RECONSTRUCTION OF GROUPOIDS AND C^*-RIGIDITY OF DYNAMICAL SYSTEMS

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Abstract. We show how to construct a graded locally compact Hausdorff étale groupoid from a C^*-algebra carrying a coaction of a discrete group, together with a suitable abelian subalgebra. We call this groupoid the extended Weyl groupoid. When the coaction is trivial and the subalgebra is Cartan, our groupoid agrees with Renault's Weyl groupoid. We prove that if G is a second-countable locally compact étale groupoid carrying a grading of a discrete group, and if the interior of the trivially graded isotropy is abelian and torsion free, then the extended Weyl groupoid of its reduced C^*-algebra is isomorphic as a graded groupoid to G. In particular, two such groupoids are isomorphic as graded groupoids if and only if there is an equivariant diagonal-preserving isomorphism of their reduced C^*-algebras. We introduce graded equivalence of groupoids, and establish that two graded groupoids are graded equivalent if and only if there is an equivariant diagonal-preserving Morita equivalence between their reduced C^*-algebras. We use these results to establish rigidity results for a number of classes of dynamical systems, including all actions of the natural numbers by local homeomorphisms of locally compact Hausdorff spaces.

Introduction

Background. The use of operator algebras to encode dynamics goes all the way back to the foundational results of Murray and von Neumann on the group von Neumann algebra construction [34]. Crossed-product algebras and their generalisations have played a crucial role in both von Neumann algebra theory and C^*-algebra theory ever since. Recently, particularly since the work of Cuntz and Krieger [15] on operator-algebraic representations of shifts of finite type, and connections with Bowen–Franks theory [3], significant strides have been made in the direction of C^*-rigidity of dynamical systems. In broad terms this is the principle that dynamical systems can be recovered, up to a suitable notion of equivalence, from associated C^*-algebraic data.

A seminal result in this direction was Krieger's celebrated theorem [27] showing that nonsingular ergodic actions of $\mathbb{Z}$ are classified up to orbit equivalence by isomorphism of the associated von Neumann factors. This was soon followed by Cuntz and Krieger's construction [15] of C^*-algebras from irreducible shifts of finite type and Rørdam's proof

Date: November 6, 2017.

1991 *Mathematics Subject Classification.* 46L05 (primary); 20M20, 22A22, 37B05, 37B10, 46L55.

Key words and phrases. Groupoid; C^*-algebra; Cartan subalgebra; orbit equivalence.

This research was initiated while the authors were participating in the research program *Classification of operator algebras: complexity, rigidity, and dynamics* at the Mittag-Leffler Institute, January–April 2016. It also benefitted significantly from the support of the MATRIX@Melbourne Research Program *Refining C^*-algebraic invariants for dynamics using KK-theory, July 18–29, 2016* and from the Intensive Research Program *Operator algebras: dynamics and interactions* at the Centre de Recerca Matemàtica in Barcelona. This research was supported by Australian Research Council grant DP150101598 and by grants from the Simons Foundation (#279369 to Efren Ruiz and #527708 to Mark Tomforde).

[41] that stable isomorphism of Cuntz–Krieger algebras classifies irreducible shifts of finite type up to equivalence via the combination of flow equivalence and the so-called Cuntz splice on directed graphs. Later, building on work of Boyle [5], Giordano–Putnam–Skau [22] proved the remarkable result that for minimal homeomorphisms of the Cantor set, flip conjugacy, continuous orbit equivalence, and diagonal-preserving isomorphism of the associated crossed-product C^*-algebras are equivalent. Tomiyama [46] and Boyle–Tomiyama [6] subsequently proved that topologically free homeomorphisms of the Cantor set (minimal or not), are continuously orbit equivalent if and only if they each decompose as a disjoint union of two subsystems one pair of which are conjugate and the other pair of which are flip-conjugate. These results introduced, in particular, *diagonal-preserving* C^*-isomorphisms as a key ingredient in C^*-algebraic rigidity of topological dynamics.

The importance of diagonal-preserving isomorphism in operator algebras associated to groupoids goes back further. Feldman and Moore [18, 19, 20] proved that a Borel equivalence relation R can be reconstructed from the pair consisting of its associated von Neumann algebra M and the canonical Cartan subalgebra $D \subseteq M$, and that every Cartan pair of von Neumann algebras arises from such an R and a Borel 2-cocycle on R. In his thesis [38], Renault introduced a notion of a Cartan subalgebra of a C^*-algebra, and proved that a topologically principal étale groupoid G and a continuous cocycle c on G can be recovered from the Cartan pair $(C^*(G), C_0(G^{(0)}))$, and that every Cartan pair arises this way. Subsequently Kumjian [28] refined Renault's notion of a twisted groupoid C^*-algebra and showed that Renault's theorem extended to these more-general twists, in the setting of principal étale groupoids. Later in [39], Renault further extended Kumjian's results to topologically principal groupoids. Renault's machinery, and techniques from groupoid homology, underpinned Matsumoto and Matui's remarkable recent results [30] that irreducible (two-sided) shifts of finite type are flow equivalent if and only if there is a diagonal-preserving isomorphism of the stabilisations of the associated Cuntz–Krieger algebras, and that the corresponding one-sided shifts are continuously orbit equivalent if and only if the Cuntz–Krieger algebras are isomorphic in a diagonal-preserving way.

These results all require topologically freeness of actions, or topologically principal groupoids. While seemingly fairly natural, these conditions are not generic. General C^*-algebraic rigidity theorems for homeomorphisms, local homeomorphisms, and more general group actions, require a version of Renault's theory for non-topologically-principal groupoids. Ad hoc results in this direction have been achieved recently for graph C^*-algebras [9, 7], but there is no general theory available.

Our results. A version of Renault's theory is impossible for general groupoids: for example, there is no way to distinguish the groupoids $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ using their C^*-algebras. Our key observation, inspired by techniques developed in [9, 10] is that this obstruction disappears if we insist that the fibres of the interior of the isotropy bundle of the groupoid are torsion-free and abelian; in the 1-unit case, we can then recover the groupoid as the quotient of the unitary group of its C^*-algebra by the connected component of the identity. Groupoids of this sort include all groupoids arising from actions of $\mathbb{Z}^k$ by homeomorphisms or of $\mathbb{N}^k$ by local homeomorphisms: so all Cantor systems, all groupoids associated to graphs and k-graphs and their topological analogues, and many other natural examples.

Our key results, following the idea introduced in [1], deal with groupoids G graded by cocycles c into discrete groups Γ. The reduced C^*-algebra $C^*_r(G)$ then carries a natural coaction δ_c of Γ. We prove that if the interior of the isotropy in $c^{-1}(e)$ is torsion-free and abelian, then G and c can be reconstructed from $C^*_r(G)$, the subalgebra $C_0(G^{(0)})$ and the coaction δ_c. We also obtain a C^*-algebraic characterisation of groupoid equivalences that respect gradings in an appropriate sense.

Incorporating gradings and coactions has significant advantages [11], but the reader may, in the first instance, wish to keep in mind the case where Γ is the trivial group, so c is trivial: our results in this situation are special cases of our general theorems, but the statements are simpler, and still have substantial new content (we summarise our results in this setting in Section 3).

We detail the consequences of our results for dynamical systems, demonstrating the breadth of their applicability: we develop C^*-algebraic characterisations of appropriate notions of stabiliser-preserving orbit equivalence and topological conjugacy for group actions whose essential stabilisers are torsion-free and abelian, generalising Li's continuous-orbit-equivalence-rigidity theorem [29]; we characterise stabiliser-preserving continuous orbit equivalence both of local homeomorphisms and of the associated stabilised systems in the same terms, generalising Matsumoto and Matui's theorem; we generalise Boyle and Tomiyama's theorem to arbitrary homeomorphisms of compact Hausdorff spaces; and we give a new proof that dilations of local homeomorphisms of Cantor sets are conjugate if and only if the associated stabilised Deaconu–Renault C^*-algebras are isomorphic in a gauge-equivariant diagonal-preserving way, and characterise this by an appropriate notion of orbit equivalence. Our results have many other potential applications, particularly to topological graphs, to k-graphs and their topological analogues, to actions of $\mathbb{Z}^k$ on locally compact spaces, and to actions of $\mathbb{N}^k$ by local homeomorphisms.

Précis. The paper is laid out as follows. We give some very brief background in Section 1. In Section 2, we introduce the notion of a weakly Cartan subalgebra D of a C^*-algebra A and establish some technical results about these subalgebras and their normalisers. Our main results are about C^*-algebras A carrying coactions δ of discrete groups, and containing a subalgebra D that is weakly Cartan in the generalised fixed-point algebra A^δ, but in Section 3, we state these results as they apply to ungraded groupoids and C^*-algebras. In Section 4 we show how to construct a locally compact Hausdorff étale groupoid $\mathcal{H}(A, D, \delta)$ from a separable C^*-algebra A, a coaction δ of a discrete group on A and a weakly Cartan subalgebra D of the generalised fixed-point algebra A^δ. Our construction builds on those of Kumjian [28] and Renault [38, 39], but extends them by incorporating the structure of the unitary groups of the fibres of the relative commutant D' in A^δ. Our main application is to groupoid C^*-algebras, but this general construction yields an interesting invariant for general systems (A, D, δ). In Section 5 we prove that if G is an étale groupoid in which the interior of the isotropy is abelian, then the C^*-algebra of the interior of the isotropy is a maximal abelian subalgebra of $C^*_r(G)$; this is needed in Section 6, but also answers a question left open in [8].

In Section 6 we prove our main theorem: if (G, c) is a graded groupoid and the interior of the isotropy in $c^{-1}(e)$ is torsion-free and abelian, then $\mathcal{H}(C^*_r(G), C_0(G^{(0)}), \delta_c) \cong G$ via an isomorphism that intertwines gradings; and any such isomorphism of graded groupoids induces an isomorphism of triples $(C^*_r(G), C_0(G^{(0)}), \delta_c)$. Sections 7–9 detail the consequences of Section 6 for group actions, for local homeomorphisms, and for homeomorphisms, including extensions of Li's rigidity theorem, Matsumoto and Matui's theorem about continuous orbit equivalence, and Boyle and Tomiyama's theorem.

The final two sections deal with Morita equivalence. In Section 10 we introduce equivariant Morita equivalence of triples (A_i, D_i, δ_i) as above, and prove that such a Morita equivalence induces a graded equivalence of extended Weyl groupoids. In Section 11, we apply this result to triples $(C^*_r(G_i), C_0(G_i^{(0)}), \delta_{c_i})$ corresponding to graded groupoids (G_i, c_i) such that the interior of the isotropy in each $c_i^{-1}(e)$ is torsion-free and abelian. We prove that (G_1, c_1) and (G_2, c_2) are graded equivalent if and only if the associated triples $(C^*_r(G_i), C_0(G_i^{(0)}), \delta_{c_i})$ are equivariantly Morita equivalent. Restricting attention to ample groupoids, we use the results of [13] to relate these notions to versions of graded Kakutani equivalence and to graded stable isomorphism of groupoids. We finish by detailing our applications to surjective local homeomorphisms and their dilations.

1. Background

We establish some brief background, and notational conventions, for étale groupoids and their reduced C^*-algebras, $C_0(X)$-algebras, coactions on C^*-algebras, and C^*-algebraic Morita equivalence. For more details see [43], [47, Appendix C], [16] and [37].

Étale groupoids. A groupoid G is the set of morphisms of a small category with inverses; the space of identity morphisms is called the *unit space* and denoted $G^{(0)}$, and the set of composable pairs of morphisms is denoted $G^{(2)}$. A locally compact Hausdorff groupoid is a groupoid G with a locally compact Hausdorff topology under which the inverse and multiplication maps are continuous. A map c from G to a discrete group Γ is a *cocycle* if $c(\eta_1\eta_2) = c(\eta_1)c(\eta_2)$ for $(\eta_1, \eta_2) \in G^{(2)}$ (this forces $c(G^{(0)}) = \{e\}$ and $c(\eta^{-1}) = c(\eta)^{-1}$).

We write r, s for the range and source maps $r(\eta) = \eta\eta^{-1}$ and $s(\eta) = \eta^{-1}\eta$ from G to $G^{(0)}$. A subset $X \subseteq G^{(0)}$ is *full* if $\{r(\eta) : s(\eta) \in X\} = G^{(0)}$. The *isotropy* of G is $\operatorname{Iso}(G) := \{\eta \in G : r(\eta) = s(\eta)\}$. We say that G is *étale* if r (equivalently s) is a local homeomorphism from G to $G^{(0)}$, and that it is *ample* if it is étale and $G^{(0)}$ is totally disconnected. For $x \in G^{(0)}$, we write $G_x := s^{-1}(x)$ and $G^x = r^{-1}(x)$. Using that r, s are local homeomorphisms and that G is Hausdorff, one can check that $G^{(0)}$ is clopen in $C^*_r(G)$, and also that G has a basis of open sets U such that $r|_U$ and $s|_U$ are homeomorphisms of U onto $r(U)$ and $s(U)$ respectively; we call such sets *bisections*.

Given a locally compact Hausdorff étale groupoid G, the space $C_c(G)$ is a *-algebra under the operations $f^*(\gamma) = \overline{f(\gamma)}$ and $(f*g)(\gamma) = \sum_{\alpha\in G^{r(\gamma)}} f(\alpha)g(\alpha^{-1}\gamma)$. For each $u \in G^{(0)}$, there is a *-representation λ_u of $C_c(G)$ on $\ell^2(G_u)$ defined by $\lambda_u(f)\delta_\gamma = \sum_{\alpha\in G_{r(\gamma)}} f(\alpha)\delta_{\alpha\gamma}$. We call λ_u the *regular representation* of $C_c(G)$ at u. The *reduced C^*-algebra* $C^*_r(G)$ is the completion of $C_c(G)$ with respect to the norm $\|f\| = \sup_{u\in G^{(0)}} \|\lambda_u(f)\|$. This norm agrees with the supremum norm on functions f supported on bisections. Since $G^{(0)}$ is clopen, $C_c(G^{(0)})$ includes in $C_c(G)$ in the canonical way, and this extends to an injection $C_0(G^{(0)}) \hookrightarrow C^*_r(G)$. The representations λ_u extend to representations $\lambda_u : C^*_r(G) \to \mathcal{B}(\ell^2(G_u))$. So there is a norm-decreasing map $a \mapsto f_a$ from $C^*_r(G)$ to $C_0(G)$ given by

$$f_a(\gamma) = \big(\lambda_{s(\gamma)}(a)\delta_{s(\gamma)} \mid \delta_\gamma\big) \quad \text{for all } a \in C^*_r(G) \text{ and } \gamma \in G, \tag{1.1}$$

and $a \mapsto f_a$ restricts to the identity map on $C_c(G)$.

$C_0(X)$-algebras. Let X be a locally compact Hausdorff space. A $C_0(X)$-algebra is a C^*-algebra A together with a nondegenerate inclusion $\iota : C_0(X) \to ZM(A)$ of $C_0(X)$ into the centre of the multiplier algebra of A. We obtain a family of ideals $I_x := \overline{\iota(\{f \in C_0(X) : f(x) = 0\})A}$ of A (these subsets are automatically linear subspaces), and then a bundle of C^*-algebras $\{A_x : x \in X\}$ over X given by $A_x := A/I_x$. Each $a \in A$ determines a section $f_a : X \to \mathcal{A} = \bigsqcup_{x\in X} A_x$ such that $f_a(x) = a + I_x \in A_x$. There is a unique topology on $\mathcal{A}$ under which these sections are all continuous. With respect to this topology, $\mathcal{A}$ is an upper-semicontinuous bundle of C^*-algebras in the sense that $b \mapsto \|b\|$ is upper semicontinuous from $\mathcal{A}$ to $[0, \infty)$.

Given any upper-semicontinuous bundle $\mathcal{A}$ of C^*-algebras over X, the space $\Gamma_0(X, \mathcal{A})$ of continuous sections of $\mathcal{A}$ that vanish at infinity is a C^*-algebra under pointwise operations and the supremum norm, and it becomes a $C_0(X)$-algebra with respect to the map $\iota : C_0(X) \to ZM(\Gamma_0(X, \mathcal{A}))$ given by $(\iota(f)\xi)(x) = f(x)\xi(x)$ for $f \in C_0(X)$ and $\xi \in \Gamma_0(X, \mathcal{A})$. If $\mathcal{A}$ is the bundle coming from a $C_0(X)$-algebra A as above, the map $a \mapsto f_a$ is an isomorphism $A \cong \Gamma_0(X, \mathcal{A})$.

Coactions. Given a discrete group Γ, we write λ_g for the image of $g \in \Gamma$ in the left regular representation of Γ on $\ell^2(\Gamma)$. We write $\delta_\Gamma : C^*_r(\Gamma) \to C^*_r(\Gamma) \otimes C^*_r(\Gamma)$ (we use the minimal tensor product) for the comultiplication such that $\delta_\Gamma(\lambda_g) = \lambda_g \otimes \lambda_g$ for $g \in \Gamma$. Given a C^*-algebra A, a *coaction* of Γ on A is a nondegenerate homomorphism $\delta : A \to A \otimes C^*_r(\Gamma)$ satisfying the coaction identity $(\delta \otimes 1) \circ \delta = (1 \otimes \delta_\Gamma) \circ \delta$. The spectral subspaces of A are the spaces $A_g := \{a \in A : \delta(a) = a \otimes \lambda_g\}$; we write A^δ for the neutral spectral subspace A_e, and call it the *generalised fixed-point algebra* for δ. Since we are dealing with reduced coactions, they automatically satisfy $A = \overline{\operatorname{span}}\big(\bigcup_g A_g\big)$.

For each $g \in \Gamma$ there is a norm-decreasing linear map $\Phi_g : A \to A_g$ that fixes A_g pointwise and annihilates A_h for $h \neq g$. Specifically, writing Tr for the canonical trace on $C^*_r(\Gamma)$, the map Φ_g is given by $\Phi_g(a) = (\mathrm{id}_A \otimes \mathrm{Tr})(\delta(a)(1_A \otimes \lambda_{g^{-1}}))$ for $a \in A$. In particular $\Phi^\delta := \Phi_e : A \to A^\delta$ is a conditional expectation.

1.1. **Morita equivalence.** Throughout the paper, we say that an element or a subset of a C^*-algebra A is A-*full* (or just *full*) if it generates A as an ideal. Given C^*-algebras A and B, an A–B-imprimitivity bimodule is an A–B-bimodule X carrying a left A-linear A-valued inner-product ${}_A\langle\cdot,\cdot\rangle$ and a right B-linear B-valued inner-product $\langle\cdot,\cdot\rangle_B$ such that ${}_A\langle x, y \cdot b\rangle = {}_A\langle x \cdot b^*, y\rangle$ and $\langle a \cdot x, y\rangle_b = \langle x, a^* \cdot y\rangle_B$ for all $x, y \in X$, $a \in A$ and $b \in B$, and such that $x \cdot \langle y, z\rangle_B = {}_A\langle x, y\rangle \cdot z$ for all $x, y, z \in X$. We say that C^*-algebras A and B are *Morita equivalent* if there exists an A–B-imprimitivity bimodule. For any C^*-algebra A and any positive element (in particular, any projection) $a \in M(A)$, the space Aa is an imprimitivity bimodule from the ideal $AaA := \overline{\operatorname{span}}\{bac : b, c \in A\}$ generated by a to the hereditary subalgebra $\overline{aAa}$ under ${}_{AaA}\langle x, y\rangle = xy^*$ and $\langle x, y\rangle_{aAa} = x^*y$. So if a is A-full, then A is Morita equivalent to $\overline{aAa}$. If X is an A–B-imprimitivity bimodule, then the conjugate module X^* is a B–A-imprimitivity bimodule, and $L := A \oplus X \oplus X^* \oplus B$ becomes a C^*-algebra called the *linking algebra*, under the natural operations obtained by regarding elements (a, x, y^*, b) as 2×2-matrices $\left(\begin{smallmatrix} a & x \\ y^* & b \end{smallmatrix}\right)$: products of the form xy^* and y^*x are given by ${}_A\langle x, y\rangle$ and $\langle y, x\rangle_B$ respectively. The projections $P = 1_{M(A)}$ and $Q = 1_{M(B)}$ are complementary full multiplier projections such that $PLP \cong A$, $QLQ \cong B$ and $PLQ \cong X$. The Brown–Green–Rieffel theorem implies that if A and B admit countable approximate units, then they are Morita equivalent if and only if they are stably isomorphic.

2. Weakly Cartan pairs and normalisers

In this section we introduce *weakly Cartan subalgebras*, recall the notion of *normalisers*, and prove some fundamental results that we will need later (mainly in Section 4).

Given a C^*-algebra A and an abelian subalgebra D of A, we write D'_A for the relative commutant $D'_A = \{a \in A : ad = da \text{ for all } d \in D\}$ and $\widehat{D}$ for the set of characters of D.

Definition 2.1. A C^*-subalgebra D of a separable C^*-algebra A is *weakly Cartan* if

1. D is abelian,
2. D contains an approximate unit for A,
3. for each $\phi \in \widehat{D}$, the quotient D'/J_ϕ by the ideal $J_\phi := \overline{\ker(\phi)D'}$ of D' is unital, and
4. for each $\phi \in \widehat{D}$, there exist $d \in D$ and an open neighbourhood U of ϕ such that $d + J_\psi = 1_{D'/J_\psi}$ for all $\psi \in U$.

We call the pair (A, D) a *weakly Cartan pair* of C^*-algebras.

Remark 2.2. If $D \subseteq A$ is a Cartan subalgebra in the sense of Renault [39], then it satisfies (1) and (2) by definition. Since Cartan subalgebras are by definition maximal abelian, we have $D' = D \cong C_0(\widehat{D})$, so each $J_\phi = \ker(\phi)$ and each $D'/J_\phi \cong \phi(D) = \mathbb{C}$. So D satisfies (3) and (4) by the Gelfand–Naimark theorem. Hence every Cartan subalgebra is weakly Cartan, justifying our terminology.

Lemma 2.3. *Let A be a separable C^*-algebra and let $D \subseteq A$ be a subalgebra satisfying (1)–(3) of Definition 2.1. Then D is a weakly Cartan if and only if*

(4′) *For each $\phi \in \widehat{D}$ and for each $d \in D$, we have $d + J_\phi = \phi(d)1_{D'/J_\phi}$.*

Proof. Suppose D satisfies (4). Fix $\phi \in \widehat{D}$ and $d \in D$. Choose $d_0 \in D$ and an open $U \ni \phi$ such that $d_0 + J_\psi = 1_{D'/J_\psi}$ for all ψ in U. Since $(d_0^2 + J_\phi)^2 = (1_{D'/J_\phi})^2 = 1_{D'/J_\phi} = d_0 + J_\phi$, we have $\phi(d_0^2) = \phi(d_0)$. Thus, $\phi(d_0) = 1$ and $\phi(\phi(d)d_0) = \phi(d)$. Consequently, $d - \phi(d)d_0 \in J_\phi$, giving

$$d + J_\phi = \phi(d)d_0 + J_\phi = \phi(d)(d_0 + J_\phi) = \phi(d)1_{D'/J_\phi}.$$

Conversely, suppose that D satisfies (4′). Fix $\phi \in \widehat{D}$. Since $\widehat{D}$ is a locally compact Hausdorff space, there exist an open $U \ni \phi$ and $d \in D$ such that $\psi(d) = 1$ for all $\psi \in U$. Thus $d + J_\psi = \psi(d)1_{D'/J_\psi} = 1_{D'/J_\psi}$ for all $\psi \in U$. □

Definition 2.4 (See [28, 38, 39]). Given a C^*-algebra A and a subalgebra D of A containing an approximate unit for A, a *normaliser* n of D in A is an element $n \in A$ such that $nDn^* \cup n^*Dn \subseteq D$. We write $N_A(D)$, or just $N(D)$ if A is clear from context, for the set of normalisers of D in A.

Notation 2.5. For the remainder of the section, A is a C^*-algebra and $D \subseteq A$ is an abelian subalgebra containing an approximate unit for A. For $d \in D$, let $\operatorname{supp}^\circ(d) := \{\phi \in \widehat{D} : \phi(d) \neq 0\}$ and $I(d) := \{d' \in D : \operatorname{supp}^\circ(d') \subseteq \operatorname{supp}^\circ(d)\}$. So $\operatorname{supp}^\circ(d)$ is an open subset of $\widehat{D}$, and $I(d)$ is an ideal of D.

We establish some basic properties of normalisers, several of which also appear in [28].

Lemma 2.6 (Kumjian, Renault). *Let A be a separable C^*-algebra and D an abelian C^*-subalgebra containing an approximate unit for A. For $m, n \in N(D)$,*

1. *$n^*n, nn^* \in D$;*
2. *there is a unique homeomorphism $\alpha_n : \operatorname{supp}^\circ(n^*n) \to \operatorname{supp}^\circ(nn^*)$ such that $\phi(n^*n)\alpha_n(\phi)(d) = \phi(n^*dn)$ for all $d \in D$;*

(3) *there is a unique isomorphism* $\alpha_n^\# : I(nn^*) \to I(n^*n)$ *such that* $\phi(\alpha_n^\#(d)) = \alpha_n(\phi)(d)$ *for* $d \in I(nn^*)$ *and* $\phi \in \operatorname{supp}^\circ(n^*n)$*;*
(4) *if* $d \in I(nn^*)$*, then* $dn = n\alpha_n^\#(d)$*;*
(5) $mn \in N(D)$*,* $\operatorname{supp}^\circ((mn)^*(mn)) = \alpha_n^{-1}(\operatorname{supp}^\circ(m^*m) \cap \operatorname{supp}^\circ(nn^*))$*, and on this domain,* $\alpha_m \circ \alpha_n = \alpha_{mn}$*;*
(6) $\alpha_{n^*} = \alpha_n^{-1}$*; and*
(7) *if* $U \subseteq \operatorname{supp}^\circ(n^*n) \cap \operatorname{supp}^\circ(m^*m) \subseteq \widehat{D}$ *is open and satisfies* $\alpha_n|_U = \alpha_m|_U$*, then* $dn^*m = n^*md$ *for all* $d \in D$ *with* $\operatorname{supp}^\circ(d) \subseteq U$*, and* $\phi(m^*nn^*m) = \phi(m^*m)\phi(n^*n)$*.*

Proof. (1) Let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A in D. Then each $n^*e_\lambda n, ne_\lambda n^* \in D$ because $n \in N(D)$. So $n^*n = \lim_\lambda n^*e_\lambda n \in D$ and $nn^* = \lim_\lambda ne_\lambda n^* \in D$.

(2) and (3) This is proved in [28, Proposition 1.6]. We summarise the points of the proof that we need for the remaining statements. Let $n = v|n|$ be the polar decomposition of n in A^{**}. Then $vv^*d = dvv^* = d$ for $d \in I(nn^*)$, and $v^*vd = dv^*v = d$ for $d \in I(n^*n)$. Kumjian shows that $vI(n^*n)v^* \subseteq I(nn^*)$ and $v^*I(nn^*)v \subseteq I(n^*n)$. So $d \mapsto v^*dv$ defines an isomorphism $\alpha_n^\# : I(nn^*) \to I(n^*n)$, and there is a homeomorphism $\alpha_n : \operatorname{supp}^\circ(n^*n) \to \operatorname{supp}^\circ(nn^*)$ such that $\alpha_n(\phi)(d) = \phi(\alpha_n^\#(d))$ for all $d \in I(nn^*)$. If $d \in D$, then

$$\phi(n^*n)\alpha_n(\phi)(d) = \phi(|n|)\phi(\alpha_n^\#(d))\phi(|n|) = \phi(|n|v^*dv|n|) = \phi(n^*dn).$$

That each of $\widehat{D}$ and D separates elements of the other gives uniqueness of α_n and $\alpha_n^\#$.

(4) As above, let $n = v|n|$ be the polar decomposition of n. By (1), we have $|n| = (n^*n)^{1/2} \in D$. It follows that if $d \in I(nn^*)$, then

$$dn = dv|n| = vv^*dv|n| = v\alpha_n^\#(d)|n| = v|n|\alpha_n^\#(d) = n\alpha_n^\#(d).$$

(5) Fix $d \in D$ and $m, n \in N(D)$. Then $(mn)^*d(mn) = n^*(m^*dm)n^* \in n^*Dn \subseteq D$; likewise $(mn)d(mn)^* \in D$, so $mn \in N(D)$. We have $n^*m^*mnn^*n = n^*n\alpha_n^\#(m^*mnn^*)$ by (4). So $\operatorname{supp}^\circ((mn)^*(mn)) = \alpha_n^{-1}(\operatorname{supp}^\circ(m^*m) \cap \operatorname{supp}^\circ(nn^*))$. For $\phi \in \operatorname{supp}^\circ((mn)^*(mn))$,

$$\begin{aligned}\phi(n^*m^*mn)\alpha_{mn}(\phi)(d) &= \phi(n^*m^*dmn) = \phi(n^*n)\alpha_n(\phi)(m^*dm)\\ &= \phi(n^*n)\alpha_n(m^*m)\alpha_m(\alpha_n(\phi))(d) = \phi(n^*m^*mn)\alpha_m(\alpha_n(\phi))(d).\end{aligned}$$

This shows that $\alpha_{mn}(\phi) = \alpha_m(\alpha_n(\phi))$.

(6) Suppose $\phi \in \operatorname{supp}^\circ((n^*n)^*(n^*n))$ and $d \in D$. Then

$$\phi((n^*n)^*(n^*n))\alpha_{n^*n}(\phi)(d) = \phi((n^*n)^*d(n^*n)) = \phi(n^*n)\phi(d)\phi(n^*n) = \phi((n^*n)^*(n^*n))\phi(d).$$

So $\alpha_{n^*n} = \mathrm{id}_{\operatorname{supp}^\circ(n^*n)}$, and then (5) gives $\alpha_{n^*} \circ \alpha_n = \mathrm{id}_{\operatorname{supp}^\circ(n^*n)}$. Thus, $\alpha_{n^*} = \alpha_n^{-1}$.

(7) Fix $d \in D$ with $\operatorname{supp}^\circ(d) \subseteq U$. As $\alpha_n|_U = \alpha_m|_U$, statement (3) and (6) give $\alpha_{n^*}^\#(d) = \alpha_{m^*}^\#(d)$. Two applications of (4) then give $dn^*m = n^*\alpha_{n^*}(d)m = n^*md$. The definition of $\alpha\#_n$ shows that $\alpha_n^\#(nn^*) = n^*n$, so $\alpha_n(\phi)(nn^*) = \phi(n^*n)$ for all ϕ. So for $\phi \in U$, we have $\phi(m^*nn^*m) = \phi(m^*m)\alpha_m(\phi)(nn^*) = \phi(m^*m)\alpha_n(\phi) = \phi(m^*m)\phi(n^*n)$. □

3. Results for ungraded groupoids and C^*-algebras

To maximise their generality, we formulate our key results later in the paper for graded groupoids and C^*-algebras carrying coactions, under suitable hypotheses on the trivially-graded subgroupoids and generalised fixed-point subalgebras. In this section, we summarise the consequences of our main results for trivial gradings and coactions.

