\}$$

is acyclic and that each of the following are satisfied:

- (T1) *each vertex in F^0 is the range of at most one $y \in E^\infty$ such that $s_E(y_i) \in T^0$ for all $i \geq 1$;*

and for each $x \in T^\infty$,

- (T2) $r_E(x) \geq F^0$
- (T3) $|s_E^{-1}(r_E(x_i))| = 1$ *for all i; and*
- (T4) *whenever $s_E(e) = r_E(x)$, we have $|r_E^{-1}(r_E(e))| < \infty$.*

Let F be the graph with vertex set F^0 and one edge e_β for each path $\beta \in E^* \setminus E^0$ with $s_E(\beta), r_E(\beta) \in F^0$ and $r_E(\beta_i) \in T^0$ for $1 \leq i < |\beta|$ such that $s_F(e_\beta) = s_E(\beta)$ and $r_F(e_\beta) = r_E(\beta)$. Let $G \subseteq G_E$ denote the pointed groupoid with respect to F^0. Then

1. *G and G_E are equivalent groupoids and*
2. *G is isomorphic to G_F.*

Remark 6.3. We will be using [7, Lemma 3.3], which says that if a graph E has no heads, satisfies (T1), (T2) and (T3), and T and F are as above, then $F^0 \geq v$ for all $v \in T^0$. Note that this Lemma also implies that $r_E^{-1}(v) = \emptyset$ if and only if $r_F^{-1}(v) = \emptyset$.

Proof. To prove (1), we will apply Lemma 6.1 with $X = G^{(0)} = F^0 E^\infty$. First notice that

$$G^{(0)} = \bigcup_{v \in F^0} Z(v) = G_E^{(0)} \setminus \Big(\bigcup_{w \in T^0} Z(w) \Big).$$

Since each $Z(v)$ is open, we deduce that $G^{(0)}$ is clopen in $G_E^{(0)}$. Now consider $x \in G_E^{(0)} \setminus G^{(0)}$. We must show that $[x] \cap G^{(0)} \neq \emptyset$. Since $x \notin G^{(0)}$, $r_E(x) \in T^0$. We consider 2 cases. For the first case, suppose that $\sigma^n(x) \in T^\infty$ for some n. Then (T2) implies that there exists $\mu \in E^*$ such that $s_E(\mu) = r_E(x_{n+1})$ and $r_E(\mu) \in F^0$. So $\mu(\sigma^n(x)) \in [x] \cap G^{(0)}$. For the second case, suppose that $\sigma^n(x) \notin T^\infty$ for all n. Since $E^0_{\mathrm{sing}} \subseteq F^0$, there exists n such that $s_E(x_n) \in F^0$. Hence $\sigma^n(x) \in [x] \cap G^{(0)}$. Now Lemma 6.1 implies that XG_E is a (G, G_E)-equivalence.

To prove (2), we first define a map $\phi : G_F^{(0)} \to G^{(0)}$, which will take a little preparation. By construction, F^1 is a subset of E^*; we write $\phi_{\mathrm{fin}} : F^1 \to E^*$ for the inclusion map. Since ϕ_{fin} preserves ranges and sources, we can extend ϕ_{fin} to an injection from F^* to E^* by

$$\phi_{\mathrm{fin}}(\mu) = \phi_{\mathrm{fin}}(\mu_1)\phi_{\mathrm{fin}}(\mu_2) \dots \phi_{\mathrm{fin}}(\mu_{|\mu|}).$$

Again by construction of F, we have

$$\phi_{\mathrm{fin}}(F^*) = \{\mu \in E^* : r_E(\mu), s_E(\mu) \in F^0\}.$$

We claim that if $v \in F^0$ satisfies $|r_F^{-1}(v)| = \infty$ but $|r_E^{-1}(v)| < \infty$, then there is a unique infinite path $y_v \in T^\infty$ with $r_E(y_v) = v$. Indeed, the set

$$B_v := \{\beta \in E^* \setminus E^0 \mid r_E(\beta) = v, s_E(\beta) \in F^0 \text{ and } r_E(\beta_i) \in T^0 \text{ for } 1 \leq i \leq |\beta|\} \tag{6.1}$$

is infinite, and so [7, Lemma 3.4(d)] gives such a y_v. That there is a unique such path follows from (T1).

Define $\phi : G_F^{(0)} \to G^{(0)}$ by

$$\phi(x) = \begin{cases} \phi_{\mathrm{fin}}(x_1)\phi_{\mathrm{fin}}(x_2) \dots & \text{if } x \in F^\infty; \\ \phi_{\mathrm{fin}}(x) & \text{if } x \in F^* \text{ and } s_F(x) \in E^0_{\mathrm{sing}}; \text{ and} \\ \phi_{\mathrm{fin}}(x) y_{s_F(x)} & \text{if } x \in F^*,\ |r_F^{-1}(s_F(x))| = \infty, \text{ and } 0 < |r_E^{-1}(s_F(x))| < \infty. \end{cases}$$

To see that this defines ϕ on all $G_F^{(0)}$ observe that if $x \in G_F^{(0)}$ belongs to F^* and $s_F(x) \notin E^0_{\mathrm{sing}}$, then we have $s_F(x) \in F^0_{\mathrm{sing}} \setminus E^0_{\mathrm{sing}}$, and since $r_E^{-1}(v) = \emptyset$ if and only if $r_F^{-1}(v) = \emptyset$, we deduce that $|r_F^{-1}(s_F(x))| = \infty$ and $0 < |r_E^{-1}(s_F(x))| < \infty$.

