arXiv:1305.6388v1 [math.OA] 28 May 2013

THE PRIMITIVE IDEALS OF THE CUNTZ-KRIEGER ALGEBRA OF A ROW-FINITE HIGHER-RANK GRAPH WITH NO SOURCES

TOKE MEIER CARLSEN, SOORAN KANG, JACOB SHOTWELL, AND AIDAN SIMS

Abstract. We catalogue the primitive ideals of the Cuntz-Krieger algebra of a row-finite higher-rank graph with no sources. Each maximal tail in the vertex set has an abelian periodicity group of finite rank at most that of the graph; the primitive ideals in the Cuntz-Krieger algebra are indexed by pairs consisting of a maximal tail and a character of its periodicity group. The Cuntz-Krieger algebra is primitive if and only if the whole vertex set is a maximal tail and the graph is aperiodic.

1. Introduction

Graph C^*-algebras were introduced in the early 1980's by Enomoto and Watatani [6] as an alternative description of the Cuntz-Krieger algebras invented in [4]. Enomoto and Watatani considered only finite graphs, but since the late 1990's substantial work has gone into describing and understanding the analogous construction for infinite directed graphs in various levels of generality (to name just a few, [2, 8, 10, 14, 18, 23]). A gem in this program is Hong and Szymański's description [9] of the primitive ideal space of the C^*-algebra of a directed graph. Hong and Szymański catalogue the primitive ideals of $C^*(E)$ in terms of elementary structural features of E. They also describe the closure operation in the hull-kernel topology in terms of this catalogue.

In 1999, Robertson and Steger discovered a class of higher-rank Cuntz-Krieger algebras arising from $\mathbb{Z}^k$ actions on buildings [21]. Shortly afterwards, Kumjian and Pask introduced higher-rank graphs and the associated C^*-algebras [12] as a simultaneous generalisation of the graph C^*-algebras of [13] and Robertson and Steger's higher-rank Cuntz-Krieger algebras. Kumjian and Pask's k-graph algebras have since attracted a fair bit of attention. But their structure is more subtle and less well understood than that of graph C^*-algebras. In particular, while large portions of the gauge-invariant theory of graph C^*-algebras can be generalised readily to k-graphs, higher-rank analogues of other structure theorems for graph C^*-algebras are largely still elusive.

Our main result here is a complete catalogue of the primitive ideals in the C^*-algebra of a row-finite k-graph with no sources. Our methods are very different from Hong and Szymański's, and require a different set of technical tools. We have been unable to describe the hull-kernel topology on $\mathrm{Prim}(C^*(\Lambda))$, and we leave this question open for the present.

We begin in Section 2 by introducing P-graphs Γ and their C^*-algebras $C^*(\Gamma)$, where P is the image of $\mathbb{N}^k$ under a homomorphism f of $\mathbb{Z}^k$. We prove a gauge-invariant uniqueness theorem (Proposition 2.7) and a Cuntz-Krieger uniqueness theorem (Corollary 2.8) for these C^*-algebras. In Section 3, we consider the primitive ideal space of the pullback k-graph arising from a P-graph. We first show that if f and P are as above, then the

Date: May 28, 2013.

2010 *Mathematics Subject Classification.* 46L05 (primary); 46L45 (secondary).

Key words and phrases. Higher-rank graph; k-graph; primitive ideal; $C(X)$-algebra; irreducible representation.

This research was supported by the Australian Research Council.

pullback $f^*\Gamma$ of a P-graph Γ over $f : \mathbb{N}^k \to P$ and $d : \Gamma \to P$ is a k-graph. We then prove in Theorem 3.5 that the irreducible representations of $C^*(f^*\Gamma)$ are in bijection with pairs (π, χ) where π is an irreducible representation of $C^*(\Gamma)$, and χ is a character of $\ker f$.

In Section 4, we study maximal tails in k-graphs. Building on an idea from [5], we show that each maximal tail T has a well-defined *periodicity group* $\operatorname{Per}(T)$, and contains a large hereditary subset H such that the subgraph $H\Lambda T$ consisting of paths whose range and source both belong to H is isomorphic to a pullback of an $(\mathbb{N}^k/\operatorname{Per}(T))$-graph. The algebra $C^*(H\Lambda T)$ can be identified with a corner in $C^*(\Lambda T)$. Combining this with our earlier results, we describe a bijection $(T, \chi) \mapsto I_{T,\chi}$ from pairs (T, χ) where T is a maximal tail of Λ and χ is a character $\operatorname{Per}(T)$ to primitive ideals of $C^*(\Lambda)$. We conclude by showing that $C^*(\Lambda)$ is primitive if and only if Λ^0 is a maximal tail and Λ is aperiodic.

As usual in our subject, our convention is that in the context of C^*-algebras, "homomorphism" always means "*-homomorphism," and "ideal" always means "closed two-sided ideal".

2. P-graphs

We introduce P-graphs over finitely generated cancellative abelian monoids P, and an associated class of C^*-algebras. Our treatment is very brief since these objects are introduced as a technical tool, and the ideas are essentially identical to those developed in [12]. Specialising to $P = \mathbb{N}^k$ in this section also serves to introduce the notation used later for k-graphs.

Definition 2.1. Let P be a finitely generated cancellative abelian monoid, which we also regard as a category with one object. A *P-graph* is a countable small category Γ equipped with a functor $d : \Gamma \to P$ which has the factorisation property: whenever $\xi \in \Gamma$ satisfies $d(\xi) = p+q$, there exist unique elements $\eta, \zeta \in \Gamma$ such that $d(\eta) = p$, $d(\zeta) = q$ and $\xi = \eta\zeta$.

Observe that if $P = \mathbb{N}^k$, then a P-graph is precisely a k-graph in the sense of [12].

If Γ is a P-graph and $p \in P$, then we write Γ^p for $d^{-1}(p)$. When ξ is a morphism of Γ, we write $s(\xi)$ for the domain of ξ and $r(\xi)$ for the codomain of ξ. If $d(\xi) = 0$, then the factorisation property combined with the identities $\xi = \mathrm{id}_{r(\xi)}\,\xi = \xi\,\mathrm{id}_{s(\xi)}$ imply that $\xi = \mathrm{id}_{r(\xi)} = \mathrm{id}_{s(\xi)}$. Thus $\Gamma^0 = \{\mathrm{id}_o : o \text{ is an object of } \Gamma\}$, and we can regard r, s as maps from Γ to Γ^0; we then have $r(\xi)\xi = \xi = \xi s(\xi)$ for all $\xi \in \Gamma$.