Lemma 4.5 below applied to a weakly Cartan subalgebra $D \subseteq A$ and the trivial coaction on A yields an equivalence relation on $\{(n, \phi) : n \in N(D),\ \phi \in \operatorname{supp}^\circ(n^*n)\}$ such that $(n, \phi) \sim (m, \psi)$ if and only if $\phi = \psi$, α_n and α_m agree on a neighbourhood U of ϕ, and there exists $d \in D$ with $\operatorname{supp}^\circ(d) \subseteq U$ and $\phi(d) = 1$ such that $\phi(m^*m)^{-\frac{1}{2}}\phi(n^*n)^{-\frac{1}{2}} dn^*md + J_\phi$ is homotopic to the identity in the unitary group of D'/J_ϕ. Our first main theorem says that the quotient space can be made into an étale groupoid.

Theorem 3.1 (see Theorem 4.9). *Let A be a separable C^*-algebra and D a weakly Cartan subalgebra of A. Then*

$$\mathcal{H}(A, D) := \{[n, \phi] : n \in N(D),\ \phi \in \operatorname{supp}^\circ(n^*n)\}$$

is a second-countable locally compact locally Hausdorff étale groupoid with composable pairs $\mathcal{H}(A, D)^{(2)} = \{([m, \phi], [n, \psi]) : \phi = \alpha_n(\psi)\}$, multiplication and inverses given by

$$[m, \alpha_n(\psi)][n, \psi] = [mn, \psi] \quad \text{and} \quad [n, \psi]^{-1} = [n^*, \alpha_n(\psi)],$$

and topology with basic open sets

$$Z(n, U) = \{[n, \phi] : \phi \in U\}$$

indexed by $n \in N(D)$ and open $U \subseteq \operatorname{supp}^\circ(n^*n) \subseteq \widehat{D}$.

If $D \subseteq A$ is Cartan as in [39], then $\mathcal{H}(A, D)$ is the Weyl groupoid of [39]. If $A = C^*(E)$ and $D = \mathcal{D}(E)$ where E is a countable graph, then $\mathcal{H}(A, D)$ is the extended Weyl groupoid $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ of [9].

Proposition 3.2 (see Proposition 6.5). *Let G be a second-countable locally compact Hausdorff étale groupoid such that $\operatorname{Iso}(G)^\circ$ is abelian. Then $C_0(G^{(0)})$ is a weakly Cartan subalgebra of $C_r^*(G)$. If $\operatorname{Iso}(G)^\circ$ is torsion free, then there is an isomorphism $G \cong \mathcal{H}(C_r^*(G), C_0(G^{(0)}))$ that carries $\gamma \in G$ to $[f, \widehat{s(\gamma)}]$ for any $f \in C_c(G)$ supported on a bisection with $f(\gamma) \neq 0$.*

As an almost immediate consequence of Proposition 3.2, we obtain the following.

Theorem 3.3 (see Theorem 6.2). *Let G_1, G_2 be second-countable locally compact Hausdorff étale groupoids such that each $\operatorname{Iso}(G_i)^\circ$ is torsion-free and abelian.*

1. *Any isomorphism $\kappa : G_2 \to G_1$ induces an isomorphism $\phi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\phi(f) = f \circ \kappa$ for $f \in C_c(G_1)$, and in particular $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$.*
2. *Any isomorphism $\phi : C_r^*(G_1) \to C_r^*(G_2)$ satisfying $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ induces an isomorphism $\kappa : G_2 \to G_1$ such that $f \circ \kappa = \phi(f)$ for $f \in C_0(G_1^{(0)})$.*

If $D_i \subseteq A_i$ is a nested pair of C^*-algebras for $i = 1, 2$, we say that (A_1, D_1) and (A_2, D_2) are Morita equivalent if there is an A_1–A_2-imprimitivity bimodule X such that

$$X = \overline{\operatorname{span}}\{x \in X : \langle x, D_1 \cdot x\rangle_{A_2} \subseteq D_2 \text{ and } {}_{A_1}\langle x \cdot D_2, x\rangle \subseteq D_1\}.$$

Theorem 3.4 (see Theorem 10.6). *Let A_1, A_2 be separable C^*-algebras. Suppose, for $i = 1, 2$, that D_i is a weakly Cartan subalgebra of A_i. Suppose that X is a Morita equivalence between (A_1, D_1) and (A_2, D_2). Let A be the linking algebra of X and let $D := D_1 \oplus D_2 \subseteq A$. Then D is weakly Cartan in A. The groupoid $\mathcal{H} := \mathcal{H}(A, D)$ contains $\widehat{D}_1$ and $\widehat{D}_2$ as complementary full clopen subsets of $H^{(0)}$, $\widehat{D}_i\mathcal{H}\widehat{D}_i \cong \mathcal{H}(A_i, D_i)$ for $i = 1, 2$, and $\widehat{D}_1\mathcal{H}\widehat{D}_2$ is an equivalence from $\mathcal{H}(A_1, D_1)$ to $\mathcal{H}(A_2, D_2)$.*

Theorem 3.5 (see Theorem 11.1). *Let G_1, G_2 be second-countable locally compact Hausdorff étale groupoids such that each $\mathrm{Iso}(G_i)^\circ$ is torsion-free and abelian. The following are equivalent:*

1. *G_1 and G_2 are equivalent;*
2. *there exist a second-countable locally compact Hausdorff étale groupoid G such that $\mathrm{Iso}(G)^\circ$ is torsion-free and abelian, and a pair of complementary G-full open subsets $K_1, K_2 \subseteq G^{(0)}$ such that $K_i G K_i \cong G_i$ for $i = 1, 2$.*
3. *$(C_r^*(G_1), C_0(G_1^{(0)}))$ and $(C_r^*(G_2), C_0(G_2^{(0)}))$ are Morita equivalent; and*
4. *there are a separable C^*-algebra A, a weakly Cartan $D \subseteq A$ satisfying $\overline{\mathrm{span}} N(D) = A$, a pair of complementary A-full projections $P_1, P_2 \in M(D)$, and isomorphisms $\phi_i : P_i A P_i \to C_r^*(G_i)$ such that $\phi_i(P_i D P_i) = D_i$.*

As in [13], we let $\mathcal{R} = \mathbb{N} \times \mathbb{N}$ regarded as a discrete groupoid; we have $C_r^*(\mathcal{R}) \cong \mathcal{K}$, the compact operators on $\ell^2(\mathbb{N})$, with canonical diagonal subalgebra c_0. As in [13], we say G_1 and G_2 are *weakly Kakutani equivalent* if there are full open subsets $U_i \subseteq G_i^{(0)}$ such that $U_1 G_1 U_1 \cong U_2 G_2 U_2$. If U_1 and U_2 are *compact* open, G_1 and G_2 are *Kakutani equivalent* as defined in [31]. Combining our results with [13], we obtain the following.

Corollary 3.6 (see Corollary 11.3). *Let G_1, G_2 be second-countable ample Hausdorff groupoids such that each $\mathrm{Iso}(G)^\circ$ is torsion-free and abelian. Then the equivalent conditions of Theorem 3.5 are also equivalent to the following:*

1. *$G_1 \times \mathcal{R} \cong G_2 \times \mathcal{R}$;*
2. *G_1 and G_2 are Kakutani equivalent;*
3. *G_1 and G_2 are weakly Kakutani equivalent;*
4. *there is an isomorphism $\phi : C_r^*(G_1) \otimes \mathcal{K} \to C_r^*(G_2) \otimes \mathcal{K}$ satisfying $\phi(C_0(G_1^{(0)}) \otimes c_0) = C_0(G_2^{(0)}) \otimes c_0$;*
5. *there exist $C_r^*(G_i)$-full projections $p_i \in M(C_0(G_1^{(0)}))$ and an isomorphism ϕ of $p_1 C_r^*(G_1) p_1$ onto $p_2 C_r^*(G_2) p_2$ such that $\phi(p_1 C_0(G_1^{(0)})) = p_2 C_0(G_1^{(0)})$; and*
6. *there are ideals $I_i \subseteq C_0(G_i^{(0)})$ that are full in $C_r^*(G_i)$, and an isomorphism $\phi : I_1 C_r^*(G_1) I_1 \to I_2 C_r^*(G_2) I_2$ such that $\phi(I_1) = I_2$.*

4. The extended Weyl groupoid

We construct a graded groupoid $(\mathcal{H}(A, D, \delta), c_\delta)$ from a separable C^*-algebra A, a coaction of a discrete group Γ on A, and a weakly Cartan subalgebra $D \subseteq A^\delta$. The reader may wish to keep in mind the case where δ is trivial, so $A^\delta = A$; in this case, if D is a Cartan subalgebra of A, then $\mathcal{H}(A, D, \delta)$ agrees with Renault's Weyl groupoid [39]. Given a countable directed graph, then $\mathcal{H}(C^*(E), C_0(\partial E), \delta)$ is the extended Weyl groupoid $\mathcal{G}_{(C^*(E), \mathcal{D}(E))}$ of [9], both when δ is the trivial coaction, and when δ is the coaction of $\mathbb{Z}$ dual to the gauge action.

Throughout this section A is a separable C^*-algebra, Γ is a discrete group, δ is a coaction of Γ on A, and D is a weakly Cartan subalgebra of A^δ. We write

$$D'_{A^\delta} := \{a \in A^\delta : ad = da \text{ for all } d \in D\},$$

for the relative commutant of D in A^δ, and

$$\pi_\phi : D'_{A^\delta} \to D'_{A^\delta} / J_\phi$$

for the canonical quotient map.

Lemma 4.1. *Let δ be a coaction of a discrete group Γ on a separable C^*-algebra A, and suppose that D is weakly Cartan in A^δ. Then D contains an approximate unit for A.*

Proof. Corollary 1.6 of [36] shows that A^δ contains an approximate unit for A. Thus any approximate unit for A^δ is an approximate unit for A, so the result follows from condition (2) of Definition 2.1. $\square$

Following [1], we say that a normaliser n of D is a *homogeneous normaliser* if $n \in A_g$ for some $g \in \Gamma$. We write $N_g(D) := N(D) \cap A_g$, and we write $N_\star(D) := \bigcup_{g\in\Gamma} N_g(D)$. The groupoid $\mathcal{H}(A, D, \delta)$ consists of equivalence classes $[n, \phi]$ where $n \in N_\star(D)$ and $\phi \in \operatorname{supp}^\circ(n^*n)$. To define the appropriate equivalence relation, we first need two lemmas.

Lemma 4.2. *Let A be a separable C^*-algebra, δ a coaction of a discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Let $n, m \in N(D)$ and $\phi \in \operatorname{supp}^\circ(n^*n) \cap \operatorname{supp}^\circ(m^*m)$, and suppose that there is an open neighbourhood U of ϕ such that $U \subseteq \operatorname{supp}^\circ(n^*n) \cap \operatorname{supp}^\circ(m^*m)$ and $\alpha_n|_U = \alpha_m|_U$. Fix $d \in D$ with $\operatorname{supp}^\circ(d) \subseteq U$ and $\phi(d) = 1$, and let*

$$w := \phi(m^*m)^{-1/2}\phi(n^*n)^{-1/2}dn^*md. \tag{4.1}$$

Then $w \in D'_{A^\delta}$ and $\pi_\phi(w)$ is unitary in D'_{A^δ}/J_ϕ. We have

$$\pi_\phi\big(\phi(n^*n)^{-1/2}\phi(m^*m)^{-1/2}dm^*nd\big) = \pi_\phi(w^*),$$

and $\pi_\phi(w)$ is independent of the choices of U and d.

Proof. We have $w \in D'_{A^\delta}$ by Lemma 2.6(7). By Lemma 2.3 and Lemma 2.6, and since $\operatorname{supp}^\circ(d) \subseteq U$, $\phi(d) = 1$, and $\alpha_m|_U = \alpha_n|_U$, we have

$$\begin{aligned}\phi(m^*m)\phi(n^*n)\pi_\phi(w^*w) &= \pi_\phi((dn^*md)^*(dn^*md)) = \pi_\phi(d^*d)^2\pi_\phi(m^*nn^*m)\\ &= \phi(m^*nn^*m)1_{D'_{A^\delta}/J_\phi} = \phi(m^*m)\phi(n^*n)1_{D'_{A^\delta}/J_\phi},\end{aligned}$$

so $\pi_\phi(w^*w) = 1_{D'_{A^\delta}/J_\phi}$. Switching the roles of m, n gives $\pi(ww^*) = 1_{D'_{A^\delta}/J_\phi}$. We clearly have $\pi_\phi(\phi(m^*m)^{-1/2}\phi(n^*n)^{-1/2}dm^*nd) = \pi_\phi(w^*)$.

Now fix open $U_1, U_2 \subseteq \widehat{D}$ and $d_1, d_2 \in D$ with $\phi \in U_1 \cap U_2$, $\alpha_n|_{U_i} = \alpha_m|_{U_i}$, $\operatorname{supp}^\circ(d_i) \subseteq U_i$ and $\phi(d_i) = 1$. It follows from Lemma 2.6(7) that $n^*md_1d_2 = d_1d_2n^*m$. Thus

$$\pi_\phi(d_1n^*md_1d_2m^*nd_2) = \pi_\phi(d_1^2d_2n^*mm^*nd_2) = \pi_\phi(n^*mm^*n) = \phi(n^*mm^*n)1_{D'_{A^\delta}/J_\phi},$$

and we deduce that

$$\pi_\phi\big(\phi(m^*m)^{-1/2}\phi(n^*n)^{-1/2}d_1n^*md_1\big) = \pi_\phi\big(\phi(m^*m)^{-1/2}\phi(n^*n)^{-1/2}d_2n^*md_2\big). \qquad \square$$

Notation 4.3. Let A be a separable C^*-algebra, δ a coaction of a discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Suppose that $n, m \in N(D)$, $\phi \in \operatorname{supp}^\circ(n^*n) \cap \operatorname{supp}^\circ(m^*m)$, and that there is an open $U \ni \phi$ such that $\alpha_m|_U = \alpha_n|_U$. We write

$$U^\phi_{n^*m} := \pi_\phi(w)$$

for any w of the form (4.1). If ϕ is clear from context, we just write U_{n^*m} for $U^\phi_{n^*m}$

Lemma 4.4. *Let A be a separable C^*-algebra, δ a coaction of a discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Take $n_1, n_2, m \in N(D)$, $\phi \in \operatorname{supp}^\circ(n_1^*n_1) \cap \operatorname{supp}^\circ(n_2^*n_2) \cap \operatorname{supp}^\circ(m^*m)$, and open $U_i \ni \phi$ with $\alpha_m|_{U_i} = \alpha_{n_i}|_{U_i}$. Then:*

(1) $U_{n_i^*n_i} = 1_{D'_{A^\delta}/J_\phi}$,

(2) $U^*_{n_i^* m} = U_{mn_i^*}$, *and*
(3) $U_{n_1^* m} U_{m^* n_2} = U_{n_1^* n_2}$.

Proof. By normalising, we can assume that $\phi(n_i^* n_i) = \phi(m^* m) = 1$. For (1), just calculate: $\pi_\phi(dn_i^* n_i d) = \phi(d)^2 \phi(n_i^* n_i) \cdot 1_{D'_{A^\delta}/J_\phi} = 1_{D'_{A^\delta}/J_\phi}$. Statement (2) follows from Lemma 4.2 because π_ϕ is a *-homomorphism. For (3), take d_i supported on U_i with $\phi(d_i) = 1$. A quick calculation using that $\alpha_{n_i}(\phi)(mm^*) = \phi(m^* m)$ and that $\alpha_{n_i^* m}(\phi) = \phi$ gives $U_{n_1^* m} U_{m^* n_2} = \phi(m^* m)\phi(d_1 d_2)\pi_\phi(d_1 n_1^* n_2 d_2) = U_{n_1^* n_2}$. □

We are now ready to describe the elements of $\mathcal{H}(A, D, \delta)$.

Lemma 4.5. *Let A be a separable C^*-algebra, δ a coaction of a discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Define $\sim$ on $\{(n, \phi) : n \in N_\star(D), \phi \in \operatorname{supp}^\circ(n^* n)\}$ by $(n, \phi) \sim (m, \psi)$ if and only if*

(R1) $\phi = \psi$,
(R2) $n^* m \in A^\delta$,
(R3) *there exists an open neighbourhood U of ϕ in $\widehat{D}$ such that $U \subseteq \operatorname{supp}^\circ(n^* n) \cap \operatorname{supp}^\circ(m^* m)$ and $\alpha_m|_U = \alpha_n|_U$, and*
(R4) *the unitary $U^\phi_{n^* m}$ of Notation 4.3 belongs to the connected component $\mathcal{U}_0(D'_{A^\delta}/J_\phi)$ of the identity in the unitary group of D'_{A^δ}/J_ϕ.*

Then $\sim$ is an equivalence relation.

Proof. Reflexivity follows from Lemma 4.4(1). Symmetry follows from Lemma 4.4(2) and that if $U^\phi_{n^* m} \in \mathcal{U}_0(D'_{A^\delta}/J_\phi)$, then $(U^\phi_{n^* m})^* \in \mathcal{U}_0(D'_{A^\delta}/J_\phi)$.

For transitivity, suppose that $(n_1, \phi) \sim (m, \psi) \sim (n_2, \rho)$. Then $\phi = \psi = \rho$, $n_1^* m, m^* n_2 \in A^\delta$, there are open sets $\phi \in U_i \subseteq \operatorname{supp}^\circ(n_i^* n_i) \cap \operatorname{supp}^\circ(m^* m)$ such that $\alpha_{n_i}|_{U_i} = \alpha_m|_{U_i}$, and $U_{n_1^* m}, U_{m^* n_2} \in \mathcal{U}(D'_{A^\delta}/J_\phi)_0$.

The set $V := U_1 \cap U_2 \subseteq \operatorname{supp}^\circ(n_1^* n_1) \cap \operatorname{supp}^\circ(n_2^* n_2)$ is open, contains ϕ and satisfies $\alpha_{n_1}|_V = \alpha_{n_2}|_V$. We have $n_1 \in A_{g_1}$, $m \in A_h$ and $n_2 \in A_{g_2}$ for some $g_1, h, g_2 \in \Gamma$. Since $\alpha_{n_1}(\phi) = \alpha_m(\phi)$, Lemma 2.6 gives $\phi(n_1^* m m^* n_1) = \phi(n_1^* n_1)\alpha_m(\phi)(mm^*) = \phi(n_1^* n_1)\phi(m^* m)$, so $n_1^* m \neq 0$. So $n_1^* m \in A^\delta$ forces $g_1 = h$. Likewise $h = g_2$, so $n_1^* n_2 \in A^\delta$.

Finally, $U_{n_1^* n_2} = U_{n_1^* m} U_{m^* n_2}$ by Lemma 4.4(3), so $U_{n_1^* m}, U_{m^* n_2} \in \mathcal{U}_0(D'_{A^\delta}/J_\phi)$ forces $U_{n_1^* n_2} \in \mathcal{U}_0(D'_{A^\delta}/J_\phi)$, and thus $(n_1, \phi) \sim (n_2, \rho)$. □

Remark 4.6. In the situation of Lemma 4.5, if $n \in N_\star(D)$ and $\phi \in \operatorname{supp}^\circ(n^* n)$, then $m := \phi(n^* n)^{-1/2} n$ satisfies $(n, \phi) \sim (m, \phi)$ and $\phi(m^* m) = 1$. So we can and frequently will, without loss of generality, choose representatives (n, ϕ) of the equivalence classes for $\sim$ such that $\phi(n^* n) = 1$. If $\phi(m^* m) = \phi(n^* n) = 1$ and α_m and α_n agree on a neighbourhood $U \ni \phi$, then $U_{n^* m} = \pi_\phi(dn^* md)$ for any d supported on U with $\phi(d) = 1$.

We now construct our extended Weyl groupoid.

Proposition 4.7. *Let A be a separable C^*-algebra, δ a coaction of a discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Let $\sim$ be the equivalence relation of Lemma 4.5, and for $n \in N_\star(D)$ and $\phi \in \operatorname{supp}^\circ(n^* n)$, let $[n, \phi]$ denote the equivalence class of (n, ϕ) under $\sim$. Define*

$$\mathcal{H}(A, D, \delta) := \{[n, \phi] : n \in N_\star(D), \phi \in \operatorname{supp}^\circ(n^* n)\}.$$

There are maps

$$r, s\colon \mathcal{H}(A, D, \delta) \to \widehat{D},$$
$$M\colon \mathcal{H}(A, D, \delta)\ {}_s{\times}_r\ \mathcal{H}(A, D, \delta) \to \mathcal{H}(A, D, \delta), \text{ and}$$
$$I\colon \mathcal{H}(A, D, \delta) \to \mathcal{H}(A, D, \delta)$$

such that

$$r([n, \phi]) = \alpha_n(\phi), \quad s([n, \phi]) = \phi,$$
$$M([n, \phi], [m, \psi]) = [nm, \psi], \quad \text{and} \quad I([n, \phi]) = [n^*, \alpha_n(\phi)].$$

Moreover, $\mathcal{H}(A, D, \delta)$ is a groupoid under these operations, and there is a cocycle c_δ : $\mathcal{H}(A, D, \delta) \to \Gamma$ such that $c_\delta([n, \phi]) = g$ if and only if $n \in A_g$.

We need the following lemma for the proof of Proposition 4.7.

Lemma 4.8. *Let A be a separable C^*-algebra, δ a coaction of a discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Suppose $m \in N(D)$, $\phi \in \operatorname{supp}^\circ(m^*m)$, and $\phi(m^*m) = 1$. Then there is an isomorphism $\iota_m : D'_{A^\delta}/J_{\alpha_m(\phi)} \to D'_{A^\delta}/J_\phi$ such that $\iota_m(\pi_{\alpha_m(\phi)}(a)) = \pi_\phi(m^*am)$ for $a \in D'_{A^\delta}$.*

Proof. By Lemma 2.6, $a \in J_{\alpha_m(\phi)} \implies m^*am \in J_\phi$. Thus $a \mapsto m^*am$ descends to a linear *-preserving map $\iota_m : D'_{A^\delta}/J_{\alpha_m(\phi)} \to D'_{A^\delta}/J_\phi$. For $a_1, a_2 \in D'_{A^\delta}$,

$$\begin{aligned}\pi_\phi(m^*a_1m)\pi_\phi(m^*a_2m) &= \pi_\phi(m^*a_1mm^*a_2m) = \pi_\phi(m^*mm^*a_1a_2m)\\ &= \phi(m^*m)\pi_\phi(m^*a_1a_2m) = \pi_\phi(m^*a_1a_2m)\end{aligned}$$

so ι_m is multiplicative and hence a *-homomorphism. Symmetry gives a *-homomorphism $\iota_{m^*} : D'_{A^\delta}/J_\phi \to D'_{A^\delta}/J_{\alpha_m(\phi)}$ such that $\iota_{m^*}(\pi_\phi(a)) = \pi_{\alpha_m(\phi)}(mam^*)$ for $a \in D'_{A^\delta}$. It is easy to check that ι_{m^*} is an inverse to ι_m. □

Proof of Proposition 4.7. If $(n, \phi) \sim (m, \psi)$, then $\phi = \psi$, and since $\alpha_n = \alpha_m$ on a neighbourhood of ϕ, we have $\alpha_n(\phi) = \alpha_m(\phi)$; so r and s are well defined. Suppose that $[n, \phi] = [n', \phi]$ and $r([m, \psi]) = s([n, \phi])$. Then $nm, n'm \in N_\star(D)$. We claim that $(nm, \psi) \sim (n'm, \psi)$. Indeed, (R1) is clear, and (R2) is immediate because $(n, \phi) \sim (n', \phi)$ forces $(nm)^*n'm = m^*n^*n'm \in A^\delta$. Take an open U with $\phi \in U \subseteq \operatorname{supp}^\circ(n^*n) \cap ((n')^*n')$ and $\alpha_n|_U = \alpha_{n'}|_U$. Fix d supported on U with $\phi(d) = 1$. Since $(n, \phi) \sim (n', \phi)$, we have $U^\phi_{n^*n'} \in \mathcal{U}_0(D'_{A^\delta}/J_\phi)$. Now $V := \alpha_m^{-1}(U \cap \operatorname{supp}^\circ(mm^*))$ is a neighbourhood of ψ, and Lemma 2.6 gives $\alpha_{nm}|_V = \alpha_{n'm}|_V$, giving (R3). For (R4), we may assume that $\psi(m^*m) = 1$. The isomorphism ι_m of Lemma 4.8 satisfies $U^\psi_{(nm)^*(n'm)} = \iota_m(U^\phi_{n^*n'})$, so $U^\phi_{n^*n'} \in \mathcal{U}_0(D'_{A^\delta}/J_\phi)$ forces $U^\psi_{(nm)^*(n'm)} \in \mathcal{U}_0(D'_{A^\delta}/J_\psi)$, giving (R4). So $(nm, \psi) \sim (n'm, \psi)$. Therefore

$$\text{if } [n, \phi] = [n', \phi] \text{ and } s([n, \phi]) = r([m, \psi]), \text{ then } [nm, \psi] = [n'm, \psi]. \tag{4.2}$$

Now suppose that $[m, \psi] = [m', \psi]$ and $r([m, \psi]) = s([n, \phi])$. We have $nm, nm' \in N_\star(D)$. We claim that $(nm, \psi) \sim (nm', \psi)$. Both (R1) and (R2) are immediate as above. The argument for (R3) is very similar to that in the preceding paragraph. For (R4), we may assume that $\psi(m^*m) = \phi(n^*n) = 1$. Choose $V \ni \psi$ open such that $V \subseteq$

$\mathrm{supp}^\circ((nm)^*nm) \cap \mathrm{supp}^\circ((nm')^*nm')$, and $d \in D$ with $\mathrm{supp}^\circ(d) \subseteq V$ and $\psi(d) = 1$. Then

$$\begin{aligned} U^{\psi}_{(nm)^*nm'} &= \pi_\psi(d(nm)^*nm'd) = \pi_\psi(\alpha^{\#}_m(\alpha^{\#}_{m^*}(d)n^*n)m^*m'd) = \phi(n^*n)\pi_\psi(dm^*m'd) \\ &= \pi_\psi(dm^*m'd) = U^{\psi}_{m^*m'} \in \mathcal{U}_0(D'_{A^\delta}/J_\psi). \end{aligned}$$

Thus, $(nm, \psi) \sim (nm', \psi)$. Hence

$$\text{if } [m, \psi] = [m', \psi] \text{ and } r([n, \phi]) = s([m, \psi]), \text{ then } [nm, \psi] = [nm', \psi]. \tag{4.3}$$

By (4.2) and (4.3), if $[n, \phi] = [n', \phi]$, $[m, \psi] = [m', \psi]$, and $r([m, \psi]) = s([n, \phi])$, then $[nm, \psi] = [n'm, \psi] = [n'm', \psi]$, so M is well-defined.

To see that I is well-defined, suppose that $[n, \phi] = [m, \phi]$. We claim that $(n^*, \alpha_n(\phi)) \sim (m^*, \alpha_m(\phi))$. Again (R1) and (R2) are clear, and (R3) is routine because $\alpha_{n^*} = \alpha_n^{-1}$ and similarly for α_{m^*}. For (R4), we may assume that $\psi(m^*m) = \phi(n^*n) = 1$. For an open $U \ni \phi$ with $\alpha_n|_U = \alpha_m|_U$ and d supported on U with $\phi(d) = 1$,

$$\begin{aligned} U^{\alpha_n(\phi)}_{nm^*} &= \pi_{\alpha_n(\phi)}\big((ndn^*)nm^*(ndn^*)\big) \\ &= \iota_{n^*}(\pi_\phi(dm^*nd)) = \iota_{n^*}(U^\phi(m^*n)) \in \mathcal{U}_0(D'_{A^\delta}/J_{\alpha_n(\phi)}). \end{aligned}$$

So (R4) is also satisfied, and $(n^*, \psi) \sim (m^*, \psi)$. Thus I is well-defined.

The multiplication defined by M is associative because multiplication in A is associative. By construction, we have $[n, \phi]^{-1}[n, \phi] = [n^*n, \phi] = [d, \phi]$ for any $d \in D$ with $\phi(d) > 0$. Similarly, $[n, \phi][n, \phi]^{-1} = [c, \alpha_n(\phi)]$ for any $c \in D$ with $\alpha_n(\phi)(c) > 0$. Since $\pi_\phi(d) = \phi(d)1_{D'_{A^\delta}/J_\phi}$ for $d \in D$, it is routine to check using the definition of M that

$$[c, \alpha_m(\psi)][m, \psi] = [m, \psi] = [m, \psi][d, \psi]$$

for any $c, d \in D$ such that $\alpha_m(\psi)(c) > 0$ and $\psi(d) > 0$. So $\mathcal{H}(A, D, \delta)$ is a groupoid.

Finally, the formula $c_\delta([n, \phi]) = g$ if $n \in N_g(D)$ is well defined by (R2), and it is multiplicative because $n \in A_g$ and $m \in A_h$ implies $nm \in A_{g+h}$. □

We now show how to make $(\mathcal{H}(A, D, \delta), c_\delta)$ into a graded Hausdorff étale groupoid.

Theorem 4.9. *Let A be a separable C^*-algebra, δ a coaction of a countable discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Let $\mathcal{H}(A, D, \delta)$ be the groupoid of Proposition 4.7. For $n \in N_\star(D)$ and an open set $X \subseteq \widehat{D}$ contained in* $\mathrm{supp}^\circ(n^*n)$ *let*

$$Z(n, X) := \{[n, \phi] : \phi \in X\} \subseteq \mathcal{H}(A, D, \delta).$$

Then

$$\{Z(n, X) : n \in N_\star(D),\ X \subseteq \widehat{D} \text{ is open and } X \subseteq \mathrm{supp}^\circ(n^*n)\} \tag{4.4}$$

constitutes a countable basis for a locally compact locally Hausdorff étale topology on $\mathcal{H}(A, D, \delta)$, and c_δ is continuous with respect to this topology.

To prove the theorem, we need some preliminary lemmas.

Lemma 4.10. *Let A be a separable C^*-algebra, δ a coaction of a countable discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Let $\mathcal{H}(A, D, \delta)$ be the groupoid of Proposition 4.7. If $[n, \phi], [m, \phi] \in \mathcal{H}(A, D, \delta)$ and $[n, \phi] = [m, \phi]$, then there is an open $U \subseteq \widehat{D}$ with $\phi \in U \subseteq \mathrm{supp}^\circ(m^*m) \cap \mathrm{supp}^\circ(n^*n)$ such that $[n, \psi] = [m, \psi]$ for all $\psi \in U$.*

Proof. Since $[n,\phi]=[m,\phi]$, there is an open U' with $\phi\in U'\subseteq \operatorname{supp}^\circ(n^*n)\cap\operatorname{supp}^\circ(m^*m)$ such that $\alpha_n|_{U'}=\alpha_m|_{U'}$. By scaling m and n by appropriate elements of D, we may assume that $\psi(m^*m)=\psi(n^*n)=1$ for all $\psi\in U'$. Since $[n,\phi]=[m,\phi]$ we have $n^*m\in A^\delta$, so (R1), (R2) and (R3) are satisfied for (n,ψ) and (m,ψ) for each $\psi\in U'$.