Since ϕ_{fin} is injective, ϕ is also injective. We have

$$\phi(F^\infty) = \{x \in F^0 E^\infty \mid s_E(x_n) \in F^0 \text{ for infinitely many } n\} \text{ and}$$
$$\phi(\{\mu \in F^* : s_F(x) \in E^0_{\mathrm{sing}}\}) = \{\mu \in F^0 E^* : s_E(\mu) \in E^0_{\mathrm{sing}}\}$$

because $E^0_{\text{sing}} \subseteq F^0$. The complement of these two sets in $G^{(0)}$ is

$$\begin{aligned}\{x \in F^0E^\infty \mid s_E(x_i) \notin F^0 \text{ eventually}\}\\ &= \{x \in F^0E^\infty \mid s_E(x_i) \in T^0 \text{ eventually}\}\\ &= \{\mu y \mid \mu \in F^0E^*F^0, y \in s_E(\mu)E^\infty, \sigma^1(y) \in T^\infty\}.\end{aligned} \tag{6.2}$$

Let μy be an element of the set (6.2). To see that ϕ is surjective, it suffices to show that $|r_F^{-1}(r_E(y))| = \infty$, and $0 < |r_E^{-1}(r_E(y))| < \infty$. For then $\phi(\phi_{\text{fin}}^{-1}(\mu)) = \mu y$. Condition (T4) applied to $e = y_1$ implies that $r_E(y_1)$ is not an infinite receiver in E. We must now show that $r_F^{-1}(r_E(y_1))$ is infinite. Since T is acyclic, y has no repeating edges or vertices. Lemma 3.3 of [7] yields a path $\mu^1 \in E^*$ with $r_E(\mu^1) = s_E(y_1)$ and $s_E(\mu^1) = v_1 \in F^0$. Since $s_E(\mu^1) \in F^0$, (T3) implies that there exists $m_1 < |\mu^1|$ such that $y_j \notin \{\mu^1_{m_1}, \dots, \mu^1_{|\mu^1|}\}$ for all j.

Repeating this process for each $n \in \mathbb{N}$, we obtain distinct paths μ^n such that $r_E(\mu^n) = s_E(y_{k_n})$ where $k_n = \sum_{i=1}^n (m_i + 2)$ and $s_E(\mu^n) \in F^0$. Now $y_1 \dots y_{k_n}\mu^n \in r_F^{-1}(r_E(y))$ for all n, and these are distinct elements of F^1, so that $r_F^{-1}(r(y))$ is infinite as required. Therefore, ϕ is surjective. Notice that ϕ also preserves concatenation of paths.

Next we show that ϕ is a homeomorphism. It takes cylinder sets $Z(\mu)$ in $G_F^{(0)}$ onto cylinder sets $Z(\phi_{\text{fin}}(\mu))$ of $G^{(0)}$, and since it is bijective, it is therefore open.

To see that ϕ is continuous, suppose $x^n \to x$ in $G_F^{(0)}$. We consider the three possibilities for x. First, if $x \in F^\infty$, then the collection $\{Z(x_1), Z(x_1x_2), \dots\}$ is a neighbourhood base at x and the collection

$$\{\phi(Z(x_1)), \phi(Z(x_1x_2)), \dots\} = \{Z(\phi(x_1)), Z(\phi(x_1x_2)), \dots\}$$

is a neighbourhood base for $\phi(x)$. So $\phi(x^n)$ converges to $\phi(x)$.

Second, if $x \in F^*$ and $s_F(x)$ is a source, then $\{x\}$ is open in $G_F^{(0)}$ and hence $x^n = x$ eventually. Therefore $\phi(x^n) = \phi(x)$ eventually and hence $\phi(x^n)$ converges to $\phi(x)$.