Given $\xi, \eta \in \Gamma$ and a subset $E \subseteq \Gamma$, we define

$$\xi E = \{\xi\zeta : \zeta \in E, r(\zeta) = s(\xi)\}, \quad E\eta = \{\zeta\eta : \zeta \in E, s(\zeta) = r(\eta)\}, \quad \text{and}$$
$$\xi E\eta = \xi E \cap E\eta.$$

For $X, E, Y \subseteq \Gamma$, we write XEY for $\bigcup_{\xi\in X,\eta\in Y} \xi E\eta$. We say that the P-graph Γ is *row-finite with no sources* if $0 < |v\Gamma^p| < \infty$ for all $v \in \Gamma^0$ and $p \in P$.

Example 2.2. Let P be a finitely generated cancellative abelian monoid. Let

$$\Omega_P := \{(p, q) \in P \times P : \text{ there exists } r \in P \text{ such that } p + r = q\}.$$

Since P is cancellative, we may embed it in its Grothendieck group G so that it makes sense to write $p - q$ for $p, q \in P$. If $(p, q) \in \Omega_P$, then $q - p \in P$. Define $d : \Omega_P \to P$ by $d(p, q) = q - p$. Define $r, s : \Omega_P \to P$ by $r(p, q) = p$ and $s(p, q) = q$, and for $(p, q), (q, r) \in \Omega_P$, define $(p, q)(q, r) = (p, r)$; this (p, r) belongs to Ω_P because $r = p+((q-p)+(r-q))$. Then Ω_P is a category with objects P and identity morphisms (p, p), and d is a functor from Ω_P to P. Cancellativity of P implies that Ω_P is a P-graph. We identify Ω_P^0 with P via $(p, p) \mapsto p$. For $v \in \Omega_P^0$ and $p \in P$, we then have $v\Omega_P^p = \{(v, v + p)\}$, so Ω_P is row-finite with no sources.

A *P-graph morphism* is a functor $x : \Gamma \to \Sigma$ between P-graphs Γ and Σ such that $d_\Sigma(x(\xi)) = d_\Gamma(\xi)$ for all $\xi \in \Gamma$. For a P-graph Γ, we define

$$\Gamma^\Omega := \{x : \Omega_P \to \Gamma \mid x \text{ is a } P\text{-graph morphism}\}.$$

Lemma 2.3. *Let P be a finitely generated cancellative abelian monoid and let Γ be a row-finite P-graph with no sources. For each $v \in \Gamma^0$ there exists $x \in \Gamma^\Omega$ such that $x(0) = v$.*

Proof. Fix generators $a_1, \dots, a_k$ for P. Let $\mathbf{1} = \sum_{i=1}^k a_i$. Let $v_0 := v$. Recursively for $n \geq 1$ choose $\lambda_n \in v_{n-1}\Lambda^{\mathbf{1}}$ and set $v_n := s(\lambda_n)$. Suppose that $(p, q) \in \Omega_P$. Express $q = \sum_{i=1}^k q_i a_i$ where each $q_i \in \mathbb{N}$. Let $n = \max_{i \leq k} q_i$. Then $d(\lambda_1 \dots \lambda_n) = p + (q - p) + \sum_{i=1}^k (n - q_i) a_i$. Two applications of the factorisation property give $\lambda_1 \dots \lambda_n = \xi\eta\zeta$ where $d(\xi) = p$ and $d(\eta) = q - p$. Define $x(p, q) := \eta$. The factorisation property guarantees that there is a well-defined function $x : \Omega_P \to \Gamma$ defined by this formula, and that x is a P-graph morphism. We have $x(0) = v$ by construction. □

We associate to each row-finite P-graph Γ with no sources a C^*-algebra $C^*(\Gamma)$. As we shall see in section 3, these C^*-algebras are isomorphic to quotients by primitive ideals of crucial building blocks of k-graph C^*-algebras.

Definition 2.4. Let P be a finitely generated cancellative abelian monoid and let Γ be a row-finite P-graph with no sources. A *Cuntz-Krieger* Γ*-family* in a C^*-algebra B is a collection $\{t_\xi : \xi \in \Gamma\} \subseteq B$ such that

(CK1) $\{t_v : v \in \Gamma^0\}$ is a set of mutually orthogonal projections;
(CK2) $t_\xi t_\eta = t_{\xi\eta}$ whenever $s(\xi) = r(\eta)$;
(CK3) $t_\xi^* t_\xi = t_{s(\xi)}$ for all $\xi \in \Gamma$; and
(CK4) $t_v = \sum_{\xi \in v\Lambda^p} t_\xi t_\xi^*$ for all $v \in \Lambda^0$ and $p \in P$.

Fix a Cuntz-Krieger Γ-family $\{t_\xi : \xi \in \Gamma\}$. For $\xi, \eta \in \Gamma$,

$$t_\xi^* t_\eta = t_\xi^* \sum_{\zeta \in r(\eta)\Gamma^{d(\xi)+d(\eta)}} t_\zeta t_\zeta^* t_\eta.$$

Fix $\zeta \in r(\eta)\Gamma^{d(\xi)+d(\eta)}$. By the factorisation property, there are unique factorisations $\zeta = \theta\theta' = \omega\omega'$ with $d(\theta) = d(\xi)$ and $d(\omega) = d(\eta)$. Relations (CK1) and (CK3) show that $t_\xi^* t_\zeta = 0$ unless $\theta = \xi$, and $t_\omega^* t_\eta = 0$ unless $\omega = \eta$, and we deduce that

$$t_\xi^* t_\eta = \sum_{\xi\xi' = \eta\eta' \in r(\eta)\Gamma^{d(\xi)+d(\eta)}} t_{\xi'} t_{\eta'}^*.$$

Hence $C^*(\{t_\xi : \xi \in \Gamma\}) = \overline{\operatorname{span}}\{t_\xi t_\eta^* : \xi, \eta \in \Gamma\}$.

A standard argument now shows that there is a universal C^*-algebra $C^*(\Gamma)$ generated by a Cuntz-Krieger family $\{s_\xi : \xi \in \Gamma\}$. If $P = \mathbb{N}^k$ then the relations of Definition 2.4 are those of [12], and so the universal C^*-algebra $C^*(\Gamma)$ coincides with the C^*-algebra of Γ regarded as a k-graph. To see that the generators of $C^*(\Gamma)$ are all nonzero, we introduce the groupoid $\mathcal{G}_\Gamma$.