Since $U^\phi_{n^*m}=\pi_\phi(dn^*md)\in\mathcal{U}_0(D'_{A^\delta}/J_\phi)$, there is a path of unitaries in D'_{A^δ}/J_ϕ from $\pi_\phi(dn^*md)$ to $1_{D'_{A^\delta}/J_\phi}$. Choose $0=t(0)<t(1)<\cdots<t(k)=1$ such that $\|w_{t(j)}-w_{t(j+1)}\|<\frac{1}{4}$ for all j. Let $a_0=dn^*md$ and $a_k=d$, and choose $a_j\in\pi_\phi^{-1}(w_{t(j)})$, $0<j<k$. Since $\psi\mapsto\|\pi_\psi(a)\|$ is upper semicontinuous, there is an open $U\ni\phi$ such that

$$\|\pi_\psi(a_ja_j^*-a_k)\|<\frac{1}{4},\quad \|\pi_\psi(a_j^*a_j-a_k)\|<\frac{1}{4},\quad \|\pi_\psi(a_j^*)\|<2,\quad\text{and}\quad \|\pi_\psi(a_j-a_{j+1})\|<\frac{1}{4}$$

for all $\psi\in U$ and $j<k$. We claim that $(n,\psi)\sim(m,\psi)$ for $\psi\in U$.

Fix $\psi\in U$. We have already seen that (R1)–(R3) are satisfied for (n,ψ) and (m,ψ). The first two properties of the a_j ensure that $\pi_\psi(a_ja_j^*)$, $\pi_\psi(a_ja_j^*)$, and $\pi_\psi(a_j)$ are invertible, and that $\|\pi_\psi(a_ja_j^*)^{-1}\|<4/3$. So $\|\pi_\psi(a_j-a_{j+1})\|<3/8<\|\pi_\psi(a_j)^{-1}\|^{-1}$. So [42, Proposition 2.1.11] shows that each $\pi_\psi(a_j)\sim_h\pi_\psi(a_{j+1})$ in $\big(D'_{A^\delta}/J_\psi\big)^{-1}$. Hence $U^\psi_{n^*m}\sim_h\pi_\psi(a_k)=1_{D'_{A^\delta}/J_\psi}$ in $\big(D'_{A^\delta}/J_\psi\big)^{-1}$, and then [42, Proposition 2.1.8] gives $U^\psi_{n^*m}\sim_h 1_{D'_{A^\delta}/J_\psi}$ in the unitary group of D'_{A^δ}/J_ψ. So (R4) is satisfied, and hence $(n,\psi)\sim(m,\psi)$. □

Lemma 4.11. *Let A be a separable C^*-algebra, δ a coaction of a countable discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Suppose that $n,m\in N(D)$ satisfy $\|n-m\|<\|n\|/5$. Then $\alpha_n(\phi)=\alpha_m(\phi)$ for all $\phi\in\widehat{D}$ such that $\phi(n^*n)>\|n\|^2/2$.*

Proof. Since $\alpha_n=\alpha_{n/\|n\|}$ and $\alpha_m=\alpha_{m/\|n\|}$, it suffices to prove the result for $\|n\|=1$. So, by assumption, $\|n-m\|<1/5$ giving $\|m\|\le 6/5$. For any $d\in D$ with $\|d\|\le1$, we have

$$\|n^*dn-m^*dm\|\le\|n^*dn-n^*dm\|+\|n^*dm-m^*dm\|<\tfrac{1}{5}+\tfrac{6}{25}<\tfrac{1}{2}. \tag{4.5}$$

Fix $\phi\in\widehat{D}$ with $\phi(n^*n)>\frac{1}{2}$, and $d\in D$ with $0\le d\le1$ and $\phi(d)=1$. Then $\alpha_n(\phi)(ndn^*)=\phi(d)\alpha_n(\phi)(nn^*)=\phi(n^*n)>\frac{1}{2}$, so (4.5) gives $\alpha_n(\phi)(mdm^*)>\alpha_n(\phi)(ndn^*)-\frac{1}{2}>0$. Hence $0<\alpha_n(\phi)(mdm^*)=\alpha_m^{-1}(\alpha_n(\phi))(d)\alpha_n(\phi)(mm^*)$, giving $\alpha_m^{-1}(\alpha_n(\phi))(d)\ne0$. So if $0\le d\le 1$ and $\phi(d)=1$, then $\alpha_m^{-1}(\alpha_n(\phi))(d)>0$. So Urysohn's lemma forces $\alpha_m^{-1}(\alpha_n(\phi))=\phi$. □

Proof of Theorem 4.9. To see that the $Z(n,X)$ are a basis for a topology, suppose that $[n,\phi]\in Z(n,X)\cap Z(n',Y)$. Then $\phi\in X\cap Y$, and $[n,\phi]=[n',\phi]$. By Lemma 4.10, there is an open W with $\phi\in W\subseteq X\cap Y$ such that $[n,\psi]=[n',\psi]$ for all $\psi\in W$. Hence $Z(n,W)\subseteq Z(n,X)\cap Z(n',Y)$.

To see that this topology is second countable, we first claim that each $N_g(D)$ has a countable dense subset. For this, fix a dense sequence a_i in A_g. For $i,j\in\mathbb{N}$ fix $n_{i,j}\in B(a_i,1/j)\cap N_g(D)$ if this set is nonempty, and otherwise let $n_{i,j}=0\in N_g(D)$. Fix $n\in N_g(D)$. Choose i_k with $\|a_{i_k}-n\|<\frac{1}{3k}$. Then $n_{i_k,2k}\in B(a_{i_k},1/2k)$, forcing $\|n_{i_k,2k}-n\|<\frac{1}{k}$. So $n\in\overline{\{n_{i,j}:i,j\in\mathbb{N}\}}$, proving the claim. Now since Γ is countable, $N_\star(D)$ has a countable dense sequence $(n_i)_{i=1}^\infty$. Choose a countable basis $\{U_j\}$ for $\widehat{D}$. We claim that for $n\in N_\star(D)$, an open subset $X\subseteq\operatorname{supp}^\circ(n^*n)$, and $\phi\in X$, there are $i,j\in\mathbb{N}$ such that $[n,\phi]\in Z(n_i,U_j)\subseteq Z(n,X)$. Since $[n,\phi]=[nd,\phi]\in Z(nd,X\cap\operatorname{supp}^\circ(d))\subseteq Z(n,X)$ for $d\in D$ with $0\le d\le1$ and $\phi(d)=1$, we may assume that $\phi(n^*n)=\|n\|^2$. Choose j so that $\phi\in U_j\subseteq X$ and $\psi(n^*n)>\|n\|^2/2$ for all $\psi\in U_j$. Fix a subsequence $(n_{i_k})_{k=1}^\infty$ of $(n_i)_{i=1}^\infty$ converging to n with each $n^*n_{i_k}\in A^\delta$ and $\|n_{i_k}-n\|<\|n\|/5$. So, Lemma 4.11

gives $\alpha_n|_{U_j} = \alpha_{n_{i_k}}|_{U_j}$. Since $\sup_{\psi\in U_j} \|\pi_\psi(d_\psi n^* n_{i_k} d_\psi) - \pi_\psi(n^*n)\| \to 0$ where $d_\psi \in D$ with $\|d\| = 1$, $\mathrm{supp}^\circ(d_\psi) \subseteq U_j$, and $\psi(d_\psi) = 1$, we have $\sup_{\psi\in U_j} \|U^\psi_{n^*n_{i_k}} - U^\psi_{n^*n}\| \to 0$. Since $U^\psi_{n^*n} = 1_{D'_{A^\delta}/J_\psi}$ and since unitaries in $B(1_{D'_{A^\delta}/J_\psi}; 2)$ are homotopic to $1_{D'_{A^\delta}/J_\psi}$, for large k, we have $U^\psi_{n^*n_{i_k}} \sim_h 1_{D'_{A^\delta}/J_\psi}$ for $\psi \in U_j$. Thus, $[n, \phi] \in Z(n_{i_k}, U_j) \subseteq Z(n, X)$. So the $Z(n_i, U_j)$ form a countable basis for the topology.

To see that $\mathcal{H}(A, D, \delta)$ is locally Hausdorff and étale, fix a basic open set $Z(n, X)$. The source map $[n, \psi] \mapsto \psi$ is a homeomorphism $h : Z(n, X) \to X$: it is bijective by definition of $\sim$, continuous as $h^{-1}(Y) = Z(n, Y)$ for $Y \subseteq X$, and open because each open subset of $Z(n, X)$ is a union of sets of the form $Z(n, Y)$ with $Y \subseteq X$ open, and each $h(Z(n, Y)) = Y$ is open. Similarly, the range map is a homeomorphism $Z(n, X) \to \alpha_n(X)$. Thus, since $\widehat{D}$ is Hausdorff, $\mathcal{H}(A, D, \delta)$ is locally Hausdorff and étale.

The map I is a homeomorphism because $I(Z(n, X)) = Z(n^*, \alpha_n(X))$ and α_n is a homeomorphism on $\mathrm{supp}^\circ(n^*n)$. To see that M is continuous, suppose that $[n_i, \phi_i] \to [n, \phi]$, that $[m_i, \psi_i] \to [m, \psi]$, and that each $\phi_i = \alpha_{m_i}(\psi_i)$. Then the preceding paragraph gives $s([n, \phi]) = r([m, \psi])$, and then $M([n, \phi], [m, \psi]) = [nm, \psi]$. Fix an open V with $\psi \in V \subseteq \mathrm{supp}^\circ(m^*m) \cap \alpha_m^{-1}(\mathrm{supp}^\circ(n^*n))$. Then $Z(m, V) \ni [m, \psi]$ and $Z(n, \alpha_m(V)) \ni [n, \phi]$ are open, giving $[n_i, \phi_i] \in Z(n, \alpha_m(V))$ and $[m_i, \psi_i] \in Z(m, V)$ for large i. Since $Z(m, V)$ and $Z(n, \alpha_m(V))$ are bisections, for large i we have $[n_i, \phi_i] = [n, \phi_i]$ and $[m_i, \psi_i] = [m, \psi_i]$, so $M([n_i, \phi_i], [m_i, \psi_i]) = [nm, \psi_i]$. In particular, $\psi_i \to \psi$, and as s is a homeomorphism on $Z(mn, V)$, we obtain $[nm, \psi_i] \to [nm, \psi]$. So M is continuous.

For local compactness, fix $\gamma \in \mathcal{H}(A, D, \delta)$ and an open $W \ni \gamma$. Choose $n \in N_\star(D)$ and X open with $\gamma \in Z(n, X)$. Since $\widehat{D}$ is locally compact, there is a compact neighbourhood K of $s(\gamma)$ in X. Now $\{[n, \phi] : \phi \in K\}$ is the inverse image of K under the homeomorphism $s|_{Z(n,X)}$, and hence a compact neighbourhood of γ. Finally, c_δ is continuous because it is constant on basic open sets. □

Our results so far do not require that $\overline{\mathrm{span}} N_\star(D)$ is all of A; but out next lemma indicates that that is the situation of greatest interest.

Lemma 4.12. *Let δ be a coaction of a discrete group Γ on a separable C^*-algebra A, and D a weakly Cartan subalgebra of A^δ. Let $A_N = \overline{\mathrm{span}} N_\star(D) \subseteq A$. Then A_N is a C^*-algebra, $\delta_N := \delta|_{A_N}$ is a coaction, D is weakly Cartan in $A_N^{\delta_N}$, and $\mathcal{H}(A, D, \delta) \cong \mathcal{H}(A_N, D, \delta_N)$.*

Proof. As $N_\star(D)$ is closed under multiplication and adjoints, A_N is a C^*-algebra. Since $D \subseteq N_e(D)$, we have $D \subseteq A_N$, and D clearly contains an approximate unit for $A_N^{\delta_N}$. Since $A_N = \overline{\mathrm{span}} \bigcup_g A_g$, the restriction of $\delta_N := \delta|_{A_N}$ takes values in $A_N \otimes C^*_r(G)$. It is nondegenerate because $D \subseteq A_N^{\delta_N}$ contains an approximate unit.

Put $B := A_N$ and $\epsilon := \delta_N$. We have $D'_{B^\epsilon} \subseteq D'_{A^\delta}$ because $B^\epsilon = B \cap A^\delta \subseteq A^\delta$. Fix $\phi \in \widehat{D}$. Condition (4) of Definition 2.1 shows that there exists $d \in D$ such that $\pi_\phi(d) = 1_{D'_{A^\delta}/J_\phi}$. So $\pi_\phi(d') = \phi(d')1_{D'_{A^\delta}/J_\phi}$ for all $d' \in D$. Let $K_\phi := J^N_\phi$ be the ideal of D'_{B^ϵ} generated by $\ker \phi$, and let $\pi^N_\phi : D'_{B^\epsilon} \to D'_{B^\epsilon}/K_\phi$ be the quotient map. Fix $a \in D'_{B^\epsilon}$. Since $\pi_\phi(d) = 1_{D'_{A^\delta}/J_\phi}$, we have $a - da \in J_\phi$. Since $a - da \in D'_{B^\epsilon}$, it follows that $a - da \in K_\phi$, so $\pi^N_\phi(d)\pi^N_\phi(a) = \pi^N_\phi(a)$. Thus $\pi^N_\phi(d)$ is an identity for D'_{B^ϵ}/K_ϕ. Hence D is a weakly Cartan subalgebra of B^ϵ. The final statement is trivial: the isomorphism is $[n, \phi] \mapsto [n, \phi]$. □

5. The interior of the isotropy in an étale groupoid

This section contains some technical results that we will need in order to prove our reconstruction results in the next section. Our proof of the first, Lemma 5.1, is largely due to Becky Armstrong; the key elements, in the more general situation of twisted groupoid C^*-algebras, will appear in her PhD thesis.

Lemma 5.1 (Armstrong)**.** *Let G be a locally compact Hausdorff étale groupoid such that $\mathrm{Iso}(G)^\circ$ is abelian. Then $C^*(\mathrm{Iso}(G)^\circ) = C^*_r(\mathrm{Iso}(G)^\circ)$. The inclusion $C_0(G^{(0)}) \hookrightarrow C^*(\mathrm{Iso}(G)^\circ)$ makes $C^*(\mathrm{Iso}(G)^\circ)$ into a $C_0(G^{(0)})$-algebra. The fibre homomorphisms $\pi_u : C^*(\mathrm{Iso}(G)^\circ) \to C^*(\mathrm{Iso}(G)^\circ)_u$ have the property that $u \mapsto \|\pi_u(a)\|$ is continuous for $a \in C^*(\mathrm{Iso}(G)^\circ)$. For each $u \in G^{(0)}$ there is an isomorphism $C^*(\mathrm{Iso}(G)^\circ))_u \cong C^*(\mathrm{Iso}(G)^\circ_u)$ that takes $\pi_u(d)$ to $d(u)1_{C^*(\mathrm{Iso}(G)^\circ_u)}$ for $d \in C_0(G^{(0)})$.*

Proof. Theorem 3.5 of [40] shows that $C^*(\mathrm{Iso}(G)^\circ) = C^*_r(\mathrm{Iso}(G)^\circ)$. For $f \in C_c(\mathrm{Iso}(G)^\circ)$, $d \in C_0(G^{(0)})$ and $\gamma \in \mathrm{Iso}(G)^\circ$, we have that $(df)(\gamma) = d(r(\gamma))f(\gamma) = f(\gamma)d(s(\gamma)) = (fd)(\gamma)$. So $C_0(G^{(0)})$ is central in $C^*(\mathrm{Iso}(G)^\circ)$ by continuity, and any approximate unit for $C_0(G^{(0)})$ is an approximate unit for $C^*(\mathrm{Iso}(G)^\circ)$. Thus $C^*(\mathrm{Iso}(G)^\circ)$ is a $C_0(G^{(0)})$-algebra.

Fix $u \in G^{(0)}$. For each $\gamma \in \mathrm{Iso}(G)^\circ_u$, choose $a_\gamma \in C_c(\mathrm{Iso}(G)^\circ)$ supported on a bisection with $a_\gamma(\gamma) = 1$. Then $\gamma \mapsto \pi_u(a_\gamma)$ is a unitary representation of $\mathrm{Iso}(G)^\circ_u$ in $C^*(\mathrm{Iso}(G)^\circ)_u$, and thus determines a homomorphism $\tilde{\pi}_u : C^*(\mathrm{Iso}(G)^\circ_u) \to C^*(\mathrm{Iso}(G)^\circ)_u$.

The regular representation $\lambda_u : C^*_r(\mathrm{Iso}(G)^\circ) \to \mathcal{B}(\ell^2(\mathrm{Iso}(G)^\circ_u))$ satisfies $\lambda_u(f) = 0$ for $f \in C_0(G^{(0)} \setminus \{u\})$, and hence descends to a representation

$$\tilde{\lambda}_u : C^*(\mathrm{Iso}(G)^\circ)_u \to \mathcal{B}(\ell^2(\mathrm{Iso}(G)^\circ_u)).$$

The representation $\tilde{\lambda}_u \circ \tilde{\pi}_u$ is precisely the regular representation $C^*(\mathrm{Iso}(G)^\circ_u)$ and hence faithful since $\mathrm{Iso}(G)^\circ_u$ is abelian. Identifying $C^*_r(\mathrm{Iso}(G)^\circ_u)$ with $C^*(\mathrm{Iso}(G)^\circ_u)$,

$$\tilde{\pi}_u \circ \tilde{\lambda}_u\big(\pi_u(a_\gamma)\big) = \pi_u(a_\gamma)$$

for $\gamma \in \mathrm{Iso}(G)^\circ_u$, and so $\tilde{\pi}_u \circ \tilde{\lambda}_u = \mathrm{id}_{C^*(\mathrm{Iso}(G)^\circ)_u}$. Since $\tilde{\lambda}_u(\pi_u(d)) = d(u)1_{C^*(\mathrm{Iso}(G)^\circ_u)}$, this $\tilde{\lambda}_u : C^*(\mathrm{Iso}(G)^\circ)_u \to C^*(\mathrm{Iso}(G)^\circ_u)$ is the desired isomorphism.

Fix $a \in C_c(\mathrm{Iso}(G)^\circ)$ supported on a bisection U. Write $\sigma : s(U) \to U$ for the inverse of the source map. Then $\|\pi_u(a)\| = |a(\sigma(u))|$, so $u \mapsto \|\pi_u(a)\|$ is continuous. An $\frac{\varepsilon}{3}$-argument now shows that $u \mapsto \|\pi_u(a)\|$ is continuous for all $a \in C^*(\mathrm{Iso}(G)^\circ)$. □

For our next lemma, recall from Section 1 that there is a norm-decreasing injection $a \mapsto f_a$ from $C^*_r(G)$ to $C_0(G)$ given by $f_a(\gamma) = \big(\lambda_{s(\gamma)}(a)e_{s(\gamma)} \mid e_\gamma\big)$. We show that $\{a \in C^*_r(G) : f_a \text{ is supported on } \mathrm{Iso}(G)^\circ\}$ is a $C_0(G^{(0)})$-algebra.

Lemma 5.2. *Let G be a locally compact Hausdorff étale groupoid such that $\mathrm{Iso}(G)^\circ$ is abelian. Let $D := C_0(G^{(0)}) \subseteq C^*_r(G)$. Let $A := \{a \in C^*_r(G) : \mathrm{supp}^\circ(f_a) \subseteq \mathrm{Iso}(G)^\circ\}$. Then A is a $C_0(G^{(0)})$-algebra with respect to $D \hookrightarrow A$. For each $u \in G^{(0)}$, let $J_u := \overline{\{f \in D : f(u) = 0\}A}$, the ideal of A generated by $\{f \in D : f(u) = 0\}$, and let $A_u := A/J_u$. Then there is an isomorphism $\tilde{\Phi}_u : A_u \to C^*(\mathrm{Iso}(G)^\circ_u)$ such that*

$$\tilde{\Phi}_u(f) = \sum_{\gamma \in \mathrm{Iso}(G)^\circ_u} f(\gamma)\lambda_\gamma \quad \textit{for all } f \in C_c(G) \cap A.$$

Proof. For $b \in C_r^*(G)$ and $g \in D$, we have $f_{gb}(\gamma) = g(r(\gamma))f_b(\gamma)$ and $f_{bg}(\gamma) = f_b(\gamma)g(s(\gamma))$, and so $D \hookrightarrow A$ is a central inclusion. Since D contains an approximate unit for $C_r^*(G)$, we deduce that A is a $C_0(G^{(0)})$-algebra. So we fix $u \in G^{(0)}$ and show that there is an isomorphism $\tilde{\Phi}_u : A_u \to C^*(\mathrm{Iso}(G)^\circ_u)$.

Let $P \in \mathcal{B}(\ell^2(G_u))$ be the orthogonal projection onto $\ell^2(\mathrm{Iso}(G)^\circ_u)$. Define $\Phi_u : C_r^*(G) \to \mathcal{B}(\ell^2(\mathrm{Iso}(G)^\circ_u))$ by $\Phi_u(a) := P\lambda_u(a)P$. Then for $f \in C_c(G)$ we have

$$\Phi_u(f) = \textstyle\sum_{\gamma\in \mathrm{Iso}(G)^\circ_u} f(\gamma)\lambda_\gamma \in C^*(\mathrm{Iso}(G)^\circ_u),$$

and hence $\Phi_u(C_r^*(G)) \subseteq C^*(\mathrm{Iso}(G)^\circ_u)$ by continuity.

If $a \in \overline{\{f \in D : f(u) = 0\}A}$, then $\Phi_u(a) = 0$. Hence Φ_u induces a map $\tilde{\Phi}_u : A_u \to C^*(\mathrm{Iso}(G)^\circ_u)$. Proposition 4.2 of [38] shows that $f_{ab}(\gamma) = \sum_{\alpha\beta=\gamma} f_a(\alpha)f_b(\beta) = (f_a * f_b)(\gamma)$ and $f_{a^*}(\gamma) = \overline{f_a(\gamma^{-1})} = f_a^*(\gamma)$, so $\tilde{\Phi}_u$ is a C^*-homomorphism. This $\tilde{\Phi}_u$ is surjective because its image contains the canonical generators of $C^*(\mathrm{Iso}(G)^\circ_u)$.

To see that $\tilde{\Phi}_u$ is injective, we first claim that $J_u = \{a \in A : f_a|_{\mathrm{Iso}(G)^\circ_u} = 0\}$. For $\subseteq$, observe that $\{a \in A : f_a|_{\mathrm{Iso}(G)^\circ_u} = 0\}$ is an ideal containing $\{f \in D : f(u) = 0\}$. For the reverse containment, suppose that $f_a|_{\mathrm{Iso}(G)^\circ_u} = 0$ and $\|a\| \le 1$. Since $f_a|_{\mathrm{Iso}(G)^\circ_u} = 0$, we have $\|\pi_u(a)\| = 0$. The map $G^{(0)} \ni v \mapsto \|\pi_v(a)\|$ is upper semicontinuous by [47, Proposition C.10(a)]. So for $n \in \mathbb{N}$ the set $U_n := \{v \in G^{(0)} : \|\pi_v(a)\| < 1/n\}$ is an open neighbourhood of u, and $G^{(0)} \setminus U_n$ is compact. So there exists $g_n \in C_0(G^{(0)})_+$ with $g_n \le 1$, $g_n(u) = 0$, and $g_n \equiv 1$ on $G^{(0)} \setminus U_n$. Now $\|a - g_n a\| < \frac{1}{n}$. Since each $g_n a \in J_u$, we deduce that $a = \lim_n g_n a \in J_u$. This proves the claim. Now suppose $b \in A \setminus J_u$. Then $f_b|_{\mathrm{Iso}(G)^\circ_u} \ne 0$, so $f_{b^*b}(u) \ne 0$. Hence $(\Phi_u(b^*b)\delta_u \mid \delta_u) = f_{b^*b}(u) \ne 0$, forcing $\tilde{\Phi}_u(b^*b) \ne 0$. Thus $\tilde{\Phi}_u(b) \ne 0$ because $\tilde{\Phi}_u$ is a homomorphism. Hence $\tilde{\Phi}_u$ is injective. □

For the next result, recall from [35, Proposition 1.9] that if G is an étale groupoid then the canonical inclusion $C_c(\mathrm{Iso}(G)^\circ) \hookrightarrow C_c(G)$ extends to an injective homomorphism $\iota : C_r^*(\mathrm{Iso}(G)^\circ) \hookrightarrow C_r^*(G)$.

Corollary 5.3. *Let G be a locally compact Hausdorff étale groupoid such that $\mathrm{Iso}(G)^\circ$ is abelian. Let $A := \{a \in C_r^*(G) : \mathrm{supp}^\circ(f_a) \subseteq \mathrm{Iso}(G)^\circ\}$. Then $A = C_0(G^{(0)})'_{C_r^*(G)} = \iota(C^*(\mathrm{Iso}(G)^\circ))$.*

Proof. Fix $a \in A$. For $d \in C_0(G^{(0)})$, we have $f_d = d$ and [38, Proposition 4.2] gives $f_{ad}(\gamma) = (f_a * f_d)(\gamma) = f_a(\gamma)d(s(\gamma))$ and similarly $f_{da}(\gamma) = d(r(\gamma))f_a(\gamma)$. Since $a \in A$, we have $f_a(\gamma) \ne 0$ only if $r(\gamma) = s(\gamma)$, so $f_{ad} = f_{da}$. Since $b \mapsto f_b$ is injective, it follows that $A \subseteq C_0(G^{(0)})'_{C_r^*(G)}$. Now fix $a \notin A$. Since f_a is continuous, there is then $\gamma \notin \mathrm{Iso}(G)$ such that $f_a(\gamma) \ne 0$. Fix $d \in C_0(G^{(0)})$ with $d(r(\gamma')) = 1$ and $d(s(\gamma')) = 0$. Then $f_{da}(\gamma) = d(r(\gamma))f_a(\gamma) = f_a(\gamma) \ne 0$, and $f_{ad}(\gamma) = f_a(\gamma)d(s(\gamma)) = 0$. So $da \ne ad$. Hence $C_0(G^{(0)})'_{C_r^*(G)} \subseteq A$, and thus $C_0(G^{(0)})'_{C_r^*(G)} = A$.

If $f \in C_c(\mathrm{Iso}(G)^\circ)$ and $d \in C_0(G^{(0)})$, then $(df)(\gamma) = d(r(\gamma))f(\gamma)$ and $(fd)(\gamma) = f(\gamma)d(s(\gamma))$ for every $\gamma \in G$. Since $\mathrm{supp}^\circ(f) \subseteq \mathrm{Iso}(G)^\circ$, we obtain $df = fd$. Thus $\iota(C^*(\mathrm{Iso}(G)^\circ)) \subseteq A$.

Both A and $\iota(C_r^*(\mathrm{Iso}(G)^\circ))$ are $C_0(G^{(0)})$-algebras with respect to the inclusion of $D = C_0(G^{(0)})$ in both. Write J_u for the ideal of A generated by $C_0(G^{(0)} \setminus \{u\}) \subseteq D$ and K_u for the ideal of $\iota(C^*(\mathrm{Iso}(G)^\circ))$ generated by $C_0(G^{(0)} \setminus \{u\}) \subseteq D$, so $A_u = A/J_u$ and $\iota(C^*(\mathrm{Iso}(G)^\circ))_u = \iota(C^*(\mathrm{Iso}(G)^\circ))/K_u$.

Lemma 5.2 gives isomorphisms $\tilde{\Phi}_u^{-1} : \iota(C_r^*(\mathrm{Iso}(G)^\circ))_u \to A_u$ such that $\tilde{\Phi}_u^{-1}(\iota(f)+K_u) = f+K_u$ for $f \in C_c(\mathrm{Iso}(G)^\circ)$. Thus $\{m_f : u \mapsto f+J_u \mid f \in C_c(\mathrm{Iso}(G)^\circ)\}$ is a fibrewise dense vector space of continuous sections of $\mathcal{A} := \bigsqcup_u A_u$ that are the images of a fibrewise dense vector space of continuous sections k_f of $\mathcal{I} := \bigsqcup_u \iota(C_r^*(\mathrm{Iso}(G)^\circ))_u$. So [21, Proposition 1.6] gives $\mathcal{A} \cong \mathcal{I}$ as bundles, and hence there is an isomorphism $\Gamma_0(\mathcal{A}) \cong \Gamma_0(\mathcal{I})$ carrying k_f to m_f. Since $f \mapsto k_f$ is an isomorphism $C_r^*(\mathrm{Iso}(G)^\circ) \to \Gamma_0(\mathcal{I})$ and $f \mapsto m_f$ is an isomorphism $A \to \Gamma_0(\mathcal{A})$, we obtain an isomorphism $\iota(C_r^*(\mathrm{Iso}(G)^\circ)) \cong A$ extending $\mathrm{id}_{C_c(\mathrm{Iso}(G)^\circ)}$. So $A = \overline{C_c(\mathrm{Iso}(G)^\circ)} = \iota(C_r^*(\mathrm{Iso}(G)^\circ))$. □

We take the opportunity to resolve a loose end from [8].

Corollary 5.4 (cf. [8, Theorem 4.3])**.** *Let G be a locally compact Hausdorff étale groupoid such that $\mathrm{Iso}(G)^\circ$ is abelian. Then $\iota(C^*(\mathrm{Iso}(G)^\circ)) \subseteq C_r^*(G)$ is maximal abelian.*

Proof. Lemma 4.4 of [8] says that $\iota(C^*(\mathrm{Iso}(G)^\circ)) \subseteq C_r^*(G)$ is maximal abelian if $\{a \in C_r^*(G) : \mathrm{supp}^\circ(f_a) \subseteq \mathrm{Iso}(G)^\circ\} \subseteq \iota(C^*(\mathrm{Iso}(G)^\circ))$, which follows from Corollary 5.3. □

6. Reconstruction of groupoids

Let Γ be a discrete group. If G is an étale groupoid, and $c : G \to \Gamma$ is a continuous cocycle, we call (Γ, c) a Γ-graded groupoid. To state our main theorem, we first show that c induces a coaction on $C_r^*(G)$. For $g \in \Gamma$, we write $\lambda_g \in \mathcal{B}(\ell^2(\Gamma))$ for the image of g in the left regular representation of Γ.

Lemma 6.1. *Let G be a locally compact Hausdorff étale groupoid. Suppose that $c : G \to \Gamma$ is a continuous cocycle. Then there is a coaction $\delta_c : C_r^*(G) \to C_r^*(G) \otimes C_r^*(\Gamma)$ such that $\delta_c(f) = f \otimes \lambda_g$ whenever $g \in \Gamma$ and $f \in C_c(G)$ satisfy $\mathrm{supp}(f) \subseteq c^{-1}(g)$.*

Proof. Let $\mathcal{H}$ be the Hilbert $C^*(G^{(0)})$-module completion of $C_c(G)$ under $\langle f, g\rangle_{C_0(G^{(0)})} = (f^*g)|_{G^{(0)}}$. For $g \in \Gamma$, write $C_c(G)_g := C_c(c^{-1}(g)) \subseteq C_c(G)$, and let $\mathcal{H}_g = \overline{C_c(G)_g} \subseteq \mathcal{H}$. The $\mathcal{H}_g$ are mutually orthogonal because $G^{(0)} \subseteq c^{-1}(e)$, so a calculation using inner product shows that there are isometries $V_h : \mathcal{H} \to \mathcal{H} \otimes \ell^2(\Gamma)$ such that $V_h(\xi) = \xi \otimes e_{gh}$ for $\xi \in \mathcal{H}_g$. As in [24, Appendix A] there is a faithful representation $\pi : C_r^*(G) \to \mathcal{L}(\mathcal{H})$ extending left multiplication. So $\bigoplus_h \mathrm{Ad}\, V_h \circ \pi : C_r^*(G) \to \mathcal{L}(\mathcal{H} \otimes \ell^2(\Gamma))$ is faithful.