Finally, suppose $x \in F^*$ and $s(x)$ is an infinite receiver. If x^n is eventually constant then $\phi(x^n)$ converges to $\phi(x)$ as above. So suppose otherwise. Since $x^n \in Z(x)$ eventually, we may assume that each $x^n = xz^n$ where $z^n \in G_F^{(0)}$. Also, we have that $\phi(x) = \phi_{\text{fin}}(x)y_{s_E(x)}$. Let $B := Z(\phi_{\text{fin}}(x)y_1 \dots y_m)$ be a basis element containing $\phi(x)$. Since open sets containing x include sets of the form

$$Z(x) \cap \Big(\bigcup_{e \in G} Z(xe)\Big)^c$$

for finite $G \subseteq r_F^{-1}(s_F(x))$, we may assume that $z_1^n \neq z_1^m$ for $n \neq m$; that is, the first edges of the paths z^n are distinct. Condition (T4) implies that $s_F(x)$ is not an infinite receiver in E, so we may also assume that $\phi(z_1^n) \in E^* \setminus E^1$ for each n. So the $\phi(z_1^n)$ are paths in E with range and source in F^0 but all other vertices in T^0. We claim that the distinct paths $\phi(z^n)$ eventually belong to $Z(y_1y_2 \dots y_m)$. Note that [7, Lemma 3.3] and (T3) imply that $|B_{s_E(y_1)}|$ is infinite. Further, for any $e \in r_E^{-1}(s_F(x)) \setminus \{y_1\}$ we have $|B_{s_E(e)}| < \infty$; for otherwise [7, Lemma 3.4(d)] yields an infinite path that violates (T1). Hence $\phi(z^n) \in Z(y_1)$ eventually. Similarly, $|B_{s_E(y_2)}|$ is infinite and for any $e \in E^1$ with $r_E(e) = r_F(y_2)$ we have $|B_{s_E(e)}| < \infty$ so $\phi(z^n) \in Z(y_1y_2)$ for large n. Proceeding in this way, we deduce that for any m we have $\phi(z^n) \in Z(y_1 \dots y_m)$ for large n as claimed. So $\phi(x(z_n)) \in B$ for large n. Thus, ϕ is continuous and hence ϕ is a homeomorphism.

Define $\Phi : G_F \to G$ by

$$\Phi(\mu x, |\mu| - |\nu|, \nu x) = (\phi(\mu x), |\phi_{\text{fin}}(\mu)| - |\phi_{\text{fin}}(\nu)|, \phi(\nu x)).$$

Since ϕ preserves concatenation of paths, Φ is a groupoid homomorphism and it is straightforward to show that Φ is bijective using that ϕ is bijective. We have

$$\Phi(Z(\mu, \nu)) = Z(\phi_{\text{fin}}(\mu), \phi_{\text{fin}}(\nu))$$

for all $\mu, \nu \in F^*$. So Φ takes basic open sets in G_F to basic open sets in G, and hence Φ is open.

To see that Φ is continuous, suppose γ_n converges to $\gamma = (\mu x, k, \nu x) \in G_F$. So for a basis element

$$B := Z(\mu x_1 \dots x_m, \nu x_1 \dots x_m) \cap \Big(\bigcup_{\alpha \in F} Z(\mu x_1 \dots x_n \alpha, \nu x_1 \dots x_m \alpha) \Big)^c$$

containing $\gamma \in G_F$, we eventually have $\gamma_n \in B$. So for large n, the element γ_n has the form

$$\gamma_n = (\mu x_1 \dots x_m y^n, k, \nu x_1 \dots x_m y^n) \text{ for } y^n \in G_F^{(0)}.$$

Thus eventually we have

$$\Phi(\gamma_n) = (\phi((\mu x_1 \dots x_m y^n), |\phi_{\text{fin}}(\mu)| - |\phi_{\text{fin}}(\nu)|, \phi(\nu x_1 \dots x_m y^n)),$$

which converges to $(\phi(\mu x), |\phi_{\text{fin}}(\mu)| - |\phi_{\text{fin}}(\nu)|, \phi(\nu x)) = \Phi(\gamma)$. □

Corollary 6.4. *Suppose E and F are as in Proposition 6.2 and R is a commutative unital ring. Then*

(1) *$L_R(E)$ is Morita equivalent to $L_R(F)$; and*
(2) *If E is countable, then $C^*(E)$ is Morita equivalent to $C^*(F)$.*

Proof. Proposition 6.2 implies that G_E and G_F are equivalent groupoids.

Now for (1), Theorem 5.1 implies that $A_R(G_E)$ and $A_R(G_F)$ are Morita equivalent, and the result follows from Example 3.2.

For (2), observe that since E is countable, G_E is second countable and hence $C^*(G_E)$ is Morita equivalent to $C^*(G_F)$ by [14, Theorem 2.8]. We have $C^*(G_E) \cong C^*(E)$ and $C^*(G_F) \cong C^*(F)$ by [15, Corollary 3.9], and the result follows. □

Remark 6.5. Corollary 6.4(1) generalises [1, Proposition 1.11]. Our proof of Corollary 6.4(2) provides an alternative proof of [7, Theorem 3.1].

Remark 6.6. Sørensen's move (I) of [19, Theorem 3.5] is a special case of Bates and Pask's construction "insplitting" in [3, Theorem 5.3]; a Leavitt path algebra version of this is proved in [1, Proposition 1.14]. In this setting, the corresponding algebras are actually stably isomorphic. Both [19, Theorem 3.5] and [1, Proposition 1.14] can be proved via Steinberg algebras by showing that the corresponding groupoids are isomorphic. This was done in the row-finite case by Drinen in [8, Proposition 6.1.3].

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