For each $\xi \in \Gamma$, define $Z(\xi) \subseteq \Gamma^\Omega$ by

$$Z(\xi) := \{x \in \Gamma^\Omega : x(0, d(\xi)) = \xi\}.$$

The $Z(\xi)$ constitute a base of compact open sets for a second-countable locally compact Hausdorff topology on Γ^Ω. For $x \in \Gamma^\Omega$ and $p \in P$, there is a unique element $\sigma^p(x)$ of Γ^Ω defined by $\sigma^p(x)(q, r) = x(p + q, p + r)$. Define $\mathcal{G}_\Gamma := \{(x, p - q, y) : x, y \in \Omega^P, \sigma^p(x) = \sigma^q(y)\}$. This is a groupoid with $r(x, g, y) = (x, 0, x)$, $s(x, g, y) = (y, 0, y)$, $(x, g, y)(y, h, z) = (x, g + h, z)$ and $(x, g, y)^{-1} = (y, -g, x)$. Arguments like those of [12] show that $\mathcal{G}_\Gamma$ is a locally compact Hausdorff étale groupoid under the topology generated by the sets

$$Z(\xi,\eta) = \{(x, d(\xi) - d(\eta), y) : x \in Z(\xi), y \in Z(\eta), \sigma^{d(\xi)}(x) = \sigma^{d(\eta)}(y)\}.$$

Each $Z(\xi,\eta)$ is compact open in $\mathcal{G}_\Gamma$.

We recall from [20] the construction of $C^*_r(\mathcal{G}_\Gamma)$. The space $C_c(\mathcal{G}_\Gamma)$ is a *-algebra under the convolution product $(a * b)(x,g,y) = \sum_{(x,h,z)\in\mathcal{G}_\Gamma} a(x,h,z)b(z,h^{-1}g,y)$ and the involution $a^*(x,g,y) = \overline{a(y,g^{-1},x)}$. For $y \in \Gamma^\Omega$, let $\mathcal{H}_y := \ell^2(\{\alpha \in \mathcal{G}_\Gamma : s(\alpha) = y\})$ with orthonormal basis $\{\delta_\alpha : \alpha \in \mathcal{G}_\Gamma, s(\alpha) = y\}$. The regular representation ρ_y of $C_c(\mathcal{G}_\Gamma)$ on $\mathcal{H}_y$ is given by $\rho_y(a)\delta_{(x,g,y)} = \sum_{(z,h,x)\in\mathcal{G}_\Gamma} a(z,h,x)\delta_{(z,h+g,y)}$. The reduced groupoid C^*-algebra $C^*_r(\mathcal{G}_\Gamma)$ is the closure of the image of $C_c(\Gamma)$ under $\bigoplus_y \rho_y$; equivalently, it is the completion of $C_c(\mathcal{G}_\Gamma)$ in the norm $\|a\| = \sup_y \|\rho_y(a)\|$.

Lemma 2.5. *Let P be a finitely generated cancellative abelian monoid. Let Γ be a row-finite P-graph with no sources. For $\xi \in \Gamma$, let $t_\xi := 1_{Z(\xi,s(\xi))} \in C_c(\mathcal{G}_\Gamma) \subseteq C^*_r(\mathcal{G}_\Gamma)$. Then $\{t_\xi : \xi \in \Gamma\}$ is a Cuntz-Krieger Γ-family, and there is a homomorphism $\pi_t : C^*(\Gamma) \to C^*_r(\mathcal{G}_\Gamma)$ which carries each s_ξ to $1_{Z(\xi,s(\xi))}$. In particular, each generator of $C^*(\Gamma)$ is nonzero.*

Proof. Routine calculations show that the t_ξ form a Cuntz-Krieger Γ-family. The existence of the homomorphism π_t then follows from the universal property of $C^*(\Gamma)$. Since the $Z(\xi,s(\xi))$ are all nonempty, the t_ξ are all nonzero and hence the s_ξ are also nonzero. □

If G is the Grothendieck group of P, then for each character χ of G, the elements $\gamma_\chi(s_\xi) := \chi(d(\xi))s_\xi \in C^*(\Gamma)$ form a Cuntz-Krieger Γ family, and hence induce an endomorphism γ_χ of $C^*(\Gamma)$. Since $\gamma_{\chi^{-1}}$ is an inverse for γ_χ, both are automorphisms. An $\varepsilon/3$-argument shows that γ is a continuous action of $\widehat{G}$ by automorphisms. Averaging over γ gives a faithful conditional expectation Φ onto the fixed-point algebra $C^*(\Gamma)^\gamma$ [16, Proposition 3.2]. Since every spanning element of $C^*(\Gamma)$ belongs to one of the spectral subspaces of γ, we have

$$\Phi(s_\xi s_\eta^*) = \delta_{d(\xi),d(\eta)} s_\xi s_\eta^*. \tag{1}$$

In particular, we have $C^*(\Gamma)^\gamma = \overline{\operatorname{span}}\{s_\xi s_\eta^* : d(\xi) = d(\eta)\}$.

For a countable set X, we write $\mathcal{K}_X$ for the unique nonzero C^*-algebra generated by elements $\{\theta_{x,y} : x, y \in X\}$ such that $\theta_{x,y}^* = \theta_{y,x}$ and $\theta_{x,y}\theta_{w,z} = \delta_{y,w}\theta_{x,z}$.

Lemma 2.6. *Let P be a finitely generated cancellative abelian monoid and let Γ be a row-finite P-graph with no sources. Then $C^*(\Gamma)^\gamma$ is an AF algebra. A homomorphism $\phi : C^*(\Gamma) \to B$ restricts to an injection of $C^*(\Gamma)^\gamma$ if and only if $\phi(s_v) \neq 0$ for all v.*

Proof. Let $a_1, \dots, a_k$ be generators for P. For each $n \in \mathbb{N}$, let $n \cdot \mathbf{1} := \sum_{i=1}^k n \cdot a_i$, and let $A_n := \overline{\operatorname{span}}\{s_\xi s_\eta^* : d(\xi) = d(\eta) = n \cdot \mathbf{1}\}$. The Cuntz-Krieger relations ensure that the $s_\xi s_\eta^*$ are matrix units. By Lemma 2.5 all the s_ξ are nonzero, and hence $s(\xi) = s(\eta)$ implies $\|s_\xi s_\eta^*\|^2 = \|s_\eta s_\xi^* s_\xi s_\eta^*\| = \|s_\eta s_\eta^*\| = \|s_\eta\|^2 \neq 0$. It follows that $s_\xi s_\eta^* \mapsto \Theta_{\xi,\eta}$ determines an isomorphism $A_n \cong \bigoplus_{v\in\Gamma^0} \mathcal{K}_{\Gamma^{n\cdot\mathbf{1}}v}$. If $m \leq n$, then for $v \in \Gamma^0$ and $\xi, \eta \in \Gamma^{m\cdot\mathbf{1}}v$, we have

$$s_\xi s_\eta^* = \sum_{\zeta \in v\Gamma^{(n-m)\cdot\mathbf{1}}} s_{\xi\zeta} s_{\eta\zeta}^* \in A_n,$$

and hence $m \leq n$ implies $A_m \subseteq A_n$. Hence $\overline{\bigcup_n A_n}$ is an AF subalgebra of $C^*(\Gamma)^\gamma$. Suppose that $\xi, \eta \in \Gamma$ satisfy $s(\xi) = s(\eta) = v$ and $d(\xi) = d(\eta)$. Write $d(\xi) = \sum_{i=1}^k p_i a_i$. Let $n :=$

$\max_i p_i$ and let $q := \sum_{i=1}^k (n - p_i)a_i$. Then $n\cdot\mathbf{1} = d(\xi) + q$ and $s_\xi s_\eta^* = \sum_{\zeta\in v\Gamma^q} s_{\xi\zeta}s_{\eta\zeta}^* \in A_n$. Hence $C^*(\Gamma)^\gamma = \overline{\bigcup_n A_n}$ is AF.