A routine calculation shows that for $f \in C_c(G)_g$ and $\xi \in C_c(G)_k$ and $h, l \in \Gamma$, we have $\bigoplus_h \big(\mathrm{Ad}\, V_h \circ \pi\big)(f)(\xi \otimes e_l) = \delta_{kh,l}(\pi(f) \otimes \lambda_g)(\xi \otimes e_l)$. So $\delta_c := (\pi^{-1} \otimes \mathrm{id}) \circ (\bigoplus_h \mathrm{Ad}\, V_h \circ \pi)$ satisfies $\delta_c(f) = f \otimes \lambda_g$. It is routine to check that this is a coaction. □

Theorem 6.2. *Fix a discrete group Γ and Γ-graded second-countable locally compact Hausdorff étale groupoids $(G_1, c_1), (G_2, c_2)$ with $\mathrm{Iso}(c_i^{-1}(e))^\circ$ torsion-free and abelian.*

1. *Suppose that $\kappa : G_2 \to G_1$ is an isomorphism satisfying $c_1 \circ \kappa = c_2$. Then there is an isomorphism $\phi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\phi(f) = f \circ \kappa$ for $f \in C_c(G_1)$. We have $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ and $\delta_{c_2} \circ \phi = (\phi \otimes \mathrm{id}) \circ \delta_{c_1}$.*
2. *Suppose that $\phi : C_r^*(G_1) \to C_r^*(G_2)$ is an isomorphism satisfying $\phi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ and $\delta_{c_2} \circ \phi = (\phi \otimes \mathrm{id}) \circ \delta_{c_1}$. Then there is an isomorphism $\kappa : G_2 \to G_1$ such that $\kappa|_{G_2^{(0)}}$ is the homeomorphism induced by $\phi|_{C_0(G_1^{(0)})}$ and $c_1 \circ \kappa = c_2$.*

The first step in proving Theorem 6.2 is showing that $C_r^*(c^{-1}(e)) \cong C_r^*(G)^{\delta_c} \subseteq C_r^*(G)$.

Lemma 6.3. *Let (G, c) be a Γ-graded locally compact Hausdorff étale groupoid. The canonical inclusion $\iota : C_c(c^{-1}(e)) \to C_c(G)$ extends to an isomorphism $C_r^*(c^{-1}(e)) \cong C_r^*(G)^{\delta_c}$.*

Proof. Proposition 1.9 of [35] shows that ι extends to an injective homomorphism from $C_r^*(c^{-1}(e))$ to $C_r^*(G)$. Clearly $\iota(C_r^*(c^{-1}(e))) \subseteq C_r^*(G)^{\delta_c}$. If $f \in C_c(G)_g$, then $\Phi^{\delta_c}(\iota(f)) = \delta_{e,g}(\iota(f))$, so $C_r^*(G) = \overline{\operatorname{span}} \bigcup_g C_c(G)_g$. It follows that $\iota(C_r^*(c^{-1}(e))) = \Phi^{\delta_c}(C_r^*(G)) = C_r^*(G)^{\delta_c}$ by linearity and continuity. □

We now show that graded groupoids determine triples as in Theorem 4.9.

Lemma 6.4. *Let Γ be a locally compact group, let G be a second-countable locally compact Hausdorff étale groupoid and let $c : G \to \Gamma$ be a continuous cocycle such that $\operatorname{Iso}(c^{-1}(e))^\circ$ is abelian. Then $C_0(G^{(0)})$ is a weakly Cartan subalgebra of the generalised fixed-point algebra $C_r^*(G)^{\delta_c}$.*

Proof. Since G is second-countable, $C_r^*(G)^{\delta_c}$ is separable. Clearly $C_0(G^{(0)})$ is abelian, contains an approximate unit for $C_r^*(G)^{\delta_c}$, and is contained in $C_r^*(G)^{\delta_c}$. Lemma 6.3 implies that $(C_r^*(G)^{\delta_c}, C_0(G^{(0)})) \cong \big(C_r^*(c^{-1}(e)), C_0(c^{-1}(e)^{(0)})\big)$, so Corollary 5.3 and Lemma 5.1 give Definition 2.1(3) and (4). □

To prove Theorem 6.2 we show that $\big(\mathcal{H}(C_r^*(G), C_0(G^{(0)}), \delta_c), c_{\delta_c}\big) \cong (G, c)$ (cf. [39, Proposition 4.13(ii)], [9, Proposition 4.8], [1, Corollary 3.11] and [10, Remark 3.2]). For $u \in G^{(0)}$, we write $\hat{u} : C_0(G^{(0)}) \to \mathbb{C}$ for evaluation at u.

Proposition 6.5. *Let Γ be a discrete group, and (G, c) a Γ-graded second-countable locally compact Hausdorff étale groupoid with $\operatorname{Iso}(c^{-1}(e))^\circ$ torsion-free and abelian. There is an isomorphism $\theta : (G, c) \to \mathcal{H}(C_r^*(G), C_0(G^{(0)}), \delta_c)$ such that for $\gamma \in G$ and $n \in C_c(G)_{c(\gamma)}$ supported on a bisection $U \subseteq c^{-1}(c(\gamma))$ with $n(\gamma) > 0$, we have $\theta(\gamma) = [n, \widehat{s(\gamma)}]$.*

To prove proposition 6.5, we need two lemmas. We implicitly identify $C_0(G^{(0)})\widehat{\ }$ with $G^{(0)}$ via $\hat{u} \mapsto u$. For the rest of this section, we identify $C_0(G^{(0)})'_{C_r^*(c^{-1}(e))}$ with $C_r^*(\operatorname{Iso}(G)^\circ)$ using Corollary 5.3 and Lemma 5.1. For $u \in G^{(0)}$, we identify $\big(C_0(G^{(0)})'_{C_r^*(c^{-1}(e))}\big)_u$ with $C_r^*(\operatorname{Iso}(c^{-1}(e))^\circ_u)$, and $\pi_u : C_0(G^{(0)})'_{C_r^*(c^{-1}(e))} \to C_0(G^{(0)})'_{C_r^*(c^{-1}(e))}/J_{\hat{u}}$ with the regular representation $\lambda_u : C_r^*(\operatorname{Iso}(c^{-1}(e))^\circ) \to \mathcal{B}(\ell^2(\operatorname{Iso}(c^{-1}(e))^\circ_u))$.

Lemma 6.6 (cf. [39, Proposition 4.8], [9, Lemma 4.10(ii)])**.** *Let G be a second-countable locally compact Hausdorff étale groupoid. Let $D := C_0(G^{(0)}) \subseteq C_r^*(G)$. If $n \in N(D)$ and f_n as in* (1.1) *satisfies $f_n(\gamma) \neq 0$, then $r(\gamma) = \alpha_n(s(\gamma))$.*

Proof. Suppose for contradiction that $r(\gamma) \neq \alpha_n(s(\gamma))$. Since $nn^*(r(\gamma)) \geq |f_n(\gamma)|^2 > 0$, there exist orthogonal $d, d' \in C_c(\operatorname{supp}^\circ(nn^*))$ such that $d(r(\gamma)) = 1 = d'(\alpha_n(s(\gamma)))$. So

$$\begin{aligned} 0 &= (dd'nn^*)(r(\gamma)) = (dn(d' \circ \alpha_n)n^*)(r(\gamma)) \\ &\geq d(r(\gamma))|n\sqrt{d' \circ \alpha_n}(\gamma)|^2 = d(r(\gamma))\,\big|f_n(\gamma)\sqrt{d'(\alpha_n(s(\gamma)))}\big|^2 = |f_n(\gamma)|^2 > 0. \end{aligned}$$ □

Lemma 6.7. *Let G be a second-countable locally compact Hausdorff étale groupoid. Let $D := C_0(G^{(0)}) \subseteq C^*(G)$. Suppose that $n \in C_c(G)$ is supported on a bisection. Then $n \in N(D)$, and $\alpha_n(s(\gamma)) = r(\gamma)$ for $\gamma \in \operatorname{supp}^\circ(n)$. If $c : G \to \Gamma$ is a grading, then $C_r^*(G) = \overline{\operatorname{span}} N_\star(D)$.*

Proof. That $n \in N(D)$ and $\alpha_n(s(\gamma)) = r(\gamma)$ for $\gamma \in \operatorname{supp}^\circ(n)$ follow from [39, Proposition 4.8]. Since $C_c(G) = \operatorname{span}\bigcup_{g\in\Gamma} C_c(G)_g$, the Stone–Weierstrass theorem gives

$$C_c(c^{-1}(\gamma)) = \operatorname{span}\{f \in C_c(c^{-1}(\gamma)) : \operatorname{supp}(f) \text{ is contained in a bisection}\},$$

and hence $C^*_r(G) = \overline{\operatorname{span}} N_\star(C_0(G^{(0)}))$. □

Proof of Proposition 6.5. Since c is continuous, Γ is discrete, and G is étale, there exists a bisection $U \subseteq c^{-1}(c(\gamma))$ containing Γ, and then an element $n \in C_c(U) \subseteq C_c(G)$ with $n(\gamma) = 1$. Fix $n, m \in C_c(G)$ supported on bisections contained in $c^{-1}(c(\gamma))$ with $n(\gamma) = m(\gamma) = 1$. Choose an open $\gamma \in U \subseteq \operatorname{supp}^\circ(m) \cap \operatorname{supp}^\circ(n)$. Lemma 6.7 shows that $\alpha_n = \alpha_m$ on $s(U)$. Let $u := s(\gamma)$. For $d \in C_0(s(U))$ with $d(u) = 1$, we have $\operatorname{supp}^\circ(nd), \operatorname{supp}^\circ(md) \subseteq U$, so $(dn^*md)|_{\operatorname{Iso}(c^{-1}(e))^\circ_u}$ is just the point-mass δ_u. Hence $U_{n^*m} = \pi_u(dn^*md) = 1_{C^*(\operatorname{Iso}(G)^\circ_u)}$. Since $\alpha_{nd} = \alpha_n = \alpha_m = \alpha_{md}$ on $\operatorname{supp}^\circ(d) \subseteq U$, we have $[n, u] = [m, u]$. So θ is well-defined.

To see that θ is injective, fix $\gamma \neq \eta \in G$. Choose n, m supported on open bisections containing γ and η respectively, so $\theta(\gamma) = [n, \widehat{s(\gamma)}]$ and $\theta(\eta) = [m, \widehat{s(\eta)}]$. If $s(\gamma) \neq s(\eta)$, then clearly $\theta(\gamma) \neq \theta(\eta)$, so suppose that $s(\gamma) = s(\eta) =: u$. If $r(\gamma) \neq r(\eta)$, then $\alpha_n(\hat{u}) = r(\gamma) \neq r(\eta) = \alpha_m(\hat{u})$ by Lemma 6.7, so again $\theta(\gamma) \neq \theta(\eta)$. So suppose that $r(\gamma) = r(\eta) =: v$. Fix $d \in C_0(G^{(0)})$ with $\hat{u}(d) = 1$, and put $w = dn^*md$. Then $w|_{\operatorname{Iso}(c^{-1}(e))^\circ_u} = \delta_{\gamma^{-1}\eta} \in C_c(\operatorname{Iso}(c^{-1}(e))^\circ_u)$. Hence $U_{n^*m} = \pi_u(w) = U_{\gamma^{-1}\eta} \in C^*(\operatorname{Iso}(c^{-1}(e))^\circ_u)$. Since $\operatorname{Iso}(c^{-1}(e))^\circ_u$ is a discrete torsion-free abelian group, [23, Theorem 8.57] gives $\operatorname{Iso}(c^{-1}(e))^\circ_u \cong \mathcal{U}(C^*(\operatorname{Iso}(c^{-1}(e))^\circ_u))/\mathcal{U}_0(C^*(\operatorname{Iso}(c^{-1}(e))^\circ_u))$ via the map $\gamma \mapsto U_\gamma \mathcal{U}_0(C^*(\operatorname{Iso}(c^{-1}(e))^\circ_u))$; so $u_{\gamma^{-1}\eta} \notin \mathcal{U}_0(C^*(\operatorname{Iso}(c^{-1}(e))^\circ_u))$, and $\theta(\gamma) \neq \theta(\eta)$.

To see that θ is surjective, fix $[n, \hat{u}] \in \mathcal{H}(C^*_r(G), C_0(G^{(0)}), \delta_c)$, and let $\hat{v} := \alpha_n(\hat{u})$. Regard n as an element of $C_0(G)$ using (1.1). Since $0 < n^*n(u) = nn^*(\alpha_n(u)) = nn^*(v) = \sum_{\gamma\in G^v} |n(\gamma)|^2$, we have $n(\gamma) \neq 0$ for some $\gamma \in G^v$. So Lemma 6.6 gives $\alpha_n(\widehat{s(\gamma)}) = \widehat{r(\gamma)} = \hat{v} = \alpha_n(\hat{u})$. Since α_n is bijective, $s(\gamma) = u$. Fix an open bisection $B \ni \gamma$ with f_n nonzero on B. Fix $m \in C_0(B)$ with m identically 1 on a neighbourhood of γ. Lemma 6.6 gives $\alpha_n = \alpha_m$ on $s(\operatorname{supp}^\circ(m))$, so Lemma 2.6 yields $\alpha_{n^*m} = \operatorname{id}$ on a neighbourhood of $\hat{u}$. Thus n^*m is supported on $\operatorname{Iso}(c^{-1}(e))^\circ$ by Lemma 6.6. By [23, Theorem 8.57], $\operatorname{Iso}(c^{-1}(e))^\circ_u \cong \mathcal{U}(C^*(\operatorname{Iso}(c^{-1}(e))^\circ_u))/\mathcal{U}_0(C^*(\operatorname{Iso}(c^{-1}(e))^\circ_u))$, so $U_{n^*m} \sim_h U_\eta$ for some $\eta \in \operatorname{Iso}(c^{-1}(e))^\circ_u$. Fix a bisection neighbourhood W of η^{-1} and $h \in C_c(W)$ with $h(\eta^{-1}) = 1$. Then $\pi_u(w_{n^*mh}) = \pi_u(w_{n^*m})\pi_u(h) = \pi_u(w_{n^*m})\delta_{\eta^{-1}} \sim_h 1_{C^*(\operatorname{Iso}(c^{-1}(e))^\circ_u)}$. Since $\alpha_h = \operatorname{id} = \alpha_{n^*m}$, we have $\alpha_n = \alpha_{mh}$ on a neighbourhood of u, so $[n, \hat{u}] = [mh, \hat{u}] = \theta(\gamma\eta^{-1})$.

To see that θ is open, recall that G is a normal space, so has a basis of open bisections U with closure contained in precompact open bisections V. For such U, V, fix $n \in C_c(V)$ with $n|_U = 1$. Then $\theta(\gamma) = [n, \widehat{s(\gamma)}]$ for every $\gamma \in U$, so $\theta(U) = Z(n, U)$ is open.

Finally, to see that θ is continuous, fix $n \in N(D)_g$ and an open set $U \subseteq \operatorname{supp}^\circ(n^*n)$. Fix $u \in U$. Since θ is surjective, $[n, u] = \theta(\gamma) = [f, u]$ for some $f \in C_c(G)$ supported on a bisection $B \subseteq c^{-1}(g)$ containing γ. So Lemma 4.10 yields an open neighbourhood V of u such that $[n, v] = [f, v]$ for all $v \in V$. Then $\gamma \in BV \subseteq \theta^{-1}(Z(n, U))$. □

Proof of Theorem 6.2. Statement (1) is clear. For (2) let $h : G_2^{(0)} \to G_1^{(0)}$ be induced by $\phi|_{C_0(G_1^{(0)})}$, so $f(h(x)) = \phi(f)(x)$ for $f \in C_0(G_1^{(0)})$. The formula $\phi^*([n, x]) := [\phi^{-1}(n), h(x)]$ defines a graded isomorphism $\phi^* : \mathcal{H}(C^*_r(G_2), C_0(G_2^{(0)}), \delta_{c_2}) \to \mathcal{H}(C^*_r(G_1), C_0(G_1^{(0)}), \delta_{c_1})$. Proposition 6.5 gives graded isomorphisms $\theta_i : G_i \to \mathcal{H}(C^*_r(G_i), C_0(G_i^{(0)}), \delta_{c_i})$. So $\kappa :=$

$\theta_1^{-1} \circ \phi^* \circ \theta_2 : G_2 \to G_1$ is a graded isomorphism. For $x \in G_2^{(0)}$ and $a \in C_0(G_2^{(0)})$ with $a(x) > 0$, Proposition 6.5 gives $\kappa(x) = \theta_1^{-1} \circ \phi^*([a, x]) = \theta_1^{-1}(a \circ h^{-1}, h(x)) = h(x)$. $\square$

7. Group actions

In this section we consider transformation groupoids of continuous actions of countable discrete groups on topological spaces. We characterise isomorphism of transformation groupoids in terms of continuous orbit equivalence of the actions. If in each group, the subgroup $\{\gamma : \text{there is an open } U \subseteq X \text{ that } \gamma \text{ fixes pointwise}\}$ is torsion-free and abelian, Theorem 6.2 yields a generalisation of [29, Theorem 1.2].

Fix a countable discrete group Γ acting on the right of a second-countable locally compact Hausdorff space X. We write $x\gamma$ for the action of γ on x. Define

$$X \rtimes \Gamma = X \times \Gamma$$

under the product topology, let $(X \rtimes \Gamma)^{(2)} := \{\big((x_1, \gamma_1), (x_2, \gamma_2)\big) : x_2 = x_1\gamma_1\}$, and define $(x_1, \gamma_1)(x_1\gamma_1, \gamma_2) = (x_1, \gamma_1\gamma_2)$, and $(x, \gamma)^{-1} = (x\gamma, \gamma^{-1})$. Then $X \rtimes \Gamma$ is a second-countable locally compact Hausdorff étale groupoid. Its unit space is $X \times \{e\}$, which we identify with X, so $r(x, \gamma) = x$ and $s(x, \gamma) = x\gamma$. We have $C_r^*(X \rtimes \Gamma) \cong C_0(X) \rtimes_r \Gamma$ via an isomorphism that carries $C_0((X \rtimes \Gamma)^{(0)})$ to $C_0(X) \subseteq C_0(X) \rtimes_r \Gamma$ (see [43, Example 3.2.8]).

For $x \in X$, we write

$$\mathrm{Stab}(x) := \{\gamma \in \Gamma : x\gamma = x\}$$

for the *stabiliser subgroup* of x in Γ; observe that then $(X \rtimes \Gamma)_x^x = \{x\} \times \mathrm{Stab}(x)$. We also consider the *essential stabiliser subgroup*

$$\mathrm{Stab}^{\mathrm{ess}}(x) := \{\gamma \in \Gamma : \gamma \in \mathrm{Stab}(y) \text{ for all } y \text{ in some neighbourhood } U \text{ of } x\}.$$

Observe that $\mathrm{Iso}(X \rtimes \Gamma)^\circ = \bigcup_{x \in X} \{x\} \times \mathrm{Stab}^{\mathrm{ess}}(x)$. We say (X, Γ) is *topologically free* if each $\mathrm{Stab}^{\mathrm{ess}}(x) = \{e\}$; a Baire-category argument shows that (X, Γ) is topologically free if and only if $\overline{\{x \in X : \mathrm{Stab}(x) = \{e\}\}} = X$.

Definition 7.1. Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be actions of countable discrete groups on locally compact Hausdorff spaces. A *continuous orbit equivalence* (h, π, η) from (X, Γ) to (Y, Λ) consists of a homeomorphism $h : X \to Y$ and continuous maps $\phi : X \times \Gamma \to \Lambda$ and $\eta : Y \times \Lambda \to \Gamma$ such that $h(x\gamma) = h(x)\phi(x, \gamma)$ for all x, γ and $h^{-1}(y\lambda) = h^{-1}(y)\eta(y, \lambda)$ for all y, λ. We call h the *underlying homeomorphism* of (h, ϕ, η).

For topologically free systems, the intertwining condition appearing in Definition 7.1 has some important consequences.

Lemma 7.2. *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be topologically free actions of countable discrete groups on locally compact Hausdorff spaces. Let (h, ϕ, η) be a continuous orbit equivalence from (X, Γ) to (Y, Λ). Fix $x \in X$. We have*

$$\phi(x, \gamma\gamma') = \phi(x, \gamma)\phi(x\gamma, \gamma') \quad \textit{for all } \gamma, \gamma' \in \Gamma,$$

and each $\theta_x := \phi(x, \cdot) : \Gamma \to \Lambda$ is a bijection that carries e_Γ to e_Λ and restricts to bijections $\mathrm{Stab}(x) \to \mathrm{Stab}(h(x))$ and $\mathrm{Stab}^{\mathrm{ess}}(x) \to \mathrm{Stab}^{\mathrm{ess}}(h(x))$.

Proof. The first part is proved in the same way as [29, Lemma 2.8], the only difference is that the actions considered in [29] are left actions and here we consider right actions.

For the second, by [29, Corollary 2.11], for all $x \in X$ with $\mathrm{Stab}(x)$ trivial, θ_x is a bijection. For arbitrary $x \in X$, take $x_n \to x$ such that $\mathrm{Stab}(x_n)$ is trivial. Since ϕ is continuous and Λ is a discrete group, $\theta_{x_n} = \theta_x$ for large n. So θ_x is a bijection. That $\theta_x(e_\Gamma) = e_\Lambda$ follows from the first statement. The intertwining condition in Definition 7.1 and that h is a homeomorphism gives $\theta_x(\mathrm{Stab}(x)) = \mathrm{Stab}(h(x))$; and $\theta_x(\mathrm{Stab}^{\mathrm{ess}}(x)) = \mathrm{Stab}^{\mathrm{ess}}(h(x))$ because both are trivial. □

Lemma 7.2 prompts the following definition.

Definition 7.3. Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright$ be actions of countable discrete groups on locally compact Hausdorff spaces. Consider a map $\phi : X \times \Gamma \to \Lambda$.

(1) We call ϕ a *cocycle* if $\phi(x, \gamma\gamma') = \phi(x, \gamma)\phi(x\gamma, \gamma')$ for all x, γ, γ'.
(2) Let $h : X \to Y$ be a homeomorphism. We say that (h, ϕ) *preserves stabilisers* if $\phi(x, \cdot)$ restricts to bijections $\mathrm{Stab}(x) \to \mathrm{Stab}(h(x))$, and that (h, ϕ) *preserves essential stabilisers* if $\phi(x, \cdot)$ restricts to bijections $\mathrm{Stab}^{\mathrm{ess}}(x) \to \mathrm{Stab}^{\mathrm{ess}}(h(x))$.

Proposition 7.4. *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be actions of countable discrete groups on locally compact Hausdorff spaces. Suppose that $h : X \to Y$ is a homeomorphism and $\phi : X \times \Gamma \to \Lambda$ is continuous. The following are equivalent:*

(1) *there is an isomorphism $\Theta : X \rtimes \Gamma \to Y \rtimes \Lambda$ such that $\Theta(x, e) = (h(x), e)$ and $\Theta(x, \gamma) = (h(x), \phi(x, \gamma))$ for all $x \in X$ and $\gamma \in \Gamma$;*
(2) *ϕ is a cocycle, (h, ϕ) preserves stabilisers, and there is a map $\eta : Y \times \Lambda \to \Gamma$ such that (h, ϕ, η) is a continuous orbit equivalence; and*
(3) *ϕ is a cocycle, (h, ϕ) preserves essential stabilisers, and there is a map $\eta : Y \times \Lambda \to \Gamma$ such that (h, ϕ, η) is a continuous orbit equivalence.*

Proof. (1) $\implies$ (2): Define $\eta : Y \times \Lambda \to \Gamma$ by $\Theta^{-1}(y, \lambda) = (h^{-1}(y), \eta(y, \lambda))$. Then (h, ϕ, η) is a continuous orbit equivalence and ϕ, η are cocycles. The pair (h, ϕ) preserves stabilisers because $\Theta\big(\{(x, \gamma) : \gamma \in \Gamma_x\}\big) = \{(h(x), \lambda) : \lambda \in \Lambda_{h(x)}\}$.

(2) $\implies$ (3): It suffices to show that $\phi(x, \gamma) \in \mathrm{Stab}^{\mathrm{ess}}(h(x))$ if and only if $\gamma \in \mathrm{Stab}^{\mathrm{ess}}(x)$. First suppose that $\gamma \in \mathrm{Stab}^{\mathrm{ess}}(x)$. Fix an open neighbourhood $U \ni x$ such that $x'\gamma = x'$ for all $x' \in U$. Since ϕ is continuous and Λ is discrete, we can assume that $\phi(x', \gamma) = \phi(x, \gamma) =: \lambda$ for all $x' \in U$. Then $U' := h(U)$ is an open neighbourhood of $h(x)$ such that for $u \in U'$,

$$y\lambda = h(h^{-1}(y))\lambda = h(h^{-1}(y))\phi(h^{-1}(y), \gamma) = h(h^{-1}(y)\gamma) = h(h^{-1}(y)) = y.$$

Hence $\phi(x, \gamma) \in \mathrm{Stab}^{\mathrm{ess}}(h(x))$.

Now suppose that $\phi(x, \gamma) \in \mathrm{Stab}^{\mathrm{ess}}(h(x))$. There is an open $U' \ni h(x)$ such that $\lambda := \phi(x, \gamma)$ satisfies $y\lambda = y$ for all $y \in U'$. By continuity of h and ϕ, there is an open $U \ni x$ such that $h(U) \subseteq U'$ and $\phi(x', \gamma) = \lambda$ for all $x' \in U$. So $x'\gamma = x'$ for all $x' \in U$, giving $\gamma \in \mathrm{Stab}^{\mathrm{ess}}(x)$.

(3) $\implies$ (1): Define $\Theta : X \rtimes \Gamma \to Y \rtimes \Lambda$ by $\Theta(x, \gamma) = (h(x), \phi(x, \gamma))$. Then Θ is open, continuous homomorphism. To see that Θ is injective, suppose that $\Theta(x_1, \gamma_1) = \Theta(x_2, \gamma_2)$. Then $x_1 = x_2$, $\gamma_1\gamma_2^{-1} \in \Gamma_{x_1}$, and $\Theta(x_1, \gamma_1\gamma_2^{-1}) \in (Y \rtimes \Lambda)^{(0)}$. Since $Y \rtimes \Lambda$ is étale, $(Y \rtimes \Lambda)^{(0)}$ is open in $Y \rtimes \Lambda$. Hence $(Y \rtimes \Lambda)^{(0)} \subseteq \mathrm{Iso}(Y \rtimes \Lambda)^\circ$. Since $\Theta^{-1}(\mathrm{Iso}(Y \rtimes \Lambda)) \subseteq \mathrm{Iso}(X \rtimes \Gamma)$ and Θ is continuous, $\Theta^{-1}(\mathrm{Iso}(Y \rtimes \Lambda)^\circ) \subseteq \mathrm{Iso}(X \rtimes \Gamma)^\circ$. Hence $(x_1, \gamma_1\gamma_2^{-1}) \in \mathrm{Iso}(X \rtimes \Gamma)^\circ$. Since $\phi(x_1, \cdot) : \mathrm{Stab}^{\mathrm{ess}}(x_1) \to \mathrm{Stab}^{\mathrm{ess}}(h(x_1))$ is bijective, $\gamma_1 = \gamma_2$, so $(x_1, \gamma_1) = (x_2, \gamma_2)$.

For surjectivity, fix $(y, \lambda) \in Y \rtimes \Lambda$. Then $y\phi(h^{-1}(y), \eta(y, \lambda)) = h(h^{-1}(y)\eta(y, \lambda)) = h(h^{-1}(y\lambda)) = y\lambda$, so $\lambda\phi(h^{-1}(y), \eta(y, \lambda))^{-1} \in \mathrm{Stab}(y)$. By continuity of ϕ and η there is an open neighbourhood $U \ni y$ such that $y'\phi(h^{-1}(y'), \eta(y', \lambda)) = h(h^{-1}(y')\eta(y', \lambda)) = y'\lambda$ for all $y' \in U$. For $y' \in U$ we have $\lambda'\phi(h^{-1}(y'), \eta(y', \lambda'))^{-1} \in \mathrm{Stab}(y')$. Hence

$\lambda\phi(h^{-1}(y),\eta(y,\lambda))^{-1} \in \mathrm{Stab}^{\mathrm{ess}}(y)$. Since $\phi(h^{-1}(y),\cdot) : \mathrm{Stab}^{\mathrm{ess}}(h^{-1}(y)) \to \mathrm{Stab}^{\mathrm{ess}}(y)$ is bijective, there exists $\gamma \in \mathrm{Stab}^{\mathrm{ess}}(h^{-1}(y))$ such that $\phi(h^{-1}(y),\gamma) = \lambda\phi(h^{-1}(y),\eta(y,\lambda))^{-1}$. Thus $\Theta(h^{-1}(y),\gamma\eta(y,\lambda)) = (y,\lambda)$. □

This gives the following generalisation of Li's rigidity theorem [29, Theorem 1.2].

Corollary 7.5. *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be actions of countable discrete groups on second-countable locally compact Hausdorff spaces. Suppose that $\mathrm{Stab}^{\mathrm{ess}}(x)$ and $\mathrm{Stab}^{\mathrm{ess}}(y)$ are torsion-free and abelian for all $x \in X$ and all $y \in Y$, and that h is a homeomorphism from X to Y. The following are equivalent:*

1. *there exist cocycles $\phi : X \times \Gamma \to \Lambda$ and $\eta : Y \times \Lambda \to \Gamma$ such that (h,ϕ,η) is a continuous orbit equivalence from (X,Γ) to (Y,Λ) and (h,ϕ) and (h^{-1},η) preserve essential stabilisers;*
2. *there is an isomorphism $\Theta : X \rtimes \Gamma \to Y \rtimes \Lambda$ such that $\Theta(x,e) = (h(x),e)$ for all $x \in X$; and*
3. *there is an isomorphism $\phi : C_0(X)\times_r\Gamma \to C_0(Y)\times_r\Lambda$ such that $\phi(C_0(X)) = C_0(Y)$ and $\phi(f) = f \circ h^{-1}$ for $f \in C_0(X)$.*

Proof. This follows directly from Theorem 6.2 and Proposition 7.4. □

Remark 7.6. Let $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ be actions of a countable discrete group on second-countable locally compact Hausdorff spaces, and consider the reduced crossed-products $C_0(X) \times_r \Gamma$ and $C_0(Y) \times_r \Gamma$, with dual coactions δ_X and δ_Y of Γ. As $(C_0(X) \times_r \Gamma)^{\delta_X} = C_0(X)$ and likewise for Y, any equivariant isomorphism $C_0(X)\times_r\Gamma \to C_0(Y)\times_r\Gamma$ restricts to an isomorphism $C_0(X) \to C_0(Y)$. Theorem 6.2 therefore shows that $C_0(X) \times_r \Gamma$ and $C_0(Y)\times_r\Gamma$ are equivariantly isomorphic if and only if $X \rtimes \Gamma \cong Y \rtimes \Gamma$ as graded groupoids. We have $X \rtimes \Gamma \cong Y \rtimes \Gamma$ as graded groupoids if and only if there is a homeomorphism $h : X \to Y$ such that $h(x)\gamma = h(x\gamma)$ for all $x \in X$ and $\gamma \in \Gamma$, so if and only if (X,Γ) and (Y,Γ) are topologically conjugate. So we recover [26, Proposition 4.3].