Now suppose that $\phi : C^*(\Gamma) \to B$ is a homomorphism. Since each s_v is nonzero and belongs to $C^*(\Gamma)^\gamma$, if ϕ is injective on $C^*(\Gamma)^\gamma$, then each $\phi(s_v) \neq 0$. Conversely suppose that each $\phi(s_v) \neq 0$. Fix $n \in \mathbb{N}$. For $\xi, \eta \in \Gamma^{n\cdot\mathbf{1}}$, the reasoning of the first paragraph of this proof shows that $\phi(s_\xi s_\eta^*) \neq 0$. Since each $\mathcal{K}_{\Gamma^{n\cdot\mathbf{1}}v}$ is simple, it follows that ϕ is injective and hence isometric on A_n. Since the A_n are nested, it follows that ϕ is isometric on $\bigcup_n A_n$, and therefore on $\overline{\bigcup_n A_n} = C^*(\Gamma)^\gamma$ as well. □

Proposition 2.7. *Let P be a finitely generated cancellative abelian monoid, and let G be its Grothendieck group. Let Γ be a row-finite P-graph with no sources. Suppose that $\{t_\xi : \xi \in \Gamma\}$ is a Cuntz-Krieger Γ-family in a C^*-algebra B and there is an action β of $\widehat{G}$ on B such that $\beta_\chi(t_\xi) = \chi(d(\xi))t_\xi$ for all $\xi \in \Gamma$. Then the induced homomorphism $\pi_t : C^*(\Gamma) \to B$ is injective if and only if $t_v \neq 0$ for all $v \in \Gamma^0$. The homomorphism π_t of Lemma 2.5 is an isomorphism from $C^*(\Gamma)$ to $C^*_r(\mathcal{G}_\Gamma)$.*

Proof. First observe that if some $t_v = 0$ then Lemma 2.5 implies that π_t is not injective. Now suppose that each $t_v \neq 0$. By Lemma 2.6, the homomorphism π_t is injective on $C^*(\Gamma)^\gamma$. Averaging over β gives a conditional expectation $\Psi : \pi_t(C^*(\Gamma)) \to \pi_t(C^*(\Gamma)^\gamma)$ such that $\Psi\circ\pi_t = \pi_t\circ\Phi$, where Φ is the conditional expectation of (1). Now the following standard argument shows that π_t is faithful:

$$\pi_t(a) = 0 \implies \Psi(\pi_t(a^*a)) = 0 \implies \pi_t(\Phi(a^*a)) = 0 \implies \Phi(a^*a) = 0 \implies a = 0.$$

Now let $\{t_\xi : \xi \in \Gamma\}$ and $\pi_t : C^*(\Gamma) \to C^*_r(\mathcal{G}_\Gamma)$ be as in Lemma 2.5. Define $c : \mathcal{G}_\Gamma \to G$ by $c(x, g, y) = g$. For $y \in \Gamma^\Omega$, let ρ_y be the regular representation discussed prior to Lemma 2.5. For $\alpha, \beta \in \mathcal{G}_\Gamma$ with $s(\alpha) = s(\beta) = y$, let $\Theta_{\alpha,\beta} \in \mathcal{B}(\mathcal{H}_y)$ be the rank-one operator from $\mathbb{C}\delta_\beta$ to $\mathbb{C}\delta_\alpha$. There is a strongly continuous action β^y of $\widehat{G}$ on $\mathcal{B}(\mathcal{H}_y)$ such that $\beta^y_\chi(\Theta_{\alpha,\beta}) = \chi(c(\alpha) - c(\beta))\Theta_{\alpha,\beta}$. In particular, $\beta^y_\chi(\rho_y(t_\xi)) = \chi(d(\xi))\rho_y(t_\xi)$ for all $\xi \in \Gamma$ and $\chi \in \widehat{G}$. Thus $\beta = \bigoplus_y \beta^y$ is a strongly continuous action of $\widehat{G}$ on $C^*_r(\mathcal{G}_\Gamma)$ such that $\beta_\chi(t_\xi) = \chi(d(\xi))t_\xi$ for all $\xi \in \Gamma$. So π_t is injective. Lemma 2.5 implies that if $s(\xi) = s(\eta)$ in Γ, then $\pi_t(s_\xi s_\eta^*) = 1_{Z(\xi,s(\xi))}1_{s(\eta),\eta} = 1_{Z(\xi,\eta)}$. The C^*-norm on $C^*_r(\mathcal{G}_\Gamma)$ coincides with the supremum norm on each $C_c(Z(\xi,\eta))$, so the Stone-Weierstrass theorem implies that the range of π_t contains $C_c(\mathcal{G}_\Gamma)$, and hence π_t is surjective. □

Let P be a finitely generated cancellative abelian monoid and let Γ be a P-graph. We say that Γ is *aperiodic* if for every $v \in \Gamma^0$ there exists $x \in \Gamma^\Omega$ such that $p \neq q \in P$ implies $\sigma^p(x) \neq \sigma^q(x)$.

Corollary 2.8. *Let P be a finitely generated abelian monoid and let Γ be a row-finite P-graph with no sources. Suppose that Γ is aperiodic. If $\{t_\xi : \xi \in \Gamma\}$ is a Cuntz-Krieger Γ-family in a C^*-algebra B, then the induced homomorphism $\pi_t : C^*(\Gamma) \to B$ is injective if and only if $t_v \neq 0$ for every $v \in \Gamma^0$.*

Proof. The only if is clear because Lemma 2.5 implies that each $s_v \neq 0$. Suppose that each $t_v \neq 0$. Then each $t_\lambda t_\lambda^* \neq 0$. By Proposition 2.7, we can identify $C^*(\Gamma)$ with $C^*_r(\mathcal{G}_\Gamma)$, and regard π_t as a homomorphism of $C^*_r(\mathcal{G}_\Gamma)$ which carries each $1_{Z(\xi,s(\xi))}$ to t_ξ. The isomorphism $C^*(\Gamma) \cong C^*_r(\mathcal{G}_\Gamma)$ identifies $C_0(\mathcal{G}_\Gamma^{(0)})$ with a subalgebra of $C^*(\Gamma)^\gamma$, so Lemma 2.6 implies that π_t is injective on $C_0(\mathcal{G}_\Gamma^{(0)})$.

Fix a basic open set $Z(\xi)$ in $\mathcal{G}_\Gamma^{(0)} = \Gamma^\Omega$. By hypothesis, there exists $y \in \Gamma^\omega$ such that $y(0) = s(\xi)$ and $\sigma^p(y) \neq \sigma^q(y)$ whenever $p \neq q$. The factorisation property implies that there is a unique element ξy of $Z(\xi)$ such that $\sigma^{d(\xi)}(\xi y) = y$. We have

$$\sigma^p(\xi y) = \sigma^q(\xi y) \implies \sigma^{d(\xi)}(\sigma^p(\xi y)) = \sigma^{d(\xi)}(\sigma^q(\xi y)) \implies \sigma^p(y) = \sigma^q(y) \implies p = q.$$

Thus $\mathcal{G}_\Lambda$ has trivial isotropy at ξy. Thus $\mathcal{G}_\Gamma$ is topologically principal. It now follows from [7, Theorem 4.4] that every nontrivial ideal of $C^*_r(\mathcal{G}_\Gamma)$ has nontrivial intersection with $C_0(\mathcal{G}_\Gamma^{(0)})$. In particular, that $\pi_t|_{C_0(\mathcal{G}_\Gamma^{(0)})}$ is injective shows that π_t is injective. □

3. The primitive ideal space of the C^*-algebra of a pullback

In this section we consider pullback k-graphs of the form $f^*\Gamma$ where $f : \mathbb{Z}^k \to G$ is a group homomorphism and Γ is a P-graph for $P = f(\mathbb{N}^k)$. Proposition 3.3 and Lemma 3.4 combine to show that, putting $H := \ker(f)$, the C^*-algebra $C^*(f^*\Gamma)$ is a $C(\widehat{H})$-algebra with fibres identical to $C^*(\Gamma)$. We use this to give a complete listing of the irreducible representations of $C^*(f^*\Gamma)$ in terms of the irreducible representations of $C^*(\Gamma)$ and characters of $\ker f$. We begin by introducing pullbacks of P-graphs.

Definition 3.1 (cf. [12, Definition 1.9]). Let P and Q be finitely generated cancellative abelian monoids, and let $f : P \to Q$ be a monoid morphism. If (Γ, d) is a Q-graph, we define the P-graph $f^*\Gamma$ as follows: $f^*\Gamma = \{(\lambda, n) : d(\lambda) = f(n)\}$ with $d(\lambda, n) = n$, $s(\lambda, n) = s(\lambda)$ and $r(\lambda, n) = r(\lambda)$. Composition is given by $(\mu, m)(\nu, n) = (\mu\nu, m + n)$.

For $g, h \in \mathbb{Z}^k$, we write $g \vee h$ for the coordinatewise maximum of g and h and $g \wedge h$ for the coordinatewise minimum. Given $h \in \mathbb{Z}^k$ we define $h_+ := h \vee 0$ and $h_- := -(h \wedge 0)$. We then have $h = h_+ - h_-$ with $h_+ \wedge h_- = 0$.

Lemma 3.2. *Let H be a subgroup of $\mathbb{Z}^k$, let $G = \mathbb{Z}^k/H$ and let $f : \mathbb{Z}^k \to G$ be the quotient map. Let $P = f(\mathbb{N}^k) \subseteq G$. Suppose that Γ is a row-finite P-graph with no sources. Then $f^*\Gamma$ is a row-finite k-graph with no sources. Suppose that $h \in H$ and $\lambda \in \Gamma$ satisfy $d(\lambda) = f(h_+)$. Then (λ, h_+) and (λ, h_-) both belong to $f^*\Gamma$.*

Proof. Clearly $f^*\Gamma$ is a countable category and $(\lambda, n) \mapsto n$ is a functor. We check the factorisation property; if $(\lambda, m + n) \in f^*\Gamma$, then $d(\lambda) = f(m) + f(n)$. So λ factorises uniquely as $\lambda = \mu\nu$ with $d(\mu) = f(m)$, $d(\nu) = f(n)$, and then $(\lambda, m+n) = (\mu, m)(\nu, n)$ is the unique factorisation with $d(\mu, m) = m$ and $d(\nu, n) = n$.

For the second assertion, observe that $f(h_+) = f(h_- + h) = f(h_-) + f(h) = f(h_-)$ since $h \in H = \ker(f)$. □

In the following proposition, $\{U_h : h \in H\}$ denotes the canonical collection of unitary generators of the group C^*-algebra $C^*(H)$. We write $\mathcal{ZM}(A)$ for the centre of the multiplier algebra of a C^*-algebra A.

Proposition 3.3. *Let G, f, P, H be as in Lemma 3.2. Let Γ be a row-finite P-graph with no sources. For each $h \in H$ and $v \in \Gamma^0$, let $u_{v,h} := \sum_{\lambda \in v\Gamma^{f(h_+)}} s_{(\lambda,h_+)} s^*_{(\lambda,h_-)}$. Then $\sum_{v \in \Gamma} u_{v,h}$ converges strictly to a central unitary multiplier $V_h := \sum_{d(\lambda)=f(h_+)} s_{(\lambda,h_+)} s^*_{(\lambda,h_-)}$ of $C^*(f^*\Gamma)$. Moreover, there is an injective homomorphism $\rho : C^*(H) \to \mathcal{ZM}(C^*(f^*\Gamma))$ such that $\rho(U_h) = V_h$ for all $h \in H$.*

Proof. For $v \in \Gamma^0$ and $h \in H$, we claim that $u_{v,h}$ is a partial isometry whose initial and final projections are both equal to $s_{(v,0)}$. Indeed

$$\begin{aligned} u^*_{v,h} u_{v,h} &= \sum_{\lambda,\mu \in v\Gamma^{f(h_+)}} s_{(\lambda,h_-)} s^*_{(\lambda,h_+)} s_{(\mu,h_+)} s^*_{(\mu,h_-)} \\ &= \sum_{\lambda,\mu \in v\Gamma^{f(h_+)}} \delta_{\lambda,\mu} s_{(\lambda,h_-)} s^*_{(\mu,h_-)} = \sum_{\eta \in (v,0)(f^*\Gamma)^{h_-}} s_\eta s^*_\eta = s_{(v,0)}, \end{aligned}$$

and a similar calculation shows that $u_{v,h}u_{v,h}^* = s_{(v,0)}$ also.