8. Local homeomorphisms of locally compact Hausdorff spaces

In this section we adapt the ideas of Section 7 to actions of $\mathbb{N}$ by local homeomorphisms. We first characterise an appropriate notion of continuous orbit equivalence in terms of weak-Cartan-preserving isomorphism of C^*-algebras. We then characterise eventual conjugacy in terms of gauge-equivariant diagonal-preserving isomorphisms.

A *Deaconu–Renault system* is a pair (X,σ) consisting of a locally compact Hausdorff space X and a local homeomorphism $\sigma : \mathrm{dom}(\sigma) \to \mathrm{ran}(\sigma)$ from an open set $\mathrm{dom}(\sigma) \subseteq X$ to an open set $\mathrm{ran}(\sigma) \subseteq X$. Inductively define $\mathrm{dom}(\sigma^n) := \sigma^{-1}(\mathrm{dom}(\sigma^{n-1}))$, so each $\sigma^n : \mathrm{dom}(\sigma^n) \to \mathrm{ran}(\sigma^n)$ is a local homeomorphism and $\sigma^m \circ \sigma^n = \sigma^{m+n}$ on $\mathrm{dom}(\sigma^{m+n})$. We write $D_n := \mathrm{dom}(\sigma^n)$ and $\sigma^0 := \mathrm{id}_X$. For $x \in X$ we define the *stabiliser group* at x by

$$\mathrm{Stab}(x) := \{m - n : m,n \in \mathbb{N},\ x \in D_m \cap D_n,\ \text{and } \sigma^n(x) = \sigma^m(x)\} \subseteq \mathbb{Z},$$

and we define the *essential stabiliser group* at x by

$$\mathrm{Stab}^{\mathrm{ess}}(x) := \{m - n : m,n \in \mathbb{N} \text{ and there is an open neighbourhood } U \subseteq D_m \cap D_n \text{ of } x \text{ such that } \sigma^n|_U = \sigma^m|_U\} \subseteq \mathrm{Stab}(x).$$

With the convention that $\min(\emptyset) = \infty$, we define the *minimal stabiliser* of x to be

$$\mathrm{Stab}_{\min}(x) := \min\{n \in \mathrm{Stab}(x) : n \geq 1\},$$

and the *minimal essential stabiliser* at x to be

$$\mathrm{Stab}^{\mathrm{ess}}_{\min}(x) := \min\{n \in \mathrm{Stab}^{\mathrm{ess}}(x) : n \geq 1\}.$$

The Deaconu–Renault groupoid of (X, σ) is

$$G = G(X,\sigma) = \bigcup_{n,m\in\mathbb{N}} \big\{(x, n-m, y) \in D_n \times \{n-m\} \times D_m : \sigma^n(x) = \sigma^m(y)\big\},$$

under the topology with basic open sets $Z(U, n, m, V) := \{(x, n-m, y) : x \in U,\ y \in V, \text{ and } \sigma^n(x) = \sigma^m(y)\}$ indexed by quadruples (U, n, m, V) where $n, m \in \mathbb{N}$, $U \subseteq D_n$ and $V \subseteq D_m$ are open, and $\sigma^n|_U$ and $\sigma^m|_V$ are homeomorphisms. Each $Z(U, n, m, V)$ can be written as $Z(U', n, m, V')$ with $\sigma^n(U') = \sigma^m(V')$ (put $U' = U \cap (\sigma^n)^{-1}(V)$ and $V' = V \cap (\sigma^m)^{-1}(U)$). This G is a locally compact Hausdorff étale groupoid, with $G^{(0)} = \{(x, 0, x) : x \in X\}$ identified with X. It is also amenable by the argument of [45, Lemma 3.5], so $C^*_r(G) = C^*(G)$. The isotropy subgroupoid of G is $\{(x, n, x) : x \in X,\ n \in \mathrm{Stab}(x)\}$, and the interior of the isotropy is $\mathrm{Iso}(G)^\circ = \{(x, n, x) : x \in X,\ n \in \mathrm{Stab}^{\mathrm{ess}}(x)\}$. So $\mathrm{Iso}(G)^\circ$ is torsion-free and abelian. Taking $\Gamma = \{e\}$ and $c : G \to \Gamma$ the trivial cocycle, we obtain a (trivially) graded groupoid G.

8.1. **Continuous orbit equivalence.** We show that stabiliser-preserving continuous orbit equivalence of Deaconu–Renault systems characterises isomorphism of their groupoids.

Definition 8.1. Let (X, σ) and (Y, τ) be Deaconu–Renault systems. We say that (X, σ) and (Y, τ) are *continuous orbit equivalent* if there exist a homeomorphism $h : X \to Y$ and continuous maps $k, l : \mathrm{dom}(\sigma) \to \mathbb{N}$ and $k', l' : \mathrm{dom}(\tau) \to \mathbb{N}$ such that

$$\tau^{l(x)}(h(x)) = \tau^{k(x)}(h(\sigma(x))) \quad \text{ and } \quad \sigma^{l'(y)}(h^{-1}(y)) = \sigma^{k'(y)}(h^{-1}(\tau(y)))$$

for all x, y. We call (h, l, k, l', k') a continuous orbit equivalence and we call h the *underlying homeomorphism.* We say that (h, l, k, l', k') *preserves stabilisers* if $\mathrm{Stab}_{\min}(h(x)) < \infty \iff \mathrm{Stab}_{\min}(x) < \infty$, and

$$\Big|\sum_{n=0}^{\mathrm{Stab}_{\min}(x)-1} l(\sigma^n(x)) - k(\sigma^n(x))\Big| = \mathrm{Stab}_{\min}(h(x)) \text{ and}$$
$$\Big|\sum_{n=0}^{\mathrm{Stab}_{\min}(y)-1} l'(\tau^n(y)) - k'(\tau^n(y))\Big| = \mathrm{Stab}_{\min}(h^{-1}(y))$$

whenever $\mathrm{Stab}(x), \mathrm{Stab}(y)$ are nontrivial, $\sigma^{\mathrm{Stab}_{\min}(x)}(x) = x$, and $\tau^{\mathrm{Stab}_{\min}(y)}(y) = y$.

Likewise, we say that (h, l, k, l', k') *preserves essential stabilisers* if $\mathrm{Stab}^{\mathrm{ess}}_{\min}(h(x)) < \infty \iff \mathrm{Stab}^{\mathrm{ess}}_{\min}(x) < \infty$, and

$$\Big|\sum_{n=0}^{\mathrm{Stab}^{\mathrm{ess}}_{\min}(x)-1} (l(\sigma^n(x)) - k(\sigma^n(x)))\Big| = \mathrm{Stab}^{\mathrm{ess}}_{\min}(h(x)) \text{ and}$$
$$\Big|\sum_{n=0}^{\mathrm{Stab}^{\mathrm{ess}}_{\min}(y)-1} (l'(\tau^n(y)) - k'(\tau^n(y)))\Big| = \mathrm{Stab}^{\mathrm{ess}}_{\min}(h^{-1}(y))$$

whenever $\mathrm{Stab}^{\mathrm{ess}}_{\min}(x), \mathrm{Stab}^{\mathrm{ess}}_{\min}(y) < \infty$, $\sigma^{\mathrm{Stab}^{\mathrm{ess}}_{\min}(x)}(x) = x$, and $\tau^{\mathrm{Stab}^{\mathrm{ess}}_{\min}(y)}(y) = y$.

Our definition of continuous orbit equivalence boils down to the usual notion for homeomorphisms (see for instance [6], [22], and [46]), and to orbit equivalence of graphs if (X, σ) and (Y, τ) are the shifts on their boundary-path spaces (see [9]). Our main theorem in this section generalises [9, Theorem 5.1] and [46, Theorem 2]:

Theorem 8.2. *Let (X, σ) and (Y, τ) be Deaconu–Renault systems with X, Y second countable, and suppose that $h : X \to Y$ is a homeomorphism. Then the following are equivalent:*

(1) *there is a stabiliser-preserving continuous orbit equivalence from (X, σ) to (Y, τ) with underlying homeomorphism h;*
(2) *there is a groupoid isomorphism $\Theta : G(X, \sigma) \to G(Y, \tau)$ such that $\Theta|_X = h$; and*
(3) *there is an isomorphism $\phi : C^*(G(X, \sigma)) \to C^*(G(Y, \tau))$ such that $\phi(C_0(X)) = C_0(Y)$ with $\phi(f) = f \circ h^{-1}$ for $f \in C_0(Y)$.*

To prove Theorem 8.2, we need to relate isomorphism of Deaconu–Renault groupoids to continuous orbit equivalence. Arklint, Eilers and Ruiz [2] (see also [14]) proved that isomorphism of graph groupoids (and hence diagonal-preserving isomorphism of graph C^*-algebras) is characterised by continuous orbit equivalence of the shift maps on their boundary path spaces with underlying homeomorphism h satisfying $\mathrm{Stab}^{\mathrm{ess}}(h(x)) = \{0\} \iff \mathrm{Stab}^{\mathrm{ess}}(x) = \{0\}$. The following is the analogous result for Deaconu–Renault systems.

Proposition 8.3. *Let (X, σ) and (Y, τ) be Deaconu–Renault systems. Let $h : X \to Y$ be a homeomorphism and let $l, k : \mathrm{dom}(\sigma) \to \mathbb{N}$ and $l', k' : \mathrm{dom}(\tau) \to \mathbb{N}$ be continuous maps such that $h(x) \in \mathrm{dom}(\tau^{l(x)})$ and $h(\sigma(x)) \in \mathrm{dom}(\tau^{k(x)})$ for $x \in \mathrm{dom}(\sigma)$, and $h^{-1}(y) \in \mathrm{dom}(\sigma^{l'(y)})$ and $h^{-1}(\tau(y)) \in \mathrm{dom}(\sigma^{k'(y)})$ for $y \in \mathrm{dom}(\tau)$. The following are equivalent:*

(1) *there is a groupoid isomorphism $\Theta : G(X, \sigma) \to G(Y, \tau)$ such that $\Theta|_X = h$, $\Theta(x, 1, \sigma(x)) = (h(x), l(x) - k(x), h(\sigma(x)))$ for $x \in \mathrm{dom}(\sigma)$, and $\Theta^{-1}(y, 1, \tau(y)) = (h^{-1}(y), l'(y) - k'(y), h^{-1}(\tau(y)))$ for $y \in \mathrm{dom}(\tau)$;*
(2) *(h, l, k, l', k') is an essential-stabiliser-preserving continuous orbit equivalence; and*
(3) *(h, l, k, l', k') is a stabiliser-preserving continuous orbit equivalence.*

Note: (2) $\implies$ (3) of Proposition 8.3 shows that if σ and τ are topologically free, then every continuous orbit equivalence from (X, σ) to (Y, τ) preserves stabilisers.

The rest of this subsection deals with the proofs of Proposition 8.3 and Theorem 8.2.

Lemma 8.4. *Let (X, σ) be a Deaconu–Renault system. The function $l_X : G(X, \sigma) \to \mathbb{N}$ given by $l_X(x, n, y) := \min\{l \in \mathbb{N} : l \geq n \text{ and } \sigma^l(x) = \sigma^{l-n}(y)\}$ is continuous.*

Proof. Suppose that $(x_i, n_i, y_i) \to (x, n, y) \in G(X, \sigma)$. Then $n_i = n$ for large i, so we can assume that $n_i = n$ for all i. We first show that $l_X(x_i, n, y_i) \leq l_X(x, n, y)$ for large i. To see this, fix a basic open neighbourhood $Z(U, p, p-n, V)$ of (x, n, y); so $\sigma^p(U) = \sigma^{p-n}(V)$, and $\sigma^p|_U$ and $\sigma^{p-n}|_V$ are homeomorphisms. Since $(x_i, n, y_i) \to (x, n, y)$ we have $(x_i, n, y_i) \in Z(U, p, p-n, V)$ for large i. So $\sigma^p(x_i) = \sigma^{p-n}(y_i)$ for large i. Let $l := l_X(x, n, y)$. Then $l \leq p$, say $p = l + q$. Hence $\sigma^q(\sigma^l(x_i)) = \sigma^p(x_i) = \sigma^{p-n}(y_i) = \sigma^q(\sigma^{l-n}(y_i))$ for large i. Since σ^q is locally injective and $\lim_i \sigma^l(x_i) = \sigma^l(x) = \sigma^{l-n}(y) = \lim_i \sigma^{l-n}(y_i)$, we deduce that $\sigma^l(x_i) = \sigma^{l-n}(y_i)$ for large i. So $l_X(x_i, n, y_i) \leq l$ for large i.

It now suffices to show that $l_X(x_i, n, y_i) \geq l$ for large i; equivalently, if $l_X(x_i, n, y_i) = k$ for infinitely many i, then $k \geq l$. Suppose that $l(x_{i_j}, n, y_{i_j}) = k$ for all j. Then $k \geq n$ by definition of $l_X(x_i, n, y_i)$. Since $\sigma^k(x_{i_j}) = \sigma^{k-n}(y_{i_j})$ for all j and $x_{i_j} \to x$ and $y_{i_j} \to y$, continuity forces $\sigma^k(x) = \sigma^{k-n}(y)$. So $k \geq l$. □

Given a Deaconu–Renault system (X,σ), define $c_X : G(X,\sigma) \to \mathbb{Z}$ by $c_X(x,n,y) := n$.

Lemma 8.5. *Let (X,σ) and (Y,τ) be Deaconu–Renault systems. Suppose that $\Theta : G(X,\sigma) \to G(Y,\tau)$ is an isomorphism of groupoids. Let $h : X \to Y$ be the restriction of Θ to $G(X,\sigma)^{(0)}$. For $p \in \mathbb{N}$, the functions $l_p, k_p : D_p \to \mathbb{N}$ given by*

$$\begin{aligned} &l_p(x) := \min\{l \in \mathbb{N} : \tau^l(h(x)) = \tau^{l-c_Y(\Theta(x,p,\sigma^p(x)))}(h(\sigma^p(x)))\}, \quad \textit{and} \\ &k_p(x) := l_p(x) - c_Y(\Theta(x,p,\sigma^p(x))). \end{aligned}$$

are continuous, and $\tau^{l_p(x)}(h(x)) = \tau^{k_p(x)}(h(\sigma^p(x)))$ for all $x \in D_p$. For $p \in \mathbb{N}$ and $x \in D_p$,

$$\sum_{n=0}^{p-1}(l_1(\sigma^n(x)) - k_1(\sigma^n(x))) = l_p(x) - k_p(x) = c_Y(\Theta(x,p,\sigma^p(x)))$$

Proof. Since $l_p(x) = l_Y(\Theta(x,p,\sigma^p(x)))$ and Θ is continuous, l_p is continuous by Lemma 8.4. Now k_p is continuous because l_p and c_Y are. We have $\tau^{l_p(x)}(h(x)) = \tau^{k_p(x)}(h(\sigma^p(x)))$ and $l_p(x) - k_p(x) = c_Y(\Theta(x,p,\sigma^p(x)))$ by definition of k_p. So $\sum_{n=0}^{p-1}(l_1(\sigma^n(x)) - k_1(\sigma^n(x))) = \sum_{n=0}^{p-1} c_Y(\Theta(\sigma^n(x),1,\sigma^{n+1}(x))) = c_Y(\Theta(x,p,\sigma^p(x)))$. □

Lemma 8.6. *Let (X,σ) and (Y,τ) be Deaconu–Renault systems as above. Suppose that $\Theta : G(X,\sigma) \to G(Y,\tau)$ is an isomorphism. Let h, k_p, l_p be as in Lemma 8.5. Let $x \in X$. Then $\mathrm{Stab}^{\mathrm{ess}}_{\min}(x) < \infty$ if and only if $\mathrm{Stab}^{\mathrm{ess}}_{\min}(h(x)) < \infty$, and if $\mathrm{Stab}^{\mathrm{ess}}_{\min}(x) < \infty$ and $\sigma^{\mathrm{Stab}^{\mathrm{ess}}_{\min}(x)}(x) = x$, then $|l_{\mathrm{Stab}^{\mathrm{ess}}_{\min}(x)}(x) - k_{\mathrm{Stab}^{\mathrm{ess}}_{\min}(x)}(x)| = \mathrm{Stab}^{\mathrm{ess}}_{\min}(h(x))$.*

Proof. For any Deaconu–Renault system (Z,β) and $a \in Z$, $\mathrm{Stab}^{\mathrm{ess}}_{\min}(a) = \infty$ if and only if $\mathrm{Iso}(G(Z,\beta))^\circ_a = \{(a,0,a)\}$. Thus $\mathrm{Stab}^{\mathrm{ess}}_{\min}(x) = \infty \iff \mathrm{Stab}^{\mathrm{ess}}_{\min}(h(x)) = \infty$ as Θ restricts to an isomorphism $\mathrm{Iso}(G(X,\sigma))^\circ \to \mathrm{Iso}(G(Y,\tau))^\circ$ mapping $G(X,\sigma)^{(0)}$ onto $G(Y,\tau)^{(0)}$.

Suppose that $\mathrm{Stab}^{\mathrm{ess}}_{\min}(h(x)) < \infty$. Then $\Theta(x, \mathrm{Stab}^{\mathrm{ess}}_{\min}(x), x) = (h(x), q, h(x))$ for some $q \in \mathbb{Z}$. Since $(x, \mathrm{Stab}^{\mathrm{ess}}_{\min}(x), x)$ generates $\mathrm{Iso}(G(X,\sigma))^\circ_x$, we deduce that $(h(x), q, h(x))$ generates $\mathrm{Iso}(G(Y,\tau))^\circ_{h(x)}$. Thus $q = \pm\,\mathrm{Stab}^{\mathrm{ess}}_{\min}(h(x))$. So if $\sigma^{\mathrm{Stab}^{\mathrm{ess}}_{\min}(x)}(x) = x$, then

$$|l_{\mathrm{Stab}^{\mathrm{ess}}_{\min}(x)}(x) - k_{\mathrm{Stab}^{\mathrm{ess}}_{\min}(x)}(x)| = |c_Y(\Theta(x, \mathrm{Stab}^{\mathrm{ess}}_{\min}(x), x))| = \mathrm{Stab}^{\mathrm{ess}}_{\min}(h(x)). \qquad \square$$

Given a Deaconu–Renault system (X,σ) and $l : \mathrm{dom}(\sigma) \to \mathbb{N}$, we inductively define $l_m : \mathrm{dom}(\sigma^m) \to \mathbb{N}$, $m \geq 1$ by $l_1 = l$ and $l_{m+1}(x) = l(x) + l_m(\sigma(x))$. For $m, n \geq 1$ we have

$$l_m(x) = \sum_{i=0}^{m-1} l(\sigma^i(x)) \quad \text{and} \quad l_{m+n}(x) = l_m(x) + l_n(\sigma^m(x)). \tag{8.1}$$

Lemma 8.7. *Let (X,σ) and (Y,τ) be Deaconu–Renault systems and let (h,l,k,l',k') be a continuous orbit equivalence from (X,σ) to (Y,τ). Then there is a continuous cocycle $c_{(h,l,k,l',k')} : G(X,\sigma) \to \mathbb{Z}$ such that $c_{(h,l,k,l',k')}(x, m-n, x') = l_m(x) - k_m(x) - l_n(x') + k_n(x')$.*

Proof. Suppose that $x \in D_{m+1}$, $x' \in D_{n+1}$, and $\sigma^m(x) = \sigma^n(x')$. A computation shows that $l_m(x) - k_m(x) - l_n(x') + k_n(x') = l_{m+1}(x) - k_{m+1}(x) - l_{n+1}(x') + k_{n+1}(x')$. Therefore, $c_{(h,l,k,l',k')} : G(X,\sigma) \to \mathbb{Z}$ is a well-defined map. It is easy to check that this map is a cocycle. For continuity, suppose that $\sigma^m(x) = \sigma^n(x')$. Fix open subneighbourhoods $U \ni x$ and $V \ni x'$ of D_m and D_n such that $\sigma^m|_U$ and $\sigma^n|_V$ are homeomorphisms, $l, k, \ldots, l \circ \sigma^m, k \circ \sigma^m$ are constant on U, and $l, k, \ldots, l \circ \sigma^n, k \circ \sigma^n$ are constant on V. Then $c_{(h,l,k,l',k')}$ is constant on $Z(U,m,n,V)$. □

Lemma 8.8. *Let (X,σ) and (Y,τ) be Deaconu–Renault systems, and let (h,k,l,k',l') be a continuous orbit equivalence from (X,σ) to (Y,τ). Then there is a continuous groupoid homomorphism $\Theta_{k,l} : G(X,\sigma) \to G(Y,\tau)$ such that $\Theta_{k,l}(x, m-n, x') = (h(x), l_m(x) - k_m(x) - l_n(x') + k_n(x'), h(x'))$ whenever $\sigma^m(x) = \sigma^n(x')$. For each $x \in X$ there is a group homomorphism $\pi_x : \mathrm{Stab}(x) \to \mathrm{Stab}(h(x))$ such that*

$$\pi_x(m-n) = l_m(x) - k_m(x) - l_n(x) + k_n(x) \text{ whenever } \sigma^m(x) = \sigma^n(x). \tag{8.2}$$

For $x \in X$ and $m,n \in \mathbb{N}$, we have $\mathrm{Stab}(\sigma^m(x)) = \mathrm{Stab}(\sigma^n(x))$, $\mathrm{Stab}(h(\sigma^m(x))) = \mathrm{Stab}(h(\sigma^n(x)))$ and $\pi_{\sigma^m(x)} = \pi_{\sigma^n(x)}$.

Proof. Lemma 8.7 yields a continuous homomorphism $\Theta_{k,l} : G(X,\sigma) \to G(Y,\tau)$ such that $\Theta_{k,l}(x, m-n, x') = (h(x), l_m(x) - k_m(x) - l_n(x') + k_n(x'), h(x'))$ whenever $\sigma^m(x) = \sigma^n(x')$.

For $x \in X$ the map $\pi_x : \mathrm{Stab}(x) \to \mathrm{Stab}(h(x))$ defined by $(h(x), \pi_x(p), h(x)) = \Theta(x,p,x)$ is a homomorphism satisfying (8.2). That each $\mathrm{Stab}(x) = \mathrm{Stab}(\sigma(x))$ follows from the definition of Stab, and then induction gives $\mathrm{Stab}(\sigma^m(x)) = \mathrm{Stab}(\sigma^n(x))$ for all x. Since h intertwines σ-orbits and τ-orbits, it follows immediately that $\mathrm{Stab}(h(\sigma^m(x))) = \mathrm{Stab}(h(\sigma^n(x)))$ for all x. For the final statement, let $p := l_m(\sigma^n(x)) - k_m(\sigma^n(x)) - l_n(\sigma^m(x)) + k_n(\sigma^m(x))$ and calculate:

$$\begin{aligned}
(h(\sigma^m(x)), &\pi_{\sigma^m(x)}(q), h(\sigma^m(x))) = \Theta(\sigma^m(x), q, \sigma^m(x)) \\
&= \Theta(\sigma^m(x), n-m, \sigma^n(x))\Theta(\sigma^n(x), q, \sigma^n(x))\Theta(\sigma^n(x), m-n, \sigma^m(x)) \\
&= (h(\sigma^m(x)), -p, h(\sigma^n(x)))(h(\sigma^n(x)), \pi_{\sigma^n(x)}(q), h(\sigma^n(x)))(h(\sigma^m(x)), p, h(\sigma^n(x))) \\
&= (h(\sigma^m(x)), \pi_{\sigma^n(x)}(q), h(\sigma^m(x))).
\end{aligned}$$

□

Proof of Proposition 8.3. (1) $\implies$ (2): Let k_p, l_p be as in Lemma 8.5. Then $l - k = l_1 - k_1$ on $\mathrm{dom}(\sigma)$. Likewise, if $k'_p, l'_p : \mathrm{dom}(\tau^p) \to \mathbb{N}$ are the functions obtained from Lemma 8.5 for Θ^{-1}, then $l' - k' = l'_1 - k'_1$ on $\mathrm{dom}(\tau)$. So Lemmas 8.5 and 8.6 show that (h,l,k,l',k') is an essential-stabiliser-preserving continuous orbit equivalence.

For both (2) $\implies$ (3) and (3) $\implies$ (1), fix a continuous orbit equivalence (h,k,l,k',l') from (X,σ) to (Y,τ). Let

$$\Theta : G(X,\sigma) \to G(Y,\tau), \quad \text{and} \quad \Theta' : G(Y,\tau) \to G(X,\sigma)$$

be the homomorphisms of Lemma 8.8 for (h,k,l,k',l') and for (h^{-1},k',l',k,l) respectively, and for each $x \in X$, let

$$\pi_x : \mathrm{Stab}(x) \to \mathrm{Stab}(h(x))$$

be the homomorphism (8.2).

(2) $\implies$ (3). Using that, by Lemma 8.8, $\mathrm{Stab}(\cdot)$, $\mathrm{Stab}(h(\cdot))$ and $x \mapsto \pi_x$ are constant on orbits, and that (h,l,k,l',k') preserves essential stabilisers, it is easy to check that $\pi_x(\mathrm{Stab}^{\mathrm{ess}}(x)) = \mathrm{Stab}^{\mathrm{ess}}(h(x))$ for all x. Fix $x \in X$ and $n \in \mathrm{Stab}(h(x))$. Since $\Theta \circ \Theta'$ is continuous and $\Theta(\Theta'(h(x),n,h(x))) = (h(x),m,h(x))$ for some $m \in \mathrm{Stab}(h(x))$, there exist $p,q \in \mathbb{N}$ with $p - q = n$, and open neighbourhoods U,V of $h(x)$ such that $\tau^p|_U$ and $\tau^q|_V$ are homeomorphisms, $\tau^p(U) = \tau^q(V)$, and $\Theta(\Theta'(y,n,y')) = (y,m,y')$ for $y \in U$, $y' \in V$, with $\tau^p(y) = \tau^q(y')$. So $(y, n-m, y) = (y,n,y')(y,m,y')^{-1} \in G(Y,\tau)$ for all $y \in U$, giving $n - m \in \mathrm{Stab}^{\mathrm{ess}}(h(x))$. Hence $\pi_x(r) = n - m$ for some $r \in \mathrm{Stab}^{\mathrm{ess}}(x)$. Thus $\pi_x(r+s) = n$ where $s = c_X(\Theta'(h(x),n,h(x)))$. So $\mathrm{Stab}_{\min}(h(x)) < \infty \implies \mathrm{Stab}_{\min}(x) < \infty$, and symmetry gives the reverse implication.

Suppose that $\mathrm{Stab}_{\min}(x) < \infty$ and that $\sigma^{\mathrm{Stab}_{\min}(x)}(x) = x$. Since $\mathrm{Stab}_{\min}(h(x))$ generates $\mathrm{Stab}(h(x))$ and $\mathrm{Stab}_{\min}(x)$ generates $\mathrm{Stab}(x)$, we have $l_{\mathrm{Stab}_{\min}(x)}(x) - k_{\mathrm{Stab}_{\min}(x)}(x) = \pi_x(\mathrm{Stab}_{\min}(x)) = \pm\,\mathrm{Stab}_{\min}(h(x))$. Hence (h, l, k, l', k') preserves stabilisers.

(3) $\implies$ (1): We show that Θ is bijective and Θ^{-1} is continuous. For injectivity, suppose $\Theta(x_1, n_1, x_1') = \Theta(x_2, n_2, x_2')$. As h is a homeomorphism, $x_1 = x_2$ and $x_1' = x_2'$. So $\Theta(x_1, n_1 - n_2, x_1) = \Theta(x_1, n_1, x_1')\Theta(x_1, n_2, x_1')^{-1} = (h(x_1), 0, h(x_1))$, giving $\pi_x(n_1 - n_2) = 0$. As (h, k, l, k', l') preserves stabilisers and $\mathrm{Stab}(\cdot)$, $\mathrm{Stab}(h(\cdot))$ and $x \mapsto \pi_x$ are constant on orbits, each $\pi_x : \mathrm{Stab}(x) \to \mathrm{Stab}(h(x))$ is bijective. Thus $(x_1, n_1, x_1') = (x_2, n_2, x_2')$.

For surjectivity, fix $(y, n, y') \in G(Y, \tau)$. We have $\Theta(\Theta'(y, n, y')) = (y, m, y')$ for some $m \in \mathbb{Z}$, so $n - m \in \mathrm{Stab}(y)$. Since $\pi_{h^{-1}(y)}$ is bijective, $n - m = \pi_{h^{-1}(y)}(p)$ for some $p \in \mathrm{Stab}(h^{-1}(y))$. So $\Theta\big(h^{-1}(y), p + c_X(\Theta'(y, n, y')), h^{-1}(y')\big) = (y, n, y')$.

To see that Θ^{-1} is continuous, suppose $(y_n, m_n, y_n') \to (y, m, y')$ in $G(Y, \tau)$. Fix $p, q \in \mathbb{N}$ and open $U \ni h^{-1}(y)$ and $V \ni h^{-1}(y')$ such that $\sigma^p|_U$ and $\sigma^q|_V$ are homeomorphisms, $\sigma^p(h^{-1}(y)) = \sigma^q(h^{-1}(y'))$, and $\Theta^{-1}(y, m, y') = (h^{-1}(y), p - q, h^{-1}(y'))$. Choose open subneighbourhoods $U' \ni h^{-1}(y)$ and $V' \ni h^{-1}(y')$ of U, V such that $\Theta(x, p - q, x') = (h(x), m, h(x'))$ whenever $x \in U'$, $x' \in V'$, and $\sigma^p(x) = \sigma^q(x')$. Fix N such that $y_n \in h(U')$, $y_n' \in h(V')$, and $m_n = m$ for $n \geq N$. Then $\Theta^{-1}(y_n, m_n, y_n') \in Z(U', p, q, V') \subseteq Z(U, p, q, V)$ for $n \geq N$. So $\Theta^{-1}(y_n, m_n, y_n') \to \Theta^{-1}(y, m, y')$. □

Proof of Theorem 8.2. The equivalence (1) $\iff$ (2) follows from Proposition 8.3 and Lemma 8.5, and the equivalence (2) $\iff$ (3) follows from Theorem 6.2. □

8.2. **Eventual conjugacy.** Here we generalise [11, Theorem 4.1] by showing that the isomorphism of C^*-algebras in Theorem 8.2 is gauge-equivariant if and only if the groupoid isomorphism is cocycle-preserving, which is if and only if the continuous orbit equivalence is an eventual conjugacy.

Definition 8.9. Let (X, σ) and (Y, τ) be Deaconu–Renault systems. We say that (X, σ) and (Y, τ) are *eventually conjugate* if there is a stabiliser-preserving continuous orbit equivalence (h, l, k, l', k') from (X, σ) to (Y, τ) such that $l(x) = k(x) + 1$ for all $x \in X$.

Given (X, σ), there is an action $\gamma^X : \mathbb{T} \to \mathrm{Aut}(C^*(G(X, \sigma)))$ such that $\gamma^X_z(f)(x, n, x') = z^n f(x, n, x')$ for all $z \in \mathbb{T}$, $(x, n, x') \in G(X, \sigma)$ and $f \in C_c(G(X, \sigma))$.

Theorem 8.10. *Let (X, σ) and (Y, τ) be Deaconu–Renault systems and let $h : X \to Y$ be a homeomorphism. Then the following are equivalent:*

1. *there is an eventual conjugacy from (X, σ) to (Y, τ) with underlying homeomorphism h;*
2. *there is an isomorphism $\Theta : G(X, \sigma) \to G(Y, \tau)$ such that $\Theta|_X = h$ and $c_X = c_Y \circ \Theta$; and*
3. *there is an isomorphism $\phi : C^*(G(X, \sigma)) \to C^*(G(Y, \tau))$ such that $\phi(C_0(X)) = C_0(Y)$, with $\phi(f) = f \circ h^{-1}$ for $f \in C_0(X)$, and $\phi \circ \gamma^X_z = \gamma^Y_z \circ \phi$.*

Proof. Theorem 6.2 applied to the $\mathbb{Z}$-coactions dual to γ^X and γ^Y gives (2) $\iff$ (3).

(2) $\implies$ (1). If $c_X = c_Y \circ \Theta$, then the formula for k_1 in Lemma 8.5 gives $l_1(x) - k_1(x) = 1$, so the continuous orbit equivalence constructed in the proof of (2) $\implies$ (1) in Theorem 8.2 is an eventual conjugacy.

(1) $\implies$ (2). Suppose that (h, l, k, l', k') is an eventual conjugacy from (X, σ) to (Y, τ). The formula (8.1) gives $l_p(x) - k_p(x) = p$ for all $x \in X$. Thus, $\Theta_{k,l}$ of Lemma 8.8 satisfies $c_Y \circ \Theta_{k,l} = c_X$. As in the proof of (1) $\implies$ (2) in Theorem 8.2, $\Theta_{k,l}$ an isomorphism. □

9. Homeomorphisms of compact Hausdorff spaces

We now specialise to the case where X is a compact Hausdorff space and $\sigma : X \to X$ is a homeomorphism. We combine the results and techniques developed in Sections 7 and 8 to obtain a generalisation of Boyle and Tomiyama's theorem. If $\sigma : X \to X$ is a homeomorphism, then $\alpha : G(X, \sigma) \to X \rtimes \mathbb{Z}$, $\alpha(x, n, y) \mapsto (x, n)$ is an isomorphism, so induced an isomorphism $\phi : C^*(G(X, \sigma)) \cong C(X) \times_\sigma \mathbb{Z}$ with $\phi(C(G(X, \sigma)^{(0)})) = C(X)$.

Using Theorem 8.2, we can prove the following generalisation of [6, Theorem 3.6] (and thus of [22, Theorem 2.4] and [46, Corollary]). Following [6], we say that homeomorphisms $\sigma : X \to X$ and $\tau : Y \to Y$ are *flip conjugate* if there is a homeomorphism $h : X \to Y$ such that either $h \circ \sigma = \tau \circ h$ or $h \circ \sigma = \tau^{-1} \circ h$.

Theorem 9.1. *Suppose that $\sigma : X \to X$ and $\tau : Y \to Y$ are homeomorphisms of second-countable compact Hausdorff spaces. The following are equivalent:*

1. *$G(X, \sigma)$ and $G(Y, \tau)$ are isomorphic;*
2. *$C(X) \times_\sigma \mathbb{Z} \cong C(Y) \times_\tau \mathbb{Z}$ via an isomorphism that maps $C(X)$ to $C(Y)$; and*
3. *there exist decompositions $X = X_1 \sqcup X_2$ and $Y = Y_1 \sqcup Y_2$ into disjoint open invariant sets such that $\sigma|X_1$ is conjugate to $\tau|Y_1$ and $\sigma|X_2$ is conjugate to $\tau^{-1}|Y_2$.*

If σ and τ are topologically transitive or X and Y are connected, then these conditions hold if and only σ and τ are flip-conjugate.

Our proof of (1) $\implies$ (3) closely follows [6], and requires some preliminary results. Take X, Y, σ, τ as in Theorem 9.1, and an isomorphism $\theta : G(X, \sigma) \to G(Y, \tau)$. Define $h : X \to Y$ by $\theta(x, 0, x) = (h(x), 0, h(x))$. Let $c_X : G(X, \sigma) \to \mathbb{Z}$ and $c_Y : G(Y, \tau) \to \mathbb{Z}$ be the canonical cocycles. Define $f : X \times \mathbb{Z} \to \mathbb{Z}$ by $f(n, x) := c_Y(\theta(x, n, \sigma^n(x)))$. Then

$$f(m + n, x) = f(m, x) + f(n, \sigma^m(x)) \quad \text{for all } m, n \in \mathbb{Z} \text{ and } x \in X. \tag{9.1}$$

For $x \in X$, $f(\cdot, x)$ is a bijection of $\mathbb{Z}$ with inverse $n \mapsto c_X(\theta^{-1}(h(x), n, \tau^n(h(x))))$.

For $m, n \in \mathbb{Z}$, we let $[m, n] := \{k \in \mathbb{Z} : m \leq k \leq n\}$.

Lemma 9.2. *For each $M \in \mathbb{N}$ there exists $\overline{M} \in \mathbb{N}$ such that*

$$[-M, M] \subseteq \{f(n, x) : n \in [-\overline{M}, \overline{M}]\} \quad \textit{for all } x \in X. \tag{9.2}$$

Proof. Let $x \in X$. Fix $M_x \in \mathbb{N}$ such that $[-M, M] \subseteq \{f(n, x) : n \in [-M_x, M_x]\}$. Continuity of the map $f(n, \cdot)$ for each n implies x has an open neighbourhood U_x such that $[-M, M] \subseteq \{f(n, x') : n \in [-M_x, M_x]\}$ for $x' \in U_x$. Compactness of X gives a finite $F \subseteq X$ such that $\bigcup_{x \in F} U_x = X$. So $\overline{M} := \max\{M_x : x \in F\}$ satisfies (9.2). □

Lemma 9.3. *There is a positive integer N such that*

$$X_1 := \{x \in X : f(n, x) > 0 \textit{ and } f(-n, x) < 0 \textit{ for } n > N\} \quad \textit{and}$$
$$X_2 := \{x \in X : f(n, x) < 0 \textit{ and } f(-n, x) > 0 \textit{ for } n > N\}$$

are clopen σ-invariant subsets such that $X = X_1 \sqcup X_2$.

Proof. Compactness of X implies that $M := \max\{|f(1, x)| : x \in X\}$ is finite. Lemma 9.2 gives $N \in \mathbb{N}$ with $[-M, M] \subseteq \{f(n, x) : n \in [-N, N]\}$ for all x.

To see that $X = X_1 \cup X_2$, fix $x \in X$. Choose $m > N$. Since $n \mapsto f(n, x)$ is bijective and $[-M, M] \subseteq \{f(n, x) : n \in [-N, N]\}$, we have $|f(m, x)| > M$. Since $|f(m + 1, x) - f(m, x)| = |f(1, \sigma^m(x))| \leq M$, it follows that $f(m+1, x)$ and $f(m, x)$ have the same sign.

Similarly, $f(-m-1, x)$ and $f(-m, x)$ have the same sign. Since $n \mapsto f(n, x)$ is bijective, $x \in X_1 \cup X_2$. Hence $X = X_1 \cup X_2$. Continuity of $f(n, \cdot)$ imply X_i are clopen.

We show that $\sigma(X_1) \subseteq X_1$ (a similar argument gives $\sigma(X_2) \subseteq X_2$). Fix $x \in X_1$. Choose $m > N$. Then $f(m+1, x) > M$. Since $|f(m+1, x) - f(m, \sigma(x))| = |f(1, x)| \leq M$, we have $\sigma(x) \notin X_2$; so $\sigma(x) \in X_1$. □

Lemma 9.4. *There are continuous functions $a : X_1 \to \mathbb{Z}$ and $b : X_2 \to \mathbb{Z}$ such that $f(1, x) = a(x) - a(\sigma(x)) + 1$ for $x \in X_1$, and $f(1, x) = b(x) - b(\sigma(x)) - 1$ for $x \in X_2$.*

Proof. Define $n(x) := f(1, x)$ for all $x \in X$. Fix $x \in X_1$. Since $f(n, x) > 0$ and $f(-n, x) < 0$ for $n > N$, both $|\{n \geq 0 : f(n, x) < 0\}|$ and $|\{m < 0 : f(m, x) \geq 0\}|$ are finite. Let

$$a(x) := \big|\{m < 0 : f(m, x) \geq 0\}\big| - \big|\{n \geq 0 : f(n, x) < 0\}\big|.$$

Continuity of $f(n, \cdot)$ implies that a is continuous.

Take $x \in X_1$. Using (9.1) at the third equality, we calculate

$$\begin{aligned} a(x) + 1 &= \big|\{m < 0 : f(m, x) \geq 0\}\big| - \big|\{n \geq 0 : f(n, x) < 0\}\big| + 1 \\ &= \big|\{m < 1 : f(m, x) \geq 0\}\big| - \big|\{n \geq 1 : f(n, x) < 0\}\big| \\ &= \big|\{m < 1 : f(m-1, \sigma(x)) \geq -n(x)\}\big| - \big|\{n \geq 1 : f(n-1, \sigma(x)) < -n(x)\}\big| \\ &= \big|\{m < 0 : f(m, \sigma(x)) \geq -n(x)\}\big| - \big|\{n \geq 0 : f(n, \sigma(x)) < -n(x)\}\big|. \end{aligned}$$

Suppose now that $n(x) \geq 0$. Then

$$\begin{aligned} &\big|\{m < 0 : f(m, \sigma(x)) \geq -n(x)\}\big| - \big|\{m < 0 : f(m, \sigma(x)) \geq 0\}\big| \\ &\qquad = \big|\{m < 0 : 0 > f(m, \sigma(x)) \geq -n(x)\}\big|, \text{ and} \\ &\big|\{n \geq 0 : f(n, \sigma(x)) < 0\}\big| - \big|\{n \geq 0 : f(n, \sigma(x)) < -n(x)\}\big| \\ &\qquad = \big|\{n \geq 0 : 0 > f(n, \sigma(x)) \geq -n(x)\}\big|. \end{aligned}$$

Since $\{f(m, \sigma(x)) : m \in \mathbb{Z}\} = \mathbb{Z}$, we have

$$\big|\{m < 0 : 0 > f(m, \sigma(x)) \geq -n(x)\}\big| + \big|\{n \geq 0 : 0 > f(n, \sigma(x)) \geq -n(x)\}\big| = n(x).$$

Hence

$$\begin{aligned} a(x) &= \big|\{m < 0 : f(m, \sigma(x)) \geq -n(x)\}\big| - \big|\{n \geq 0 : f(n, \sigma(x)) < -n(x)\}\big| - 1 \\ &= \big|\{m < 0 : f(m, \sigma(x)) \geq 0\}\big| - \big|\{n \geq 0 : f(n, \sigma(x)) < 0\}\big| + n(x) - 1 \\ &= a(\sigma(x)) + n(x) - 1, \end{aligned}$$

so $n(x) = a(x) - a(\sigma(x)) + 1$. A similar argument applies for $n(x) < 0$.

Similarly, $b(x) := \big|\{m < 0 : f(m, x) \leq 0\}\big| - \big|\{n \geq 0 : f(n, x) > 0\}\big|$ defines a continuous function such that $n(x) = b(x) - b(\sigma(x)) = 1$ for $x \in X_2$. □

Proof of Theorem 9.1. The equivalence of (1) and (2) follows directly from Theorem 8.2.

(3) $\Longrightarrow$ (1): Suppose $h_1 : X_1 \to Y_1$ and $h_2 : X_2 \to Y_2$ are homeomorphisms such that $h_1(\sigma(x)) = \tau(h_1(x))$ for $x \in X_1$ and $h_2(\sigma(y)) = \tau^{-1}(h_2(y))$ for $y \in X_2$. Then

$$\theta(x, n, y) = \begin{cases} (h_1(x), n, h_1(y)) & \text{if } x, y \in X_1, \\ (h_2(x), -n, h_2(y)) & \text{if } x, y \in X_2, \end{cases}$$

defines an isomorphism $\theta : G(X, \sigma) \to G(Y, \tau)$.

(1) $\implies$ (3): Let X_1 and X_2 be as in Lemma 9.3 and a and b be as in Lemma 9.4. Let $Y_1 := h(X_1)$ and $Y_2 := h(X_2)$. Define $h_1 : X_1 \to Y_1$ by $h_1(x) = \tau^{a(x)}(h(x))$ and $h_2 : X_2 \to Y_2$ by $h_2(x) = \tau^{b(x)}(h(x))$. Then $h_1(\sigma(x)) = \tau^{a(\sigma(x))}(h(\sigma(x))) = \tau^{a(x)-f(1,x)+1}(h(\sigma(x))) = \tau^{a(x)+1}(h(x)) = \tau(h_1(x))$ for $x \in X_1$ because $\tau^{f(1,x)}(h(x)) = h(\sigma(x))$, and $h_2(\sigma(x)) = \tau^{b(\sigma(x))}(h(\sigma(x))) = \tau^{b(x)-f(1,x)-1}(h(\sigma(x))) = \tau^{b(x)-1}(h(x)) = \tau^{-1}(h_1(x))$ for $x \in X_2$ because $\tau^{f(1,x)}(h(x)) = h(\sigma(x))$. □

10. Equivariant Morita equivalence for weakly Cartan pairs

In this section we define equivalence of graded groupoids and equivariant Morita equivalence of nested pairs of C^*-algebras. We show that given coactions $\delta_i : A_i \to A_i \otimes C^*_r(\Gamma)$ and weakly Cartan subalgebras $D_i \subseteq A_i$, the pairs (A_i, D_i) are equivariantly Morita equivalent if and only if their extended Weyl groupoids are graded equivalent.

Groupoids G_1 and G_2 are *equivalent* if there is a topological space Z carrying commuting free and proper actions of G_1 and G_2 on the left and right respectively such that $r : Z \to G_1^{(0)}$ and $s : Z \to G_2^{(0)}$ induce homeomorphisms $G_1\backslash Z \cong G_2^{(0)}$ and $Z/G_2 \cong G_1^{(0)}$. The associated *linking groupoid* $L(G_1, G_2)$ [44] is

$$L := L(G_1, G_2) = G_1 \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup G_2$$

with $L^{(0)} = G_1^{(0)} \cup G_2^{(0)}$, the obvious range and source maps, and multiplication determined by multiplication in G_1 and G_2, the actions of the G_i on Z and Z^{op}, and the maps ${}_{G_1}[\cdot,\cdot] : \{(z, y^{\mathrm{op}}) \in Z \times Z^{\mathrm{op}} : s(z) = r(y^{\mathrm{op}})\} \to G_1$ and $[\cdot,\cdot]_{G_2} : \{(y^{\mathrm{op}}, z) \in Z^{\mathrm{op}} \times Z : s(y^{\mathrm{op}}) = r(z)\} \to G_2$ determined by ${}_{G_1}[z, y^{\mathrm{op}}] \cdot y = z$ and $y \cdot [y^{\mathrm{op}}, z]_{G_2} = z$. Conversely, if G is a groupoid and K_1, K_2 are complementary G-full open subsets of $G^{(0)}$, then $Z := K_1 L K_2$ is a K_1GK_1–K_2GK_2-equivalence under the actions given by multiplication in G.

Definition 10.1. Let $c_i : G_i \to \Gamma$, $i = 1, 2$ be gradings of locally compact groupoids. A *graded* (G_1, c_1)–(G_2, c_2)-equivalence consists of a G_1–G_2-equivalence Z and a continuous map $c_Z : Z \to \Gamma$ satisfying $c_Z(\gamma \cdot z \cdot \eta) = c_1(\gamma)c_Z(z)c_2(\eta)$ for all γ, z, η.

Lemma 10.2. *Let Γ be a discrete group. Then graded equivalence as described in Definition 10.1 is an equivalence relation on Γ-graded groupoids.*

Proof. Suppose that (G_i, c_i) is a Γ-graded groupoid for $i = 1, 2, 3$ and that (Z_i, c_{Z_i}) is a (G_i, c_i)–(G_{i+1}, c_{i+1})-equivalence for $i = 1, 2$. Define $\sim$ on $Z_1 \,{}_r\!\times_s Z_2 := \{(z_1, z_2) \in Z_1 \times Z_2 : s(z_1) = r(z_2)\}$ by $(z_1 \cdot \gamma, \gamma^{-1} \cdot z_2) \sim (z_1, z_2)$ for $\gamma \in (G_2)^{s(z_1)}$. By [32, page 6], $Z_1 *_{G_2} Z_2 := (Z_1 \,{}_r\!\times_s Z_2)/{\sim}$ is a G_1–G_3-equivalence with $\gamma_1 \cdot [z_1, z_2] = [\gamma_1 \cdot z_1, z_2]$ and $[z_1, z_2] \cdot \gamma_3 = [z_1, z_2 \cdot \gamma_3]$. For $[z_1, z_2] \in Z_1 *_{G_2} Z_2$ and $\gamma \in G_2^{s(z_1)}$, we have

$$c_{Z_1}(z_1 \cdot \gamma)c_{Z_2}(\gamma^{-1} \cdot z_2) = c_{Z_1}(z_1)c_2(\gamma)c_2(\gamma^{-1})c_{Z_2}(z_2) = c_{Z_1}(z_1)c_{Z_2}(z_2),$$

so there is a map $\tilde{c} : Z_1 *_{G_2} Z_2 \to \Gamma$ such that $\tilde{c}([z_1, z_2]) = c_{Z_1}(z_1)c_{Z_2}(z_2)$. So for $\gamma_1 \in G_1$, $[z_1, z_2] \in Z_1 *_{G_2} Z_2$, and $\gamma_3 \in G_3$ with $s(\gamma_1) = r([z_1, z_2])$ and $s([z_1, z_2]) = r(\gamma_3)$, we have $\tilde{c}(\gamma_1 \cdot [z_1, z_2] \cdot \gamma_3) = c_{Z_1}(\gamma_1 \cdot z_1)c_{Z_2}(z_2 \cdot \gamma_3) = c_1(\gamma_1)\tilde{c}([z_1, z_2])c_3(\gamma_3)$. So $(Z_1 *_{G_2} Z_3, \tilde{c})$ is a (G_1, c_1)–(G_3, c_3)-equivalence. □

Lemma 10.3. *Let Γ be a discrete group and let (G_1, c_1), (G_2, c_2) be Γ-graded locally compact Hausdorff étale groupoids with each $\mathrm{Iso}(c_i^{-1}(e))^\circ$ torsion-free and abelian. Suppose that (Z, c_Z) is a graded equivalence from (G_1, c_1) to (G_2, c_2). Let $G = L(G_1, G_2)$, and define $c : G \to \Gamma$ by $c|_{G_i} = c_i$, $c|_Z = c_Z$ and $c(z^{\mathrm{op}}) = c_Z(z)^{-1}$ for $z \in Z$. Then (G, c) is a*

Γ-graded groupoid, $\mathrm{Iso}(c^{-1}(e))^\circ$ *is torsion-free and abelian, and* $(G_i^{(0)}GG_i^{(0)}, c) \cong (G_i, c_i)$ *for each* i. *Conversely, given a* Γ*-graded groupoid* (G, c) *such that* $\mathrm{Iso}(c^{-1}(e))^\circ$ *is torsion-free and abelian, and given complementary open* G*-full sets* $K_1, K_2 \subseteq G^{(0)}$ *such that each* $(K_iGK_i, c) \cong (G_i, c_i)$, *the pair* $(K_1GK_2, c|_{K_1GK_2})$ *is a* (G_1, c_1)–(G_2, c_2) *equivalence under the left and right actions given by multiplication in* G.

Proof. Each $c_Z(\gamma \cdot z \cdot \eta) = c_1(\gamma)c_Z(z)c_2(\eta)$ so c is a cocycle; and $\mathrm{Iso}(G)^\circ \cap c^{-1}(e) = (\mathrm{Iso}(G_1)^\circ \cap c_1^{-1}(e)) \sqcup (\mathrm{Iso}(G_2)^\circ \cap c_2^{-1}(e))$ is abelian and torsion free. Since $(G_i^{(0)}GG_i^{(0)}, c) \cong (G_i, c_i)$, the first statement follows. For the second, we saw that K_1GK_2 is a G_1–G_2-equivalence; and each $c_Z(\gamma \cdot z \cdot \eta) = c(\gamma z \eta) = c(\gamma)c(z)c(\eta) = c_1(\gamma)c_Z(z)c_2(\eta)$. □

We now turn to Morita equivalence of weakly Cartan pairs. As in Section 1, if X is an A_1–A_2 imprimitivity bimodule, then X^* is its adjoint module and the linking algebra $A = A_1 \oplus X \oplus X^* \oplus A_2$ contains A_1, A_2 as complementary full corners. Writing P_i for $1_{M(A_i)} \in M(A)$, the *multiplier module of* X is $M(X) := P_1M(A)P_2$ (see [17]); so $M(X^*) \cong P_2M(A)P_1 = M(X)^*$, and the adjoint operation in $M(A)$ determines an extension to multiplier modules of the map $\xi \mapsto \xi^*$ from X to X^*.

Definition 10.4. Suppose that for $i = 1, 2$, A_i is a C^*-algebra carrying a coaction δ_i of a discrete group Γ, and $D_i \subseteq A_i^{\delta_i}$ is a subalgebra. We say that (A_1, D_1) and (A_2, D_2) are *equivariantly Morita equivalent* if there are an A_1–A_2-imprimitivity bimodule X and a right-Hilbert bimodule morphism $\zeta : X \to M(X \otimes C_r^*(\Gamma))$ such that $(\zeta \otimes \mathrm{id}_\Gamma) \circ \zeta = (\mathrm{id}_X \otimes \delta_\Gamma) \circ \zeta$ and for each $g \in \Gamma$, the subspace $X_g := \{x \in X : \zeta(x) = x \otimes \lambda_g\}$ satisfies

$$X_g = \overline{\mathrm{span}}\{\xi \in X_g : {}_{A_1}\langle \xi, \xi \cdot D_2\rangle \subseteq D_1 \text{ and } \langle D_1 \cdot \xi, \xi\rangle_{A_2} \subseteq D_2\}. \tag{10.1}$$

If $\Gamma = \{0\}$, we say that (A_1, D_1) and (A_2, D_2) are *Morita equivalent.*

The following is well known.

Lemma 10.5. *Let* A_1, A_2 *be separable* C^**-algebras and let* X *be an* A_1–A_2*-imprimitivity bimodule. Then* X *is separable.*

Proof. Since $\langle \cdot, \cdot \rangle_{A_2}$ is full, and A_2 is separable there are sequences $(y_n), (z_n)$ in X such that $(\langle y_n, z_n\rangle_{A_2})_n$ is dense in A_2. Cohen factorisation and the imprimitivity condition give $X = \overline{\mathrm{span}}\{x \cdot \langle y_n, z_n\rangle_{A_2} : x \in X, n \in \mathbb{N}\} = \overline{\mathrm{span}}\{{}_{A_1}\langle x, y_n\rangle \cdot z_n : x \in X, n \in \mathbb{N}\} \subseteq \overline{\mathrm{span}}\{A_1 \cdot z_n : n \in \mathbb{N}\}$, which is separable because A_1 is. □

We now state the main result of the section; the proof occupies the rest of the section.

Theorem 10.6. *Let* Γ *be a discrete group, and let* A_1, A_2 *be separable* C^**-algebras. Suppose, for* $i = 1, 2$, *that* δ_i *is a coaction of* Γ *on* A_i *and* D_i *is a weakly Cartan subalgebra of* $A_i^{\delta_i}$. *Suppose that* (X, ζ) *is an equivariant Morita equivalence between* (A_1, D_1) *and* (A_2, D_2). *Let* A *be the linking algebra of* X, *let* $D := D_1 \oplus D_2 \subseteq A$ *and let* $\delta : A \to A \otimes C_r^*(\Gamma)$ *be the map that restricts to* δ_i *on each* A_i, *to* ζ *on* X *and to* $x^* \mapsto \zeta(x)^*$ *on* X^*. *Then* δ *is a coaction and* D *is weakly Cartan in* A^δ. *The sets* $\widehat{D}_i$ *are complementary full clopen subsets of the unit space of* $\mathcal{H} := \mathcal{H}(A, D, \delta)$, *we have* $(\widehat{D}_i\mathcal{H}\widehat{D}_i, c_\delta) \cong (\mathcal{H}(A_i, D_i, \delta_i), c_{\delta_i})$ *for* $i = 1, 2$, *and* $(\widehat{D}_1\mathcal{H}\widehat{D}_2, \delta)$ *is an equivalence from* $\big(\mathcal{H}(A_1, D_1, \delta_1), c_{\delta_1}\big)$ *to* $\big(\mathcal{H}(A_2, D_2, \delta_2), c_{\delta_2}\big)$.

Lemma 10.7. *For $i = 1, 2$, let A_i be a separable C^*-algebra and $D_i \subseteq A_i$ an abelian C^*-subalgebra containing an approximate unit for A_i. Let X be an A_1–A_2-imprimitivity bimodule such that*

$$X = \overline{\operatorname{span}}\{x \in X : \langle D_1 \cdot x, x\rangle_{A_2} \subseteq D_2 \text{ and } {}_{A_1}\langle x, x \cdot D_2\rangle \subseteq D_1\}.$$

Let A be the linking algebra of X. Then $D := D_1 \oplus D_2$ is an abelian subalgebra of A containing an approximate unit for A. The spaces $N_{A_i}(D_i) \subseteq A_i$ and

$$\{\xi \in X : {}_{A_1}\langle \xi, \xi \cdot D_2\rangle \subseteq D_1 \text{ and } \langle D_1 \cdot \xi, \xi\rangle_{A_2} \subseteq D_2\}$$

are all contained in $N_A(D)$.

Proof. Fix approximate units $(e^i_j)_j$ for A_i in D_i. Then $(e^1_j \oplus e^2_j)_j$ is an approximate unit for A in D, which is clearly an abelian C^*-subalgebra of A. Clearly each $N_{A_i}(D_i) \subseteq N_A(D)$.

If ${}_{A_1}\langle \xi, \xi \cdot D_2\rangle \in D_1$ and $\langle D_1 \cdot \xi, \xi\rangle_{A_2} \in D_2$, then for $d_i \in D_i$,

$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} {}_{A_1}\langle \xi, \xi \cdot d_2\rangle & 0 \\ 0 & 0 \end{pmatrix} \in D_1 \oplus D_2,$$

and a similar computation shows that

$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}^*\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \langle d_1^* \cdot \xi, \xi\rangle_{A_2} \end{pmatrix} \in D. \qquad \square$$

Lemma 10.8. *Let Γ be a discrete group. For $i = 1, 2$, let δ_i be a coaction of Γ on a separable C^*-algebra A_i, and let $D_i \subseteq A_i^{\delta_i}$ be a weakly Cartan subalgebra. Suppose that (X, ζ) is an equivariant Morita equivalence between (A_1, D_1) and (A_2, D_2). Let A be the linking algebra of X, let $D = D_1 \oplus D_2$, and define $\delta : A \to A \otimes C_r^*(\Gamma)$ by*

$$\delta\begin{pmatrix} a_1 & \xi \\ \eta^* & a_2 \end{pmatrix} = \begin{pmatrix} \delta_1(a_1) & \zeta(\xi) \\ \zeta(\eta)^* & \delta_2(a_2) \end{pmatrix}.$$

Then δ is a coaction of Γ on A, we have $D'_{A^\delta} = (D_1)'_{A_1^{\delta_1}} \oplus (D_2)'_{A_2^{\delta_2}}$, and D is a weakly Cartan subalgebra of A^δ. For $i = 1, 2$, the pair (P_iAP_i, P_iD_i) is equivariantly isomorphic to (A_i, D_i). We have $P_1AP_2 = \overline{\operatorname{span}}\{P_1nP_2 : n \in N_\star(D)\}$.

Proof. It is routine to check that δ is a coaction. Clearly $(D_1)'_{A_1^{\delta_1}} \oplus (D_2)'_{A_2^{\delta_2}} \subseteq D'_{A^\delta}$. For the reverse, fix $\left(\begin{smallmatrix} a_1 & \xi \\ \eta^* & a_2 \end{smallmatrix}\right) \in D'_{A^\delta}$. Then

$$\begin{pmatrix} a_1 \otimes 1_{C_r^*(\Gamma)} & \xi \otimes 1_{C_r^*(\Gamma)} \\ \eta^* \otimes 1_{C_r^*(\Gamma)} & a_2 \otimes 1_{C_r^*(\Gamma)} \end{pmatrix} = \begin{pmatrix} a_1 & \xi \\ \eta^* & a_2 \end{pmatrix} \otimes 1_{C_r^*(\Gamma)} = \delta\begin{pmatrix} a_1 & \xi \\ \eta^* & a_2 \end{pmatrix} = \begin{pmatrix} \delta_1(a_1) & \zeta(\xi) \\ \zeta(\eta)^* & \delta_2(a_2) \end{pmatrix},$$

so each $a_i \in A^{\delta_i}$, and for $d_i \in D_i$,

$$\begin{pmatrix} a_1d_1 & \xi \cdot d_2 \\ \eta^* \cdot d_1 & a_2d_2 \end{pmatrix} = \begin{pmatrix} a_1 & \xi \\ \eta^* & a_2 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} a_1 & \xi \\ \eta^* & a_2 \end{pmatrix} = \begin{pmatrix} d_1a_1 & d_1 \cdot \xi \\ d_2 \cdot \eta^* & d_2a_2 \end{pmatrix},$$

so each $a_i \in (D_i)'_{A_i^{\delta_i}}$. Moreover, $d_1 \cdot \xi = 0$ for all $d_1 \in D_1$, so $\xi = 0$ by Cohen factorisation, and similarly $\eta^* = 0$. So $\left(\begin{smallmatrix} a_1 & \xi \\ \eta^* & a_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right) \in (D_1)'_{A_1^{\delta_1}} \oplus (D_2)'_{A_2^{\delta_2}}$.

To see that D is weakly Cartan in A^δ, it suffices by symmetry to fix $\phi \in \widehat{D}_1$ and show that D'_{A^δ}/J_ϕ is unital and $\pi_\phi(d) = \phi(d)1_{D'_{A^\delta}/J_\phi}$ for $d \in D$. Let $j_1 : A_1 \to A$ be the inclusion map. Since $\phi = 0$ on D_2, π_ϕ factors through the corresponding representation $\pi_{1,\phi}$ of $(D_1)'_{A_1^{\delta_1}}$ which is unital and satisfies $\pi_{1,\phi}(d) = \phi(d)1_{(D_1)'_{A^\delta}/J_\phi}$.

Each $(P_iAP_i, P_iD_i, \delta|_{P_iAP_i}) \cong (A_i, D_i, \delta_i)$ by construction, and Lemma 10.7 gives

$$\{\xi \in X_g : {}_{A_1}\langle \xi, \xi \cdot D_2\rangle \subseteq D_1 \text{ and } \langle D_1 \cdot \xi, \xi\rangle_{A_2} \subseteq D_2\} \subseteq N_\star(D) \quad \text{for all } g \in G.$$

Each $X_g \subseteq X = P_1AP_2$, so $P_1AP_2 = \overline{\operatorname{span}}\{P_1nP_2 : n \in N_\star(D)\}$ by (10.1). □

We call the system (A, D, δ) of Lemma 10.8 the *linking system* for (X, ζ). The elements $P_i := 1_{M(D_i)} = 1_{M(A_i)}$ are complementary full multiplier projections of A and $(P_iAP_i, P_iD, \delta|_{P_iAP_i}) \cong (A_i, D_i, \delta_i)$ for $i = 1, 2$. So the (A_i, D_i, δ_i) are complementary full subsystems of their linking system. The converse requires additional hypotheses.