By, for example, the argument of [16, Lemma 2.10], for each $a \in C^*(f^*\Gamma)$, we have $\sum_{v\in F} s_{(v,0)}a \to a$ as F increases over finite subsets of Γ^0. So for $a \in C^*(f^*\Gamma)$ and $\varepsilon > 0$, there exists a finite $F \subseteq \Gamma^0$ such that $\|a - \sum_{v\in F} s_v a\| < \varepsilon$. Thus, for $K \subseteq L \subseteq \Gamma^0 \setminus F$ and $h \in H$, we have

$$\begin{aligned}\Big\|\sum_{v\in L} u_{v,h}a - \sum_{w\in K} u_{w,h}a\Big\| &= \Big\|\sum_{v\in L\setminus K} u_{v,h}a\Big\| \\ &= \Big\|\sum_{v\in L\setminus K} u_{v,h}\Big(a - \sum_{w\in F} s_{(w,0)}a\Big)\Big\| \quad \text{since } L \subseteq \Gamma^0 \setminus F \\ &\le \varepsilon\Big\|\sum_{v\in L\setminus K} u_{v,h}\Big\| \\ &= \varepsilon\end{aligned}$$

since the $u_{v,h}$ are partial isometries whose initial projections are mutually orthogonal and whose final projections are also mutually orthogonal. Hence the net $\{\sum_{v\in F} u_{v,h}a : F \subseteq \Gamma^0 \text{ is finite}\}$ is Cauchy, so converges. Thus, the series

$$\sum_{v\in\Lambda^0} u_{v,h} = \sum_{d(\lambda)=f(h_+)} s_{(\lambda,h_+)}s_{(\lambda,h_-)}^*$$

converges strictly to a multiplier V_h of $C^*(f^*\Gamma)$.

Since $(-h)_+ = h_-$ and $(-h)_- = h_+$ for all $h \in \mathbb{Z}^k$, we have $u_{v,h}^* = u_{v,-h}$ for all v, h. Hence $V_h^* = V_{-h}$ for all $h \in H$. The V_h are unitary because

$$(V_h^* V_h a) = \lim_{F,K} \sum_{v\in F, w\in K} u_{v,h}^* u_{v,h} a = \lim_F \sum_{v\in F} s_{(v,0)}a = a,$$

so $V_h^*V_h = 1_{\mathcal{M}(C^*(f^*\Gamma))}$, and then $V_hV_h^* = V_{-h}^*V_{-h} = 1_{\mathcal{M}(C^*(f^*\Gamma))}$ also.

To see that the V_h are central, observe that

$$V_h s_{(\lambda,m)} = \lim_F \sum_{v\in F} u_{v,h}s_{(\lambda,m)} = \lim_F \sum_{v\in F}\sum_{\mu\in v\Gamma^{f(h_+)}} s_{(\mu,h_+)}s_{(\mu,h_-)}^* s_{(\lambda,m)}.$$

Applying the Cuntz-Krieger relation, we obtain

$$V_h s_{(\lambda,m)} = \lim_F \sum_{v\in F}\sum_{\mu\in v\Gamma^{f(h_+)}} \sum_{\substack{\alpha\in s(\mu)\Gamma^{f(m)} \\ \beta\in s(\lambda)\Gamma^{f(h_-)}}} s_{(\mu\alpha,h_++m)}s_{(\mu\alpha,h_-+m)}^* s_{(\lambda\beta,h_-+m)}s_{(\beta,h_-)}^*.$$

The only nonzero terms are those where $\mu\alpha = \lambda\beta$. Since Γ has no sources, for each $\beta \in s(\lambda)\Gamma^{f(h_-)}$ there is a unique $\mu \in r(\lambda)\Gamma^{f(h_-)}$ and a unique $\alpha \in s(\mu)\Gamma^{f(m)}$ such that $\mu\alpha = \lambda\beta$. So the final sum above collapses to give

$$V_h s_{(\lambda,m)} = \sum_{\beta\in s(\lambda)\Gamma^{f(h_-)}} s_{(\lambda\beta,h_++m)}s_{(\beta,h_-)}^*.$$

On the other hand,

$$\begin{aligned}s_{(\lambda,m)}V_h &= s_{(\lambda,m)} \lim_F \sum_{v\in F}\sum_{\beta\in v\Gamma^{f(h_+)}} s_{(\beta,h_+)}s_{(\beta,h_-)}^* \\ &= \sum_{\beta\in s(\lambda)\Gamma^{f(h_+)}} s_{(\lambda,m)}s_{(\beta,h_+)}s_{(\beta,h_-)}^* = \sum_{\beta\in s(\lambda)\Gamma^{f(h_+)}} s_{(\lambda\beta,h_++m)}s_{(\beta,h_-)}^*\end{aligned}$$

as required.

The universal property of $C^*(H)$ implies that there is a homomorphism $\rho : C^*(H) \to \mathcal{ZM}(C^*(f^*\Gamma))$ such that $\rho(U_h) = V_h$ for all $h \in H$. To see that ρ is injective, define an action β of $\widehat{H}$ on $C^*(f^*\Gamma)$ by $\beta_\chi(s_{(\lambda,n)}) = \chi(n)s_{(\lambda,n)}$. Then β extends to an action of $\widehat{H}$ on $\mathcal{M}C^*(f^*\Gamma)$ such that $\beta_\chi(V_h) = \chi(h)V_h$. So ρ is nonzero and equivariant for β and the dual action of $\widehat{H}$ on $C^*(H)$. Therefore ρ is injective. □

Lemma 3.4. *Let G, f, P, H be as in Lemma 3.2. Let Γ be a row-finite P-graph with no sources. For $z \in \mathbb{T}^k$, let I_z be the ideal of $C^*(f^*\Gamma)$ generated by $\{z^{-n_1}s_{(\lambda,n_1)} - z^{-n_2}s_{(\lambda,n_2)} : \lambda \in \Gamma,\ n_1, n_2 \in \mathbb{N}^k,\ f(n_1) = f(n_2) = d(\lambda)\}$. Then there is an isomorphism $\psi_z : C^*(f^*\Gamma)/I_z \to C^*(\Gamma)$ such that $\psi_z(s_{(\lambda,n)} + I_z) = z^n s_\lambda$ for all $\lambda \in \Gamma$.*

Proof. The set $\{z^n s_\lambda : (\lambda, n) \in f^*\Gamma\}$ is a Cuntz-Krieger $f^*\Gamma$-family. Hence there is a homomorphism $\psi : C^*(f^*\Gamma) \to C^*(\Gamma)$ carrying each $s_{(\lambda,n)}$ to $z^n s_\lambda$. Each generator of I_z belongs to $\ker\psi$, and hence ψ descends to a homomorphism $\psi_z : C^*(f^*\Gamma)/I_z \to C^*(\Gamma)$ satisfying $\psi_z(s_{(\lambda,n)} + I_z) = z^n s_\lambda$.