Lemma 10.9. *Let A be a separable C^*-algebra, δ a coaction of a discrete group Γ on A, and D a weakly Cartan subalgebra of A^δ. Suppose that $P_1, P_2 \in M(D)$ are complementary full projections. Then $\delta_i := \delta|_{P_iAP_i}$ is a coaction of Γ on $A_i = P_iAP_i$ and $D_i := P_iD$ is a weakly Cartan subalgebra of A_i for each i. If $P_1AP_2 = \overline{\operatorname{span}}\{P_1nP_2 : n \in N_\star(D)\}$, then $(P_1AP_2, \delta|_{P_1AP_2})$ is an equivariant Morita equivalence from (A_1, D_1) to (A_2, D_2).*

Proof. Since $P_i \in M(D_i) \subseteq M(A_i^{\delta_i})$, the extension $\tilde\delta$ of δ to $M(A_i^{\delta_i})$ satisfies $\tilde\delta(P_i) = P_i \otimes 1_{C^*_r(\Gamma)}$, so for $g \in \Gamma$ and $a \in A_g$, we have $\delta(P_iaP_i) = \delta(P_i)(a \otimes \lambda_g)\delta(P_i) = P_iaP_i \otimes \lambda_g \in P_iAP_i \otimes C^*_r(\Gamma)$. Fix an approximate unit $(e_j)_j$ for A in D. Then $(P_ie_j)_j$ is an approximate unit for P_iAP_i in D_i, so $\delta(P_ie_jP_i) = \big(P_ie_jP_i \otimes 1_{C^*_r(\Gamma)}\big)_j$ is an approximate unit for $P_iAP_i \otimes C^*_r(\Gamma)$. So δ_i is nondegenerate. The final paragraph of Lemma 10.8 gives isomorphisms $(P_iD)'_{(P_iAP_i)^{\delta_i}}/J_{i,\phi} \cong D'_{A^\delta}/J_\phi$ that carry $\pi_{i,\phi}(P_id)$ to $\pi_\phi(d) = \phi(d)1_{D'_{A^\delta}/J_\phi}$. So $P_iD \subseteq (P_iAP_i)^{\delta_i}$ is weakly Cartan; and $(P_1AP_2, \delta|_{P_1AP_2})$ is an equivariant A_1–A_2-imprimitivity bimodule as discussed before Definition 10.4.

By definition of the inner products, $P_1AP_2 = \overline{\operatorname{span}}\{P_1nP_2 : n \in N_\star(D)\}$ if and only if

$$(P_1AP_2)_g = \overline{\operatorname{span}}\{\xi \in (P_1AP_2)_g : {}_{A_1}\langle \xi, \xi \cdot D_2\rangle \subseteq D_1 \text{ and } \langle D_1 \cdot \xi, \xi\rangle_{A_2} \subseteq D_2\}$$

for each $g \in \Gamma$. The map $\delta|_{P_1AP_2}$ is clearly a bimodule homomorphism. That $(\delta|_{P_1AP_2} \otimes \mathrm{id}_\Gamma) \circ \delta|_{P_1AP_2} = (\mathrm{id}_{P_1AP_2} \otimes \delta_\Gamma) \circ \delta|_{P_1AP_2}$ follows from the coaction identity for δ. □

Corollary 10.10. *Let Γ be a discrete group. For $i = 1, 2$, let δ_i be a coaction of Γ on a separable C^*-algebra A_i, and let $D_i \subseteq A_i^{\delta_i}$ be a weakly Cartan subalgebra such that $\overline{\operatorname{span}} N_\star(D_i) = A_i$. Then (A_1, D_1) and (A_2, D_2) are equivariantly Morita equivalent if and only if there exist a separable C^*-algebra A, a coaction δ of Γ on A, a weakly Cartan subalgebra D of A^δ such that $\overline{\operatorname{span}} N_\star(D) = A$, a pair of complementary full projections $P_1, P_2 \in M(D)$, and isomorphisms $\phi_i : P_iAP_i \to A_i$ such that*

$$\phi_i(P_iDP_i) = D_i \qquad \textit{and} \qquad \delta_i \circ \phi_i = (\phi_i \otimes \mathrm{id}_{C^*_r(\Gamma)}) \circ \delta|_{P_iAP_i} \quad \textit{for } i = 1, 2.$$

Proof. We have $P_2AP_1 = (P_1AP_2)^* \subseteq N_\star(D)$, and each $P_iAP_i \subseteq A \subseteq \overline{\operatorname{span}} N_\star(D)$. So Lemma 10.8 gives "only if." Lemma 10.9 gives "if" because $A = \overline{\operatorname{span}} N_\star(D)$ forces $P_iAP_i = \overline{\operatorname{span}}\{P_inP_i : n \in N_\star(D)\} \subseteq N_\star(D_i)$. □

Lemma 10.11. *Let Γ be a discrete group. For $i = 1, 2$, let δ_i be a coaction of Γ on a separable C^*-algebra A_i, and let $D_i \subseteq A_i^{\delta_i}$ be a weakly Cartan subalgebra. Suppose that (X, ζ) is an equivariant (A_1, D_1)–(A_2, D_2)-equivalence. Let (A, D, δ) be the linking system. Fix $g \in \Gamma$ and $n = \left(\begin{smallmatrix} a_1 & \xi \\ \eta^* & a_2 \end{smallmatrix}\right) \in N_g(D)$. Then $a_1^* \cdot \xi = \xi \cdot a_2^* = a_1 \cdot \eta = \eta \cdot a_2 = 0$. Moreover, $a_1 \in N_g(D_1)$, $a_2 \in N_g(D_2)$, and $\langle D_1 \cdot \xi, \xi\rangle_{A_2} \cup \langle D_1 \cdot \eta, \eta\rangle_{A_2} \subseteq D_2$, and ${}_{A_1}\langle \xi \cdot D_2, \xi\rangle \cup {}_{A_1}\langle \eta \cdot D_2, \eta\rangle \subseteq D_2$.*

Proof. Let $d_1 \in D_1$ and let $d_2 \in D_2$. Then

$$n^*(d_1 \oplus 0)n = \begin{pmatrix} a_1^* d_1 a_1 & a_1^* d_1 \cdot \xi \\ \xi^* \cdot d_1 a_1 & \langle d_1^* \cdot \xi, \xi\rangle_{A_2} \end{pmatrix}, \qquad n^*(0 \oplus d_2)n = \begin{pmatrix} {}_{A_1}\langle \eta, \eta \cdot d_2\rangle & \eta \cdot d_2 a_2 \\ a_2^* d_2 \cdot \eta^* & a_2^* d_2 a_2 \end{pmatrix},$$

$$n(d_1 \oplus 0)n^* = \begin{pmatrix} a_1 d_1 a_1^* & a_1 d_1 \cdot \eta \\ \eta^* \cdot d_1 a_1^* & \langle d_1^* \eta, \eta\rangle_{A_2} \end{pmatrix}, \text{ and } \quad n(0 \oplus d_2)n^* = \begin{pmatrix} {}_{A_1}\langle \xi, \xi \cdot d_2\rangle & \xi \cdot d_2 a_2^* \\ a_2 d_2 \cdot \xi^* & a_2 d_2 a_2^* \end{pmatrix}.$$

That $n^*(d_1 \oplus 0)n, n(d_1 \oplus 0)n^* \in D$, gives $a_1 \in N(D_1)$ and $(a_1^* d_1) \cdot \xi = (\xi . d_2) \cdot a_2^* = (a_1 d_1) \cdot \eta = (\eta \cdot d_2) \cdot a_2 = 0$. Taking the limit as the d_i range over approximate identities for the A_i proves the first assertion. Since $n \in N_g(D)$ we have

$$\begin{pmatrix} a_1 \otimes \lambda_g & \xi \otimes \lambda_g \\ \eta^* \otimes \lambda_g & a_2 \otimes \lambda_g \end{pmatrix} = n \otimes \lambda_g = \delta(n) = \begin{pmatrix} \delta_1(a_1) & \zeta(\xi) \\ \zeta(\eta)^* & \delta_2(a_2) \end{pmatrix},$$

so $a_i \in N_g(D_i)$. Since $n^*(d_1 \oplus 0)n, n(d_1 \oplus 0)n^* \in D_1 \oplus D_2$, we have $\langle d_1^* \cdot \xi, \xi\rangle_{A_2}, \langle d_1^* \eta, \eta\rangle_{A_2} \in D_2$ and ${}_{A_1}\langle \eta, \eta \cdot d_2\rangle, {}_{A_1}\langle \xi, \xi \cdot d_2\rangle \in D_1$, which proves the second assertion. □

In what follows, we sometimes deal with Weyl groupoids for multiple triples (A, D, δ). If $\mathcal{H} = \mathcal{H}(A, D, \delta)$ is a Weyl groupoid, and if $n \in N_\star(D) \subseteq A$ and $\phi \in \operatorname{supp}^\circ(n^*n) \subseteq \widehat{D}$, we write $[n, \phi]_{\mathcal{H}}$ for the corresponding element of $\mathcal{H}$.

Lemma 10.12. *Let Γ be a discrete group. For $i = 1, 2$, let δ_i be a coaction of Γ on a separable C^*-algebra A_i, and let $D_i \subseteq A_i^{\delta_i}$ be a weakly Cartan subalgebra. Suppose that (X, ζ) is an equivariant (A_1, D_1)–(A_2, D_2)-equivalence. Let (A, D, δ) be the linking system. Let $\mathcal{H} = \mathcal{H}(A, D, \delta)$. Suppose that*

$$n = \begin{pmatrix} n_1 & \xi \\ \eta^* & n_2 \end{pmatrix} \in N_g(D) \quad \text{ where } n_1 \in A_1,\ n_2 \in A_2, \text{ and } \xi, \eta \in X.$$

Fix $i \in \{1, 2\}$, and suppose that $\phi \in \operatorname{supp}^\circ(n^*n)$ satisfies $[n, \phi]_{\mathcal{H}} \in \widehat{D_i}\mathcal{H}\widehat{D_i}$. Then $\phi \in \operatorname{supp}^\circ(n_i^* n_i)$, and $[n, \psi]_{\mathcal{H}} = [n_i, \psi]_{\mathcal{H}}$ for all $\psi \in \operatorname{supp}^\circ(n_i^* n_i)$.

Proof. By symmetry, it suffices to prove the result for $i = 1$. By Lemma 10.11, we have $n_1 \in N_g(D_1)$ and $n_2 \in N_g(D_2)$. Since $[n, \phi]_{\mathcal{H}} \in \widehat{D_1}\mathcal{H}\widehat{D_1}$, we have $\phi, \alpha_n(\phi) \in \widehat{D}_1$.

To see that $\phi \in \operatorname{supp}^\circ(n_1^* n_1)$, fix $d_1 \in D_1$ with $\alpha_n(\phi)(d_1) = 1$. Then

$$\begin{aligned} 0 < \phi(n^*n) &= \alpha_n(\phi)(d_1)\phi(n^*n) = \phi(n^* d_1 n) \\ &= \phi \begin{pmatrix} n_1^* d_1 n_1 & n_1^* d_1 \cdot \xi \\ \xi^* \cdot d_1 n_1 & \langle \xi, d_1 \cdot \xi\rangle_{A_2} \end{pmatrix} = \phi(n_1^* d_1 n_1) = \alpha_{n_1}(\phi)(d_1)\phi(n_1^* n_1). \end{aligned}$$

Since Lemma 10.11 gives $n_1 \in N_g(D_1)$, we have $n_1^* n \in A^\delta$. Let $U := \operatorname{supp}^\circ(n_1^* n_1) \subseteq \operatorname{supp}^\circ(n^*n)$ and $\psi \in U$. Then $\psi \in \widehat{D}_1$, and therefore ψ vanishes on D_2. So for $d = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right) \in \widehat{D}_+$ with $\alpha_{n_1}(\psi)(d) = 1$,

$$\begin{aligned} \alpha_n(\psi)(d)\psi(n^*n) = \psi(n^* d n) &= \psi \begin{pmatrix} n_1^* d_1 n_1 + {}_{A_1}\langle \eta \cdot d_2, \eta\rangle & a_1^* d_1 \cdot \xi + \eta \cdot d_2 a_2 \\ \xi^* \cdot d_1 a_1 + a_2^* d_2 \cdot \eta & \langle \xi, d_1 \cdot \xi\rangle_{A_2} + n_2^* d_2 n_2 \end{pmatrix} \\ &= \psi(n_1^* d_1 n_1 + {}_{A_1}\langle \eta \cdot d_2, \eta\rangle) \geq \psi(n_1^* d_1 n_1) = \alpha_{n_1}(\psi)(d)\psi(n_1^* n_1) > 0. \end{aligned}$$

Since $\psi(n^*n) = \psi(n_1^* n_1)$ and D_+ separates points in $\widehat{D}$, this gives $\alpha_n(\psi) = \alpha_{n_1}(\psi)$, so α_{n_1} and α_n agree on U. To see that $[n, \psi]_{\mathcal{H}} = [n_1, \psi]_{\mathcal{H}}$ it remains only to establish (R4). We have $n^* n_1 = \left(\begin{smallmatrix} n_1^* n_1 & 0 \\ \xi^* n_1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} n_1^* n_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Thus for $d \in D$ such that $\psi(d) = 1$ and $\operatorname{supp}^\circ(d) \subseteq U$, we have $\pi_\psi\big((\psi(n^*n)^{-1/2}\psi(n_1^* n_1))^{-1/2}(d n^* n_1 d)\big) = 1_{D'_{A^\delta}/J_\phi}$. Hence, $[n, \psi] = [n_1, \psi]$. □

Proof of Theorem 10.6. We have that δ is a coaction and D is weakly Cartan in A^δ by Lemma 10.8. Clearly the $\widehat{D}_i$ are complementary open subsets of $\widehat{D}$. We show that $\widehat{D}_1$ is full ($\widehat{D}_2$ is full by a symmetric argument). Fix $\phi \in \widehat{D}_2$. Since X is full in A_2, there exists $\xi \in X$ such that $\langle \xi, \xi\rangle_{A_2} \in D_2$ and $\phi(\langle \xi, \xi\rangle_{A_2}) \neq 0$. Lemma 10.8 shows that δ is a coaction, so as in Section 1, we have $A = \overline{\operatorname{span}} \bigcup_{g\in\Gamma} A_g$. Let $P_i = 1_{M(D_i)} \in M(A)$. Then $X = P_1 A P_2 = \overline{\operatorname{span}} \bigcup_{g\in\Gamma} P_1 A_g P_2 = \overline{\operatorname{span}} \bigcup_{g\in\Gamma} X_g$. Hence (10.1) shows that

$$X = \overline{\operatorname{span}} \bigcup\nolimits_{g\in\Gamma} \{\eta \in X_g : {}_{A_1}\langle \eta, \eta\cdot D_2\rangle \subseteq D_1 \text{ and } \langle D_1\cdot \eta, \eta\rangle_{A_2} \subseteq D_2\}.$$

In particular, we can approximate ξ by an element $\sum_j \eta_j$ with each $\eta_j \in N_\star(D)$ satisfying $\eta_j^* D_2 \eta_j \subseteq D_1$. Since $\phi(\langle \xi, \xi\rangle_{A_2}) \neq 0$, we may assume that there exist j, k such that $\phi(\langle \eta_j, \eta_k\rangle_{A_2}) \neq 0$. Using that X is an imprimitivity bimodule at the last step, we calculate:

$$\begin{aligned} 0 \neq |\phi(\langle \eta_j, \eta_k\rangle_{A_2})|^2 &= \phi(\langle \eta_j, \eta_k\rangle_{A_2}\langle \eta_k, \eta_j\rangle_{A_2}) \\ &= \phi\big(\big\langle \langle \eta_j, \eta_k \cdot \langle \eta_k, \eta_j\rangle_{A_2}\big\rangle_{A_2}\big) = \phi\big(\big\langle \eta_j, {}_{A_1}\langle \eta_k, \eta_k\rangle \cdot \eta_j\rangle_{A_2}\big\rangle_{A_2}\big). \end{aligned}$$

Rewriting this in terms of multiplication in A, we obtain $\phi(\eta_j^*\eta_k\eta_k^*\eta_j) \neq 0$, and therefore $\eta := \eta_j$ satisfies $\phi(\eta\eta^*)\alpha_\eta(\phi)(\eta_k\eta_k^*) \neq 0$. In particular, $\eta \in N_\star(D)$ and $\phi \in \operatorname{dom}(\alpha_\eta)$. Since $\eta D_2 \eta^* \subseteq D_1$, we have $r([\eta, \phi]) = \alpha_\eta(\phi) \in \widehat{D}_1$. So $\phi = s([\eta, \phi]) \in s(\widehat{D}_1\mathcal{H})$.

We must show that each $\widehat{D}_i\mathcal{H}\widehat{D}_i \cong \mathcal{H}(A_i, D_i, \delta_i)$. It suffices to do this for $i = 1$. Let $j_1 : A_1 \to A$ be the inclusion map. For $\phi \in \widehat{D}_1$, let $\overline{\phi}$ be the extension of ϕ to D given by $\overline{\phi}(d_1) = \phi(d_1)$ for $d_1 \in D_1$ and $\overline{\phi}(d_2) = 0$ for $d_2 \in D$.

We claim that there is a map $\Theta\colon \mathcal{H}(A_1, D_1, \delta_1) \to \widehat{D}_1\mathcal{H}(A, D, \delta)\widehat{D}_1$ such that

$$\Theta([n, \phi]_{\mathcal{H}_1}) = [j_1(n), \overline{\phi}]_{\mathcal{H}} \quad \text{for all } [n, \phi] \in \mathcal{H}(A, D, \delta).$$

For this, suppose that $[n, \phi]_{\mathcal{H}_1} = [m, \psi]_{\mathcal{H}_1}$. Then $\phi = \psi$ by definition of $\sim$ in (A_1, D_1, δ_1), so $(j_1(n), \overline{\phi})$ and $(j_1(m), \overline{\psi})$ satisfy (R1). Since $n^*m \in A_1^{\delta_1}$, we have $n, m \in N_g(D_1)$ for some g. Hence $\delta(j_1(n)) = j_1(n) \otimes \lambda_g$ and $\delta(j_1(m)) = j_1(m) \otimes \lambda_g$, so $j_1(n)^*j_1(m) \in A^\delta$, giving (R2). Since $(n, \phi), (m, \psi)$ satisfy (R3), there is a neighbourhood $U \subseteq \widehat{D}_1$ of ϕ such that $U \subseteq \operatorname{supp}^\circ(n^*n) \cap \operatorname{supp}^\circ(m^*m)$, and $\alpha_n|_U = \alpha_m|_U$. Since $\widehat{D}_1$ is open in $\widehat{D}$, the set $\overline{U} := \{\overline{\psi} : \psi \in U\}$ is a neighbourhood of $\overline{\phi}$ in $\widehat{D}$. Lemma 10.12 gives $\alpha_{j_1(n)}|_{\overline{U}} = \alpha_n|_{\overline{U}} = \alpha_m|_{\overline{U}} = \alpha_{j_1(m)}|_{\overline{U}}$. So $(j_1(n), \overline{\phi})$ and $(j_1(m), \overline{\psi})$ satisfy (R3). Lemma 10.8 gives $D'_{A^\delta} = (D_1)'_{A_1^{\delta_1}} \oplus (D_2)'_{A_2^{\delta_2}}$. Since $J_{\overline{\phi}} \lhd D'_{A^\delta}$ is generated by $\{d \in D : \phi(d) = 0\}$, the corresponding ideal $J_{1,\phi} \lhd (D_1)'_{A_1^{\delta_1}}$ satisfies $J_{\overline{\phi}} = j_1(J_{1,\phi}) \oplus (D_2)'_{A_2^{\delta_2}}$. Hence the projection map $p_1 : D'_{A^\delta} \to (D_1)'_{A_1^{\delta_1}}$ descends to an isomorphism $\tilde{p}_1 : D'_{A^\delta}/J_{\overline{\phi}} \cong (D_1)'_{A_1^{\delta_1}}/J_{1,\phi}$. By definition of $U^\phi_{n^*m}$ and $U^{\overline{\phi}}_{j_n(n)^*j_1(m)}$ as in Notation 4.3,

$$\tilde{p}_1(U^{\overline{\phi}}_{j_n(n)^*j_1(m)}) = U^\phi_{n^*m}. \tag{10.2}$$

Since $U^\phi_{n^*m} \sim_h 1$, we deduce that $U^{\overline{\phi}}_{j_n(n)^*j_1(m)} \sim_h 1$. So $[j_1(n), \overline{\phi}] = [j_1(m), \overline{\psi}]$, and Θ is well defined. For any $[n, \phi]_{\mathcal{H}_1} \in \mathcal{H}_1$, we have $s(\Theta([n, \phi])) = \overline{\phi} \in \widehat{D}_1$, and $r(\Theta([n, \phi])) = \alpha_{j_1(n)}(\overline{\phi}) = \overline{\alpha_n(\phi)} \in \widehat{D}_1$ by Lemma 10.12. So $\Theta(\mathcal{H}(A_1, D_1, \delta_1)) \subseteq \widehat{D}_1\mathcal{H}\widehat{D}_1$.

To see that Θ is surjective, fix $[n, \phi]_{\mathcal{H}} \in \widehat{D}_1\mathcal{H}\widehat{D}_1$. Write $n = \left(\begin{smallmatrix} n_1 & \xi \\ \eta^* & n_2 \end{smallmatrix}\right)$. Then Lemma 10.12 gives $[n, \phi]_{\mathcal{H}} = [j_1(n_1), \overline{\phi}]_{\mathcal{H}} = \Theta([n_1, \phi]_{\mathcal{H}_1})$. For injectivity, suppose that $\Theta([n, \phi]_{\mathcal{H}_1}) = \Theta([m, \psi]_{\mathcal{H}_1})$. Then $\phi = \psi$, and $n^*m \in A_1^{\delta_1}$ because $j_1(n^*m) \in A^\delta$ and $\delta \circ j_1 = (j_1 \otimes$

$\mathrm{id}_{C^*_r(\Gamma)}) \circ \delta_1$. Lemma 10.12 gives $\alpha_{j_1(n)} = \alpha_n$ and $\alpha_{j_1(m)} = \alpha_m$. So α_n and α_m agree on a neighbourhood of ϕ. Since $U^{\overline{\phi}}_{j_1(n)^* j_1(m)} \in \mathcal{U}_0(D'_{A^\delta}/J_\phi)$, Equation (10.2) gives $U^{\phi}_{n^*m} \in \mathcal{U}_0\big((D_1)'_{A_1^{\delta_1}}/J_{1,\phi}\big)$. So $[n,\phi]_{\mathcal{H}_1} = [m,\psi]_{\mathcal{H}_1}$.

For $[n,\phi]_{\mathcal{H}_1} \in \mathcal{H}_1$, we have $s(\Theta([n,\phi]_{\mathcal{H}_1})) = s([j_1(n),\overline{\phi}]_{\mathcal{H}}) = \overline{\phi} = \overline{s([n,\phi]_{\mathcal{H}_1})}$, so Lemma 10.12 gives $r(\Theta([n,\phi]_{\mathcal{H}_1})) = \alpha_{j_1(n)}(\overline{\phi}) = \overline{\alpha_n(\phi)} = \overline{r([n,\phi]_{\mathcal{H}_1})}$. Furthermore, if $[n,\phi]_{\mathcal{H}_1}$ and $[m,\psi]_{\mathcal{H}_1}$ are composable, then since j_1 is a homomorphism,

$$\Theta([n,\phi]_{\mathcal{H}_1})\Theta([m,\psi]_{\mathcal{H}_1}) = [j_1(n),\overline{\phi}]_{\mathcal{H}}[j_1(m),\overline{\psi}]_{\mathcal{H}} = [j_1(nm),\overline{\psi}]_{\mathcal{H}} = \Theta([n,\phi]_{\mathcal{H}_1}[m,\psi]_{\mathcal{H}_1}).$$

Thus Θ is a groupoid homomorphism.

We must show that Θ is a homeomorphism. For $n \in N_\star(D_1)$ and $U \subseteq \mathrm{supp}^\circ(n^*n) \subseteq \widehat{D}_1$ open, $\Theta(Z(n,U)) = Z(j_1(n),\overline{U})$. Hence Θ is an open map. Now fix a basic open set $Z(n,U) \cap \widehat{D}_1\mathcal{H}\widehat{D}_1 \subseteq \widehat{D}_1\mathcal{H}\widehat{D}_1$. Write $n = \left(\begin{smallmatrix} n_1 & \xi \\ \eta^* & n_2 \end{smallmatrix}\right)$ and let $U' := \{\phi \in \widehat{D_1} : \overline{\phi} \in U\}$. It then follows from Lemma 10.12 that if $[n,\phi]_{\mathcal{H}} \in Z(n,U) \cap \widehat{D}_1\mathcal{H}\widehat{D}_1$, then $\Theta([n_1,\phi_1]) = [j_1(n_1),\overline{\phi_1}] = [n,\phi]$ where ϕ_1 is the element of $\widehat{D_1}$ such that $\overline{\phi_1} = \phi$. Hence $\Theta^{-1}(Z(n,U) \cap \widehat{D}_1\mathcal{H}\widehat{D}_1) = Z(n_1,U')$ is open. □

11. Equivalence of graded groupoids and equivariant Morita equivalence

In this section we show how equivalence of graded groupoids relates to equivariant Morita equivalence of pairs (A,D). Our main result in this direction is the following.

Theorem 11.1. *Let Γ be a discrete group and let $(G_1,c_1),(G_2,c_2)$ be Γ-graded second-countable locally compact Hausdorff étale groupoids such that each $\mathrm{Iso}(c_i^{-1}(e))^\circ$ is torsion-free and abelian. The following are equivalent:*

1. *the graded groupoids (G_1,c_1) and (G_2,c_2) are graded equivalent;*
2. *there exist a second-countable locally compact Hausdorff étale groupoid G, a grading c of G by Γ such that $\mathrm{Iso}(c^{-1}(e))^\circ$ is torsion-free and abelian, a pair of complementary open G-full subsets $K_1,K_2 \subseteq G^{(0)}$, and isomorphisms $\kappa_1 : K_1GK_1 \to G_1$ and $\kappa_2 : K_2GK_2 \to G_2$ such that $c_i \circ \kappa_i = c|_{K_iGK_i}$;*
3. *$(C^*_r(G_1), C_0(G_1^{(0)}))$ and $(C^*_r(G_2), C_0(G_2^{(0)}))$ are equivariantly Morita equivalent; and*
4. *there exist a separable C^*-algebra A, a coaction δ of Γ on A, a weakly Cartan subalgebra D of A^δ such that $\overline{\mathrm{span}}N_\star(D) = A$, a pair of complementary full projections $P_1,P_2 \in M(D) \subseteq M(A)$, and isomorphisms $\phi_i : P_iAP_i \to C^*_r(G_i)$ such that $\phi_i(P_iDP_i) = D_i$ and $\delta_i \circ \phi_i = (\phi_i \otimes \mathrm{id}_{C^*_r(\Gamma)}) \circ \delta|_{P_iAP_i}$.*

Proof. Lemma 10.3 gives (1) $\iff$ (2), Corollary 10.10 gives (3) $\iff$ (4), and Theorem 10.6 gives (3) $\implies$ (1). For (2) $\implies$ (4), let (G,c), K_i and κ_i be as in (2). Lemma 6.4 shows that $C_0(G^{(0)})$ is a weakly Cartan subalgebra of $C^*_r(G)^{\delta_c}$. Theorem 4.1 of [44] shows that $P_i := 1_{K_i} \in M(C_0(G^{(0)}))$ defines complementary full projections, and the inclusions $C_c(G_i) \hookrightarrow C_c(G)$ induce isomorphisms $\phi_i : P_iC^*_r(G)P_i \to C^*_r(G_i)$ that carry $P_iC_0(G^{(0)})$ to $C_0(G_i^{(0)})$. We have $P_1C^*_r(G)P_2 = \overline{C_c(G_1^{(0)}GG_2^{(0)})} \subseteq C^*_r(G)$. Since the G_i are étale, the Haar system of [44, Lemma 2.2] consists of counting measures, so G is also étale. Hence Lemma 6.7 gives $P_1C^*_r(G)P_2 = \overline{\mathrm{span}}\{P_1nP_2 : n \in N_\star(C_0(G^{(0)}))\}$. Fix $i \in \{1,2\}$. For $g \in \Gamma$ and $f \in C_c(G_i)_g$, $\mathrm{supp}(\phi_i(f)) = \mathrm{supp}(f) \subseteq c_i^{-1}(g)$, and so $\delta_i(\phi_i(f)) = \phi_i(f) \otimes \lambda_g = (\phi_i \otimes \mathrm{id}_{C^*_r(\Gamma)})(f \otimes \lambda_g) = (\phi_i \otimes \mathrm{id}_{C^*_r(\Gamma)})(\delta(f))$. Since $P_iC^*_r(G)P_i = \overline{\mathrm{span}} \bigcup_g C_c(G_i)_g$, we obtain $\delta_i \circ \phi_i = (\phi_i \otimes \mathrm{id}_{C^*_r(\Gamma)}) \circ \delta|_{P_iAP_i}$. □

In some situations the following stronger form of equivalence than the one considered in Theorem 11.1, is interesting (see Corollaries 11.3 and 11.5). If Γ is the trivial group, then Theorems 11.1 and 11.2 both reduce to Theorem 3.5.

Theorem 11.2. *Let Γ be a locally compact group, let G_1 and G_2 be second-countable locally compact Hausdorff étale groupoids, and let $c_i : G_i \to \Gamma$ be continuous cocycles such that $\mathrm{Iso}(c_i^{-1}(e))^\circ$ is torsion-free and abelian. The following are equivalent:*

(1) *there is a graded (G_1, c_1)–(G_2, c_2)-equivalence (Z, c_Z) such that $c_Z^{-1}(e)$ is a $c_1^{-1}(e)$–$c_2^{-1}(e)$-equivalence;*
(2) *there exist a second-countable locally compact Hausdorff étale groupoid G, a grading c of G by Γ such that $\mathrm{Iso}(c^{-1}(e))^\circ$ is torsion-free and abelian, a pair of complementary open $c^{-1}(e)$-full subsets $K_1, K_2 \subseteq G^{(0)}$, and isomorphisms $\kappa_1 : K_1GK_1 \to G_1$ and $\kappa_2 : K_2GK_2 \to G_2$ such that $c_i \circ \kappa_i = c|_{K_iGK_i}$;*
(3) *there exists an equivariant $(C_r^*(G_1), C_0(G_1^{(0)}))$–$(C_r^*(G_2), C_0(G_2^{(0)}))$-imprimitivity bimodule (X, ζ) such that X_e is a $C_r^*(G_1)^{\delta_{c_1}}$–$C_r^*(G_2)^{\delta_{c_2}}$-imprimitivity bimodule; and*
(4) *there exist a separable C^*-algebra A, a coaction δ of Γ on A, a weakly Cartan subalgebra D of A^δ such that $\overline{\mathrm{span}} N_\star(D) = A$, a pair of complementary A^δ-full projections $P_1, P_2 \in M(D)$, and isomorphisms $\phi_i : P_iAP_i \to C_r^*(G_i)$ such that $\phi_i(P_iDP_i) = D_i$ and $\delta_i \circ \phi_i = (\phi_i \otimes \mathrm{id}_{C_r^*(\Gamma)}) \circ \delta|_{P_iAP_i}$.*

Proof. (1) $\Longrightarrow$ (2). By Lemma 10.3, $G := L(G_1, G_2)$ and $c : G \to \Gamma$ given by $c|_{G_i} = c_i$, $c|_Z = c_Z$ and $c(z^{\mathrm{op}}) = c_Z(z)^{-1}$ for $z \in Z$ constitute a graded groupoid with $\mathrm{Iso}(c^{-1}(e))^\circ \cong \mathrm{Iso}(c_1^{-1}(e))^\circ \sqcup \mathrm{Iso}(c_2^{-1}(e))^\circ$ torsion-free and abelian, and each $(G_i^{(0)}GG_i^{(0)}, c) \cong (G_i, c_i)$. Since $c_Z^{-1}(e)$ is a $c_1^{-1}(e)$–$c_2^{-1}(e)$-equivalence, each $G_i^{(0)}$ is $c^{-1}(e)$-full.

The proof of Corollary 10.10 gives (3) $\Longleftrightarrow$ (4), and Theorem 10.6 gives (3) $\Longrightarrow$ (1).

For (2) $\Longrightarrow$ (4), the proof of (2) $\Longrightarrow$ (4) in Theorem 11.1 shows that $P_i := 1_{K_i}$ defines full multiplier projections in $M(C_r^*(G))$ such that $P_iC_r^*(G)P_i$ is equivariantly isomorphic to $C^*(G_i)$. To see that P_1 is $C_r^*(G)_e$-full, we show that $C_c(G^{(0)}) \subseteq C_r^*(G)_eP_1C_r^*(G)_e$. It suffices to show that for each $x \in G^{(0)}$ there exists $a \in C_r^*(G)_eP_1C_r^*(G)_e \cap C_0(G^{(0)})$ such that $a(x) > 0$. This is clear for $x \in X_1$, so fix $x \in X_2$. Since X_1 is $c^{-1}(e)$-full there is an open bisection $U \subseteq c^{-1}(e)$ with $x \in r(U)$ and $s(U) \subseteq X_1$. Write $U \cap r^{-1}(x) = \{\gamma\}$. Fix $f \in C_c(U)$ with $f(\gamma) = 1$. Then $ff^* = fPf^* \in C_0(X_2) \cap C_r^*(G)_eP_1C_r^*(G)_e$, and $ff^*(x) = 1$. Symmetry shows that P_2 is also full. □

Next we specialise to ample graded groupoids. Recall that $\mathcal{R}$ is the discrete groupoid $\mathbb{N} \times \mathbb{N}$, and c_0 denotes the canonical diagonal in $C^*(\mathcal{R}) \cong \mathcal{K}$. Given a grading $c : G \to \Gamma$ of a groupoid G, we define $\bar{c} : G \times \mathcal{R} \to \Gamma$ by $\bar{c}(\eta_1, \eta_2) = c(\eta_1)$.

As in [13], we say G_1 and G_2 are *weakly Kakutani c_1–c_2 equivalent* if there are open $c_i^{-1}(e)$-full subsets $X_i \subseteq G_i^{(0)}$ and an isomorphism $\kappa : X_1G_1X_1 \to X_2G_2X_2$ such that $c_2 \circ \kappa = c_1$ on $X_1G_1X_1$. As in [31], if X_1, X_2 are clopen, we say G_1 and G_2 are *Kakutani c_1–c_2 equivalent.* Theorem 11.2 combined with the results of [13] gives the following.

Corollary 11.3. *Let Γ be a discrete group, and let $(G_1, c_1), (G_2, c_2)$ be second-countable Γ-graded ample Hausdorff groupoids such that each $\mathrm{Iso}(c_i^{-1}(e))^\circ$ is torsion-free and abelian. The following are equivalent:*

(1) *there is a graded (G_1, c_1)–(G_2, c_2)-equivalence (Z, c_Z) such that $c_Z^{-1}(e)$ is a $c_1^{-1}(e)$–$c_2^{-1}(e)$-equivalence;*

(2) *there exist a second-countable ample Hausdorff groupoid G, a grading c of G by Γ such that $\operatorname{Iso}(c^{-1}(e))^\circ$ is torsion-free and abelian, a pair of complementary clopen $c^{-1}(e)$-full subsets $X_1, X_2 \subseteq G^{(0)}$, and isomorphisms $\kappa_i : X_iGX_i \to G_i$ such that $X_1 \cup X_2 = G^{(0)}$, and each $c_i \circ \kappa_i|_{X_iGX_i} = c|_{X_iGX_i}$;*
(3) *there is an isomorphism $\kappa : G_1 \times \mathcal{R} \to G_2 \times \mathcal{R}$ such that $\bar{c}_2 \circ \kappa = \bar{c}_1$;*
(4) *G_1 and G_2 are Kakutani c_1–c_2 equivalent;*
(5) *G_1 and G_2 are weakly Kakutani c_1–c_2 equivalent;*
(6) *there is an equivariant $(C^*_r(G_1), C_0(G_1^{(0)}))$–$(C^*_r(G_2), C_0(G_2^{(0)}))$-imprimitivity bimodule (X, ζ) such that X_e is a $C^*_r(c_1^{-1}(e))$–$C^*_r(c_2^{-1}(e))$-imprimitivity bimodule;*
(7) *there exist a separable C^*-algebra A, a coaction δ of Γ on A, a weakly Cartan subalgebra D of A^δ, a pair of complementary A^δ-full projections $P_1, P_2 \in M(D)$, and isomorphisms $\phi_i : P_1AP_1 \to C^*_r(G_i)$ such that $P_1AP_2 = \overline{\operatorname{span}}\{P_1nP_2 : n \in N_\star(D)\}$ and such that $\phi_i(P_iDP_i) = D_i$ and $\delta_i \circ \phi_i = (\phi_i \otimes \mathrm{id}_{C^*_r(\Gamma)}) \circ \delta|_{P_iAP_i}$;*
(8) *there is an isomorphism $\phi : C^*_r(G_1) \otimes \mathcal{K} \to C^*_r(G_2) \otimes \mathcal{K}$ satisfying $\phi(C_0(G_1^{(0)}) \otimes c_0) = C_0(G_2^{(0)}) \otimes c_0$ and $(\delta_{c_2} \otimes \mathrm{id}_{\mathcal{K}}) \circ \phi = (\phi \otimes \mathrm{id}_{C^*_r(\Gamma)}) \circ (\delta_{c_1} \otimes \mathrm{id}_{\mathcal{K}})$;*
(9) *there are $C^*_r(c_i^{-1}(e))$-full projections $p_i \in M(C_0(G_1^{(0)}))$ and an isomorphism $\phi : p_1C^*_r(G_1)p_1 \to p_2C^*_r(G_2)p_2$ such that $\phi(p_1C_0(G_1^{(0)})) = p_2C_0(G_1^{(0)})$, and $\delta_{c_2} \circ \phi = (\phi \otimes \mathrm{id}_{C^*_r(\Gamma)}) \circ \delta_{c_1}$ on $p_1C^*_r(G_1)p_1$; and*
(10) *there are $C^*_r(c_i^{-1}(e))$-fill ideals $I_i \subseteq C_0(G_i^{(0)})$ and an isomorphism $\phi : I_1C^*_r(G_1)I_1 \to I_2C^*_r(G_2)I_2$ such that $\phi(I_1) = I_2$ and $\delta_{c_2} \circ \phi = (\phi \otimes \mathrm{id}_{C^*_r(\Gamma)}) \circ \delta_{c_1}$ on $I_1C^*_r(G_1)I_1$.*

Proof. Theorem 11.2 gives (1) $\iff$ (2) $\iff$ (6) $\iff$ (7). For equivalence of (1) and (3)–(5), we just summarise the modifications needed to the arguments of [13]. The proof of (3) $\implies$ (1) follows the second paragraph of the proof of [13, Theorem 2.1]: if $\kappa : G_1 \times \mathcal{R} \to G_2 \times \mathcal{R}$ is an isomorphism satisfying $\bar{c}_2 \circ \kappa = \bar{c}_1$, then each $G_i \times (\{1\} \times \mathbb{N})$ is a (G_i, c_i)–$(G_i \times \mathcal{R}, \bar{c}_i)$-equivalence, so Lemma 10.2 shows that (G_1, c_1) and (G_2, c_2) are equivalent. The argument of [13, Theorem 2.1] gives (1) $\implies$ (3) once we prove that given (G, c) and a clopen $c^{-1}(e)$-full $K \subseteq G^{(0)}$ there is an isomorphism $G \times \mathcal{R} \cong KGK \times \mathcal{R}$ that is equivariant for $\bar{c}$ and $\bar{c}|_{KGK \times \mathcal{R}}$. For this, apply [13, Lemma 2.4] to $c^{-1}(e)$ and K to obtain a bisection $Y \subseteq c^{-1}(e) \times \mathcal{R}$ with range $K \times \mathbb{N}$ and source $G^{(0)} \times \mathbb{N}$; then conjugation by this Y implements a graded isomorphism $G \times \mathcal{R} \cong KGK \times \mathcal{R}$.

For (1) $\iff$ (4) $\iff$ (5), we follow the proof of [13, Theorem 3.2]. The implications (4) $\implies$ (5) $\implies$ (1) follow the first paragraph of that proof using Lemma 10.2 for (5) $\implies$ (1). For (1) $\implies$ (4), we follow the proof of [13, Theorem 3.2] observing that since $r(Z_e) = G_1^{(0)}$ and $s(Z_e) = G_2^{(0)}$, we can choose the bisections V_i to belong to Z_e. Thus the bisection Y obtained in the penultimate paragraph of the proof satisfies $Y \subseteq c^{-1}(e)$. As in the proof of [13, Theorem 3.2], conjugation by Y is an isomorphism $r(Y)G_1r(Y) \cong s(Y)G_2s(Y)$, and it is graded because $Y \subseteq c^{-1}(e)$.

Theorem 6.2 gives (3) $\iff$ (8), so it suffices to prove (4) $\implies$ (9) $\implies$ (10) $\implies$ (5).

For (4) $\implies$ (9), suppose that G_1 and G_2 are Kakutani c_1–c_2-equivalent with respect to $X_i \subseteq G_i^{(0)}$ and $\kappa : X_1G_1X_1 \to X_2G_2X_2$. Then $p_i := 1_{X_i} \in M(C^*(G_i))$ defines $C^*_r(G_i)$-full projections such that $p_iC^*_r(G_i)p_i \cong C^*_r(X_iG_iX_i)$, and κ induces a δ_{c_1}–δ_{c_2}-equivariant isomorphism $C^*(X_iG_iX_i) \to C^*(X_iG_iX_i)$.

For (9) $\implies$ (10), take $I_i = p_iC_0(G_i^{(0)})$. For (10) $\implies$ (5), fix $I_i \subseteq C_0(G_i^{(0)})$ and ϕ as in (10). Each $I_i = C_0(U_i)$ for some open $U_i \subseteq G_i^{(0)}$. Since each I_i is $C^*_r(c_i^{-1}(e))$-full, each

U_i is $c_i^{-1}(e)$-full by [43, Theorem 4.3.3]. As in the proof of [43, Theorem 3.4.4], restriction of functions induces isomorphisms $R_i : I_i C_r^*(G_i) I_i \to C_r^*(U_i G_i U_i)$. Now $\psi := R_2 \circ \phi \circ R_1^{-1} : C_r^*(U_1 G_1 U_1) \to C_r^*(U_2 G_2 U_2)$ is a weak-Cartan-preserving equivariant isomorphism. Hence (3) $\implies$ (2) in Theorem 11.1 yields a graded isomorphism $\kappa : U_1 G_1 U_1 \to U_2 G_2 U_2$. $\square$

11.1. **Stable continuous orbit equivalence.** Let X be a locally compact Hausdorff space, and $\sigma : X \to X$ a local homeomorphism. Let $\widetilde{X} := X \times \mathbb{N}$ with the product topology and define a (surjective) local homeomorphism $\tilde{\sigma} : \widetilde{X} \to \widetilde{X}$ by $\tilde{\sigma}(x, 0) = (\sigma(x), 0)$ and $\tilde{\sigma}(x, n+1) = (x, n)$. We call $(\tilde{X}, \tilde{\sigma})$ the *stabilisation* of (X, σ). Then $G(\widetilde{X}, \tilde{\sigma}) \cong G(X, \sigma) \times \mathcal{R}$ via $((x, m), p, (y, n)) \mapsto \big((x, p - m + n, y), (m, n)\big)$. We have the following partial generalisation of [12, Corollary 6.3]

Corollary 11.4. *Let $\sigma : X \to X$ and $\tau : Y \to Y$, be local homeomorphisms of second-countable locally compact totally disconnected Hausdorff spaces. The following are equivalent:*

(1) *there is a stabiliser-preserving continuous orbit equivalence from $(\widetilde{X}, \tilde{\sigma})$ to $(\widetilde{Y}, \tilde{\tau})$;*
(2) *$G(X, \sigma)$ and $G(Y, \tau)$ are equivalent;*
(3) *$(C^*(G(X, \sigma)), C_0(X))$ and $(C^*(G(Y, \tau)), C_0(Y))$ are Morita equivalent; and*
(4) *there is an isomorphism $C^*(G(X, \sigma)) \otimes \mathcal{K} \to C^*(G(Y, \tau)) \otimes \mathcal{K}$ that carries $C_0(X) \otimes c_0$ to $C_0(Y) \otimes c_0$.*

Proof. Theorem 8.2 shows that (1) holds if and only if $G(\widetilde{X}, \tilde{\sigma}) \cong G(\widetilde{Y}, \tilde{\tau})$, so the discussion preceding the corollary shows that (1) holds if and only if $G(X, \sigma) \times \mathcal{R} \cong G(Y, \tau) \times \mathcal{R}$. Now the result follows from (1) $\iff$ (3) $\iff$ (6) $\iff$ (8) in Corollary 11.3 applied to the trivial cocycles on $G_1 = G(X, \sigma)$ and $G_2 = G(Y, \tau)$. $\square$

Let $\sigma : X \to X$ be a surjective local homeomorphism of a compact Hausdorff space. Let $\overline{X} := \{\xi \in X^{\mathbb{Z}} : \sigma(\xi_n) = \xi_{n+1} \text{ for every } n \in \mathbb{Z}\}$, and define $\overline{\sigma} : \overline{X} \to \overline{X}$ by $\overline{\sigma}(\xi)_n = \sigma(\xi_n)$. We call σ *expansive* if there is a metrisation (X, d) of X and an $\epsilon > 0$ such that $\sup_n d(\sigma^n(x), \sigma^n(x')) < \epsilon \implies x = x'$. We call ϵ an *expansive constant for* (X, d, σ). The following generalises [11, Theorem 5.1].

Corollary 11.5. *Let X, Y be second-countable locally compact totally disconnected Hausdorff spaces and let $\sigma : \mathrm{dom}(\sigma) \to \mathrm{ran}(\sigma)$ and $\tau : \mathrm{dom}(\tau) \to \mathrm{ran}(\tau)$ be local homeomorphisms between open subsets of X, Y. The following are equivalent:*

(1) *there are continuous, open maps $f : X \to Y$ and $f' : Y \to X$, and continuous maps $a : X \to \mathbb{N}$, $k : \mathrm{dom}(\sigma) \to \mathbb{N}$, $a' : Y \to \mathbb{N}$, and $k' : \mathrm{dom}(\tau) \to \mathbb{N}$ such that $\sigma^{a(x)}(f'(f(x))) = \sigma^{a(x)}(x)$ for $x \in X$, $\tau^{k(x)}(f(\sigma(x))) = \tau^{k(x)+1}(f(x))$ for $x \in \mathrm{dom}(\sigma)$, $\tau^{a'(y)}(f(f'(y))) = \tau^{a'(y)}(y)$ for $y \in Y$, and $\sigma^{k'(y)}(f'(\tau(y))) = \sigma^{k'(y)+1}(f'(y))$ for $y \in \mathrm{dom}(\tau)$;*
(2) *there is a graded $(G(X, \sigma), c_X)$–$(G(Y, \tau), c_Y)$-equivalence (Z, c_Z) such that $c_Z^{-1}(0)$ is a $c_X^{-1}(0)$–$c_Y^{-1}(0)$-equivalence;*
(3) *there is an isomorphism $\kappa : G(X, \sigma) \times \mathcal{R} \to G(Y, \tau) \times \mathcal{R}$ such that $\overline{c}_Y \circ \kappa = \overline{c}_X$;*
(4) *there is a $\mathbb{T}$-equivariant Morita equivalence between $(C^*(G(X, \sigma)), C_0(X))$ and $(C^*(G(Y, \tau)), C_0(Y))$ with respect to the gauge actions γ^X and γ^Y whose fixed-point submodule is a $C^*(G(X, \sigma)$–$C^*(G(Y, \tau)$-imprimitivity bimodule; and*
(5) *there is an isomorphism $C^*(G(X, \sigma)) \otimes \mathcal{K}(\mathbb{N}) \to C^*(G(Y, \tau)) \otimes \mathcal{K}(\mathbb{N})$ that carries $C_0(X) \otimes c_0$ to $C_0(Y) \otimes c_0$, and intertwines the actions $\gamma^X \otimes \mathrm{id}$ and $\gamma^Y \otimes \mathrm{id}$.*

If X and Y are compact, $\mathrm{dom}(\sigma) = \mathrm{ran}(\sigma) = X$*, and* $\mathrm{dom}(\tau) = \mathrm{ran}(\tau) = Y$*, then each of the above five conditions implies*

(6) *$(\overline{X}, \overline{\sigma})$ and $(\overline{Y}, \overline{\tau})$ are conjugate.*

If in addition σ and τ are expansive, then all six conditions are equivalent.

Remark 11.6. If (X, d) is a totally disconnected metric space and $\sigma : X \to X$ is a surjective expansive local homeomorphism, then (X, σ) is conjugate to the edge shift of a finite graph with no sinks or sources (see [4, Section 1] and [25, Theorem 1]), so the equivalence of (2), (5) and (6) follows from Theorem 8.10 and [11, Theorem 5.1].

Proof of Corollary 11.5. Equivalence of (2)–(5) follows from equivalence of statements (1), (3), (6) and (8) in Corollary 11.3.

(1) $\implies$ (2). For each $n \in \mathbb{N}$, fix a countable family I_n of mutually disjoint compact open subsets of X such that $\sigma^n|_U$ is a homeomorphism for each $U \in I_n$ and $\mathrm{dom}(\sigma^n) = \bigcup_{U \in I_n} U$. Fix an injection $\iota : \bigcup_n \{n\} \times I_n \to \mathbb{N}$. For $x \in X$, let U_x be the element of $I_{a(x)}$ containing x, and let $b(x) = \iota(a(x), U_x)$. Let $Z := \{(f(x), b(x)) : x \in X\} \subseteq Y \times \mathbb{N}$. Since a is continuous and f is open, Z is open.

Fix $y \in Y$. Since $\tau^{a'(y)}(f(f'(y))) = \tau^{a'(y)}(y)$, we see that $(y, 0, f(f'(y))) \in G(Y, \tau)$. Thus Z is $\overline{c}_Y^{-1}(e)$-full in $(G(Y,\tau) \times \mathcal{R})^{(0)}$. Corollary 11.3 yields a graded $G(Y,\tau)$–$Z(G(Y,\tau) \times \mathcal{R})Z$-equivalence (W, c_W) such that $c_W^{-1}(e)$ is a $c_Y^{-1}(e)$–$\overline{c}_Y^{-1}(e)$-equivalence.

Since $\sigma^{a(x)}(f'(f(x))) = \sigma^{a(x)}(x)$ for $x \in X$, the formula $h(x) := (f(x), b(x))$ gives a homeomorphism $h : X \to Z$. Since each $\tau^{k(x)}(f(\sigma(x))) = \tau^{k(x)+1}(f(x))$, we have $(f(x), n, f(x')) \in G(Y, \tau)$ for all $(x, n, x') \in G(X, \sigma)$. There is an injective homomorphism $\phi : G(X, \sigma) \to G(Y, \tau) \times \mathcal{R}$ given by $\phi(x, n, x') := ((f(x), n, f(x')), (b(x), b(x')))$. We have $\phi(G(X, \sigma)) \subseteq Z(G(Y, \tau) \times \mathcal{R})Z$. Since h is a homeomorphism and $\sigma^{k'(y)}(f'(\tau(y))) = \sigma^{k'(y)+1}(f'(y))$ for $y \in Y$, we see that $\phi(G(X, \sigma)) = Z(G(Y, \tau) \times \mathcal{R})Z$, giving (2).

(3) $\implies$ (1). Write $\pi_X : G(X, \sigma) \times \mathcal{R} \to G(X, \sigma)$ and $\pi_Y : G(Y, \tau) \times \mathcal{R} \to G(Y, \tau)$ for the projection maps. Define $f : X \to Y$ and $f' : Y \to X$ by $\pi_Y(\kappa((x, 0, x), (0, 0))) = (f(x), 0, f(x))$ and $\pi_X(\kappa^{-1}((y, 0, y), (0, 0))) = (f'(y), 0, f'(y))$. Clearly f and f' are continuous and open. Define $l_X : G(X, \sigma) \to \mathbb{N}$ and $l_Y : G(Y, \tau) \to \mathbb{N}$ as in Lemma 8.4. Fix $x \in X$. Then $\kappa((x, 0, x), (0, 0)) = ((f(x), 0, f(x)), (n, n))$ for some $n \in \mathbb{N}$. Since $\overline{c}_Y \circ \kappa = \overline{c}_X$, it follows that $\pi_X(\kappa^{-1}((f(x), 0, f(x)), (n, 0))) = (x, 0, f'(f(x)))$. Let $a(x) := l_X(x, 0, f'(f(x)))$. Then $a : X \to \mathbb{N}$ is continuous, and $\sigma^{a(x)}(f'(f(x))) = \sigma^{a(x)}(x)$ for all x. Similarly, $a' : Y \to \mathbb{N}$ defined by $a'(y) := l_Y(y, 0, f(f'(y)))$ is continuous, and $\tau^{a'(y)}(f(f'(y))) = \tau^{a'(y)}(y)$ for all y.

Fix $x \in \mathrm{dom}(\sigma)$. Then $\pi_Y(\kappa((x, 1, \sigma(x)), (0, 0))) = (f(x), 1, f(\sigma(x)))$ because $\overline{c}_Y \circ \kappa = \overline{c}_X$. Let $k(x) := l_Y(f(x), 1, f(\sigma(x)))$. So $k : \mathrm{dom}(\sigma) \to \mathbb{N}$ is continuous, and $\tau^{k(x)}(f(\sigma(x))) = \tau^{k(x)+1}(f(x))$ for $x \in \mathrm{dom}(\sigma)$. Similarly, $k'(y) := l_Y(f'(y), 1, f'(\tau(y)))$ defines a continuous $k' : \mathrm{dom}(\tau) \to \mathbb{N}$ such that $\sigma^{k'(y)}(f'(\tau(y))) = \sigma^{k'(y)+1}(f'(y))$.

(1) $\implies$ (6). Since X is compact, $m := \sup_{x \in X} k(x)$ is finite. We have $\tau^m(f(\sigma(x))) = \tau^{m-k(x)}(\tau^{k(x)+1}(f(x))) = \tau^{m+1}(f(x))$ for all x. So $\phi := \tau^m \circ f : X \to Y$ is continuous and satisfies $\phi(\sigma(x)) = \tau^m(f(\sigma(x))) = \tau^{m+1}(f(x)) = \tau(\phi(x))$ for all x. Thus $\overline{\phi}(\xi) := (\phi(\xi_n))_{n \in \mathbb{Z}}$ defines a continuous map $\overline{\phi} : \overline{X} \to \overline{Y}$. By definition, $\overline{\tau} \circ \overline{\phi} = \overline{\phi} \circ \overline{\sigma}$. We will show that $\overline{\phi}$ is bijective and hence a conjugacy.

For injectivity, suppose that $\overline{\phi}(\xi) = \overline{\phi}(\xi')$. Then each $\tau^m(f(\xi_n)) = \phi(\xi_n) = \phi(\xi'_n) = \tau^m(f(\xi'_n))$. Let $p := \max\{\sup_{z \in X} a(z), \sup_{y \in Y} k'(y)\}$. Then $\sigma^p(x) = \sigma^p(f'(f(x))$ for $x \in$

X, and $\sigma^{p+j}(f'(y)) = \sigma^p(f'(\tau^j(y)))$ for $y \in Y$, $j \in \mathbb{N}$. So for $n \in \mathbb{Z}$,

$$\xi_{n+p+m} = \sigma^{p+m}(\xi_n) = \sigma^p(f'(\tau^m(f(\xi_n)))) = \sigma^p(f'(\tau^m(f(\xi'_n)))) = \sigma^{p+m}(\xi'_n) = \xi'_{n+p+m}.$$

Hence $\xi = \xi'$, and we deduce that $\overline{\phi}$ is injective.

For surjectivity, fix $\eta \in \overline{Y}$. Let $q := \sup_{z\in Y} \max\{a'(z), k'(z)\}$, and put $\eta' := \overline{\tau}^{-m-q}(\eta)$. For $y \in Y$, $\sigma^q(f'(\tau(y))) = \sigma^{q-k'(y)}(\sigma^{k'(y)}(f'(\tau(y)))) = \sigma^{q+1}(f'(y))$. Thus $(\sigma^q(f'(\eta'_n)))_{n\in\mathbb{Z}} \in \overline{X}$. Since $\tau^{m+j}(f(x)) = \tau^m(f(\sigma^j(x)))$ for all x, j, and since $\tau^q(f(f'(y))) = \tau^q(y)$ for $y \in Y$,

$$\phi(\sigma^q(f'(\eta'_n))) = \tau^m(f(\sigma^q(f'(\eta'_n)))) = \tau^{m+q}(f(f'(\eta'_n))) = \tau^{m+q}(\eta'_n)$$

for all $n \in \mathbb{Z}$. So $\overline{\phi}((\sigma^q(f'(\eta_n)))_{n\in\mathbb{Z}}) = \overline{\tau}^{m+q}(\eta') = \eta$.

(6) $\implies$ (1). Suppose that (X, d) and (Y, d') are metrisations of X, Y and that ϵ, ϵ' are expansive constants for (X, D, σ) and (Y, d', τ). Suppose that $\psi : \overline{X} \to \overline{Y}$ is a conjugacy. Fix $\delta > 0$ such that $\overline{d}(\xi, \xi') < \delta \implies \overline{d}(\psi(\xi), \psi(\xi')) < \epsilon$. Let $M := \sup_{x,x'\in X} d(x, x') < \infty$, and fix N with $2^{-N+1}M < \delta$. Then $\overline{d}(\xi, \xi') < \delta$ whenever $\xi_n = \xi'_n$ for all $n > -N$. So given $\xi, \xi' \in \overline{X}$, and putting $\eta = \psi(\xi)$ and $\eta' = \psi(\xi')$, if $\xi_n = \xi'_n$ for $n > -N$, then $d'(\eta_m, \eta'_m) \le \overline{d}'(\overline{\tau}^m(\eta), \overline{\tau}^m(\eta')) < \epsilon$ for all m, giving $\eta_0 = \eta'_0$. So we can define $f : X \to Y$ by $\psi(\xi) = (f(\xi_{n-N}))_{n\in\mathbb{Z}}$, and then $f \circ \sigma = \tau \circ f$.

To see that f is continuous, fix $x \in X$ and $\gamma > 0$. Choose $\xi \in \overline{X}$ such that $x = \xi_{-N}$. Choose an open neighbourhood U of ξ in $\overline{X}$ such that $\overline{d}'(\psi(\xi), \psi(\xi') < \gamma$ for every $\xi' \in U$. Since σ is surjective and open, there is an open $V \ni x$ such that $V \subseteq \{\xi'_{-N} : \xi' \in U\}$. So $d'(f(x), f(x')) \le \sup_{\xi'\in U} \overline{d}'(\psi(\xi), \psi(\xi')) < \gamma$ for all $x' \in V$.

To see that f is open, fix $V \subseteq X$ open. Then $U := \{\xi \in \overline{X} : \xi_{-N} \in V\}$ is open, so $\psi(U)$ is open. The coordinate projections $\overline{Y} \to Y$ are open maps, so $f(V)$ is open.

So f is continuous and open, $\tau \circ f = f \circ \sigma$, and $\psi(\xi) = (f(\xi_{n-N}))_{n\in\mathbb{Z}}$ for $\xi \in \overline{X}$. Symmetry gives a continuous open $f' : (Y, \tau) \to (X, \sigma)$ with $\sigma \circ f' = f' \circ \tau$ and $\psi^{-1}(\eta) = (f'(\eta_{n-N'}))_{n\in\mathbb{Z}}$ for $\eta \in \overline{Y}$. We have $f \circ f' = \sigma^{N+N'}$ and $f \circ f' = \tau^{N+N'}$, giving (1). □

Remark 11.7. We deduce an interesting "stable isomorphism implies isomorphism" statement. Let $\sigma : X \to X$ and $\tau : Y \to Y$ be homeomorphisms of second-countable totally disconnected compact Hausdorff spaces. If $(C(X) \times_\sigma \mathbb{Z}) \otimes \mathcal{K}$ and $(C(Y) \times_\tau \mathbb{Z}) \otimes \mathcal{K}$ are $(\gamma^X \otimes \mathrm{id})$–$(\gamma^Y \otimes \mathrm{id})$-equivariantly isomorphic, then there is a γ^X–γ^Y-equivariant (and diagonal-preserving by Remark 7.6) isomorphism $C(X) \times_\sigma \mathbb{Z} \to C(Y) \times_\tau \mathbb{Z}$.

To prove this, first observe that $(X, \sigma) \cong (\overline{X}, \overline{\sigma})$ and $(Y, \tau) \cong (\overline{Y}, \overline{\tau})$ and then apply (5) $\implies$ (6) of Corollary 11.5 to see that if there is a *diagonal-preserving* equivariant isomorphism $(C(X)\times_\sigma\mathbb{Z})\otimes\mathcal{K} \cong (C(Y)\times_\tau\mathbb{Z})\otimes\mathcal{K}$, then there is an equivariant isomorphism $C(X) \times_\sigma \mathbb{Z} \cong C(Y) \times_\tau \mathbb{Z}$. Next note that $((C(X) \times_\sigma \mathbb{Z}) \otimes \mathcal{K})^{\gamma^X\otimes\mathrm{id}} = C(X) \otimes \mathcal{K}$ and likewise for (Y, τ); so any equivariant isomorphism $\phi : C(X) \times_\sigma \mathbb{Z} \to C(Y) \times_\tau \mathbb{Z}$ carries $C(X) \otimes c_0$ to a maximal abelian subalgebra of $C(Y) \otimes \mathcal{K}$; that is, to $C(Y) \otimes D$ for some maximal abelian $D \subseteq \mathcal{K}$. Fix a unitary $U \in M(\mathcal{K})$ that conjugates D to c_0. Then $\phi' := (\mathrm{id} \otimes \mathrm{Ad}_U) \circ \phi$ is a diagonal-preserving equivariant isomorphism.

This result could also be obtained without recourse to groupoids using the techniques of [26, Proposition 4.3] (it is at least implicitly contained in that result).

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