To see that this ψ_z is an isomorphism, we show that it has an inverse. Fix $\lambda \in \Gamma$ and $n_1, n_2 \in \mathbb{N}^k$ such that $f(n_1) = f(n_2) = d(\lambda)$. Then $z^{-n_1}s_{(\lambda,n_1)} - z^{-n_2}s_{(\lambda,n_2)} \in I_z$. So we may define a collection $\{t_\lambda : \lambda \in \Gamma\} \subseteq C^*(f^*\Gamma)/I_z$ by $t_\lambda = z^{-n}s_{(\lambda,n)} + I_z$ for any $n \in f^{-1}(d(\lambda))$; in particular $t_v = s_{(v,0)} + I_z$ for all $v \in \Gamma^0$. We show that the t_λ form a Cuntz-Krieger Γ-family. The t_v are mutually orthogonal projections because the $s_{(v,0)}$ are. Suppose that $s(\mu) = r(\nu)$, and fix m, n such that $f(m) = d(\mu)$ and $f(n) = d(\nu)$. Then $f(m + n) = d(\mu\nu)$, and so

$$t_\mu t_\nu = z^{-m}s_{(\mu,m)}z^{-n}s_{(\nu,n)} + I_z = z^{-m-n}s_{(\mu\nu,m+n)} + I_z = t_{\mu\nu}.$$

Moreover, $t_\mu^* t_\mu = s_{(\mu,m)}^* s_{(\mu,m)} + I_z = s_{(s(\mu),0)} + I_z = t_{s(\mu)}$. Fix $v \in \Gamma^0$, $p \in P$ and $m \in \mathbb{Z}^k$ with $f(m) = p$. Then

$$t_v = s_{(v,0)} + I_z = \sum_{\alpha \in (v,0)(f^*\Gamma)^m} s_\alpha s_\alpha^* + I_z = \sum_{\lambda \in v\Gamma^p} s_{(\lambda,m)}s_{(\lambda,m)}^* + I_z = \sum_{\lambda \in v\Gamma^p} t_\lambda t_\lambda^*.$$

So $\{t_\lambda : \lambda \in \Gamma\}$ is a Cuntz-Krieger Γ-family as claimed. Hence there is a homomorphism $C^*(\Gamma) \to C^*(f^*\Gamma)/I_z$ satisfying $s_\lambda \mapsto z^{-n}s_{(\lambda,n)} + I_z$ for any $n \in f^{-1}(d(\lambda))$; this homomorphism is an inverse for ψ_z. □

In the following theorem, given a C^*-algebra A, we write $\mathrm{Irr}(A)$ for the collection of all irreducible representations of A.

Theorem 3.5. *Let G be a finitely generated abelian group, and let $f : \mathbb{Z}^k \to G$ be a homomorphism. Let $P = f(\mathbb{N}^k) \subseteq G$ and let $H = \ker(f) \subseteq \mathbb{Z}^k$. Let Γ be a row-finite P-graph with no sources. For $z \in \mathbb{T}^k$, let $q_z : C^*(f^*\Gamma) \to C^*(f^*\Gamma)/I_z$ be the quotient map, and let $\psi_z : C^*(f^*\Gamma)/I_z \to C^*(\Gamma)$ be the isomorphism of Lemma 3.4. Let π be an irreducible representation of $C^*(\Gamma)$. Then $\pi \circ \psi_z \circ q_z$ is an irreducible representation of $C^*(f^*\Gamma)$. Fix a map $\gamma \mapsto z_\gamma$ from $\widehat{H}$ to $\mathbb{T}^k$ such that $z_\gamma^h = \gamma(h)$ for all $\gamma \in \widehat{H}$ and $h \in H$. Then $(\gamma, \pi) \mapsto \pi \circ \psi_{z_\gamma} \circ q_{z_\gamma}$ is a bijection of $\widehat{H} \times \mathrm{Irr}(C^*(\Gamma))$ onto $\mathrm{Irr}(C^*(f^*\Gamma))$.*

Remark 3.6. Theorem 3.5 together with the definition of I_z implies that if π is an irreducible representation of $C^*(f^*\Gamma)$, then there is a unique character γ of $\widehat{H}$ such that $s_{(\lambda,n_1)} - \gamma(n_1 - n_2)s_{(\lambda,n_2)} \in \ker(\pi)$ for all $\lambda \in \Gamma$ and $n_1, n_2 \in f^{-1}(d(\lambda))$.

We collect some more technical lemmas before proving Theorem 3.5.

Lemma 3.7. *Let G, f, P, H be as in Lemma 3.2. Let Γ be a row-finite P-graph with no sources. For $z \in \mathbb{T}^k$, let I_z be the ideal of $C^*(f^*\Gamma)$ generated by $\{z^{-n_1}s_{(\lambda,n_1)} - z^{-n_2}s_{(\lambda,n_2)} :$*

$\lambda \in \Gamma,\ n_1, n_2 \in \mathbb{N}^k,\ f(n_1) = f(n_2) = d(\lambda)\}$. *For* $w, z \in \mathbb{T}^k$ *we have* $I_w = I_z$ *if and only if* $w^h = z^h$ *for all* $h \in H$.

Proof. If $w^h = z^h$ for all $h \in H$, then $I_w = I_z$ because they have the same generators. Now suppose that $I_w = I_z$. Fix $h \in H$. For $v \in \Gamma^0$ we have $w^h s_{(v,0)} - u_{v,h} \in I_w = I_z \ni z^h s_{(v,0)} - u_{v,h}$. Subtracting, we obtain $(w^h - z^h)s_{(v,0)} \in I_z$. The isomorphism ψ_z of Lemma 3.4 carries each $s_{(v,0)} + I_z$ to s_v which is nonzero by Proposition 2.7. Hence $s_{(v,0)} \notin I_z$, which forces $w^h - z^h = 0$. Hence $w^h = z^h$ for all $h \in H$. □

Lemma 3.8. *Let* G, f, P, H *be as in Lemma 3.2. Let* Γ *be a row-finite* P*-graph with no sources. Let* φ *be an irreducible representation of* $C^*(f^*\Gamma)$ *on a Hilbert space* $\mathcal{H}$. *Then* φ *has a unique strict extension to a representation* $\widetilde{\varphi}$ *of* $\mathcal{M}C^*(f^*\Gamma)$ *and there is a unique character* γ_φ *of* H *such that* $\widetilde{\varphi}(V_h) = \gamma_\varphi(h)1_{\mathcal{H}}$ *for all* $h \in H$. *For any* $z \in \mathbb{T}^k$ *such that* $z^h = \gamma_\varphi(h)$ *for all* $h \in H$, *we have* $I_z \subseteq \ker(\varphi)$. *There is an irreducible representation* $\widehat{\varphi}$ *of* $C^*(f^*\Gamma)/I_z$ *such that* $\widehat{\varphi}(a + I_z) = \varphi(a)$ *for all* a, *and* $\widehat{\varphi} \circ \psi_z^{-1}$ *is an irreducible representation of* $C^*(\Gamma)$.

Proof. The irreducible representation $\varphi : C^*(f^*\Gamma) \to B(\mathcal{H})$ is extendible since it is nondegenerate. Since φ is irreducible, $\widetilde{\varphi}$ is also. Since the V_h are central in $\mathcal{M}C^*(f^*\Gamma)$, the $\widetilde{\varphi}(V_h)$ are central in $\widetilde{\varphi}(\mathcal{M}C^*(f^*\Gamma))$. Therefore the $\widetilde{\varphi}(V_h)$ are all scalar multiples of identity operator $1_{\mathcal{H}}$ by, for example, [15, Theorem 4.1.12]. That is, for each $h \in H$, there exists $\gamma_\varphi(h) \in \mathbb{T}$ such that $\widetilde{\varphi}(V_h) = \gamma_\varphi(h)1_{\mathcal{H}}$. Since $\widetilde{\varphi}$ is multiplicative, $\gamma_\varphi : H \to \mathbb{T}$ is a homomorphism, so $\gamma_\varphi \in \widehat{H}$. The uniqueness of γ_φ is obvious.

Fix $z \in \mathbb{T}^k$ such that $z^h = \gamma_\phi(h)$ for all $h \in H$. For $h \in H$, we have

$$\varphi(z^h s_{(v,0)} - u_{v,h}) = \varphi((z^h V_0 - V_h)s_{(v,0)}) = \widetilde{\varphi}(z^h V_0 - V_h)\varphi(s_{(v,0)}) = 0.$$

So φ descends to a representation $\widehat{\varphi}$ of $C^*(f^*\Gamma)/I_z$, which is irreducible because φ is. □

Proof of Theorem 3.5. Since the homomorphism $\psi_z : C^*(f^*\Gamma)/I_z \to C^*(\Gamma)$ of Lemma 3.4 is an isomorphism, the composition $\psi_z \circ q_z$ is surjective, and hence $\pi \circ \psi_z \circ q_z$ is an irreducible representation of $C^*(f^*\Gamma)$.

To see that $(\gamma, \pi) \mapsto \pi \circ \psi_{z_\gamma} \circ q_{z_\gamma}$ is surjective, fix an irreducible representation φ of $C^*(f^*\Gamma)$. Let γ_φ be the corresponding character of H given in Lemma 3.8, and let $\psi_{z_{\gamma_\varphi}} : C^*(f^*\Gamma)/I_{z_\varphi} \to C^*(\Gamma)$ be the isomorphism of Lemma 3.4. By Lemma 3.8, there is an irreducible representation $\widehat{\varphi}$ of $C^*(f^*\Gamma)/I_{z_{\gamma_\varphi}}$ such that $\widehat{\varphi} \circ q_{z_{\gamma_\varphi}} = \varphi$, and then $\widehat{\varphi} \circ \psi_{z_{\gamma_\varphi}}^{-1}$ is an irreducible representation of $C^*(\Gamma)$. Now

$$(\widehat{\varphi} \circ \psi_{z_{\gamma_\varphi}}^{-1}) \circ \psi_{z_{\gamma_\varphi}} \circ q_{z_{\gamma_\varphi}} = \widehat{\varphi} \circ q_{z_{\gamma_\varphi}} = \varphi,$$

so $(\gamma, \pi) \mapsto \pi \circ \psi_{z_\gamma} \circ q_{z_\gamma}$ is surjective.

To see that it is injective, fix π and γ. We claim that for $z \in \mathbb{T}^k$, we have $I_z \subseteq \ker(\pi \circ \psi_{z_\gamma} \circ q_{z_\gamma})$ if and only if $z^h = \gamma(h)$ for all $h \in H$. First suppose that $I_z \subseteq \ker(\pi \circ \psi_{z_\gamma} \circ q_{z_\gamma})$. Fix $h \in H$, and let $\theta = \pi \circ \psi_{z_\gamma} \circ q_{z_\gamma}$. Then $\theta(u_{v,h}) = z_\gamma^h \pi(s_v)$. Hence

$$0 = \theta(z^h s_{(v,0)} - u_{v,h}) = (z^h - \gamma(h))\pi(s_v).$$

Since π is nonzero, there exists $v \in \Gamma^0$ such that $\pi(s_v) \neq 0$ forcing $z^h = \gamma(h)$ for all $h \in H$. Now suppose that $z^h = \gamma(h)$ for all $h \in H$. Then every generator of I_z belongs to $\ker\theta$, giving $I_z \subseteq \ker\theta$.

Fix $(\gamma_1, \pi_1), (\gamma_2, \pi_2) \in \widehat{H} \times \mathrm{Irr}(C^*(\Gamma))$, and write $\theta_i := \pi_i \circ \psi_{z_{\gamma_i}} \circ q_{z_{\gamma_i}}$ for $i = 1, 2$. Suppose that $\theta_1 = \theta_2$. Then $I_z \subseteq \ker(\theta_1) = \ker(\theta_2)$ for all $z \in \mathbb{T}^k$. Thus $\gamma_1(h) = \gamma_2(h)$ for all $h \in H$ by the preceding paragraph; that is $\gamma_1 = \gamma_2$. This forces $\psi_{z_{\gamma_1}} \circ q_{z_{\gamma_1}} = \psi_{z_{\gamma_2}} \circ q_{z_{\gamma_2}}$. Since these are surjections onto $C^*(\Gamma)$, the equality $\theta_1 = \theta_2$ forces $\pi_1 = \pi_2$. □

4. Maximal tails, periodicity, and pullbacks

Theorem 3.12 of [11] implies that the maximal tails in a strongly aperiodic k-graph index the primitive gauge-invariant ideals in its C^*-algebra. These maximal tails are also a key ingredient in our catalogue of the primitive ideals of an arbitrary k-graph C^*-algebra. In this section we show that every maximal tail T in a k-graph contains a hereditary subset H for which the subgraph $H\Lambda T$ is isomorphic to a pullback of a P-graph as described in Section 2. This implies that for gauge-invariant ideals associated to maximal tails, the quotient contains a corner whose primitive-ideal space is described by Theorem 3.5. This in turn is the key to our main theorem in Section 5.

We recall the definition of a maximal tail from [11].

Definition 4.1 ([11, Definition 3.10])**.** Let Λ be a row-finite k-graph with no sources. A nonempty subset T of Λ^0 is called a *maximal tail* if

(a) for every $v_1, v_2 \in T$ there is $w \in T$ such that $v_1\Lambda w \neq \emptyset$ and $v_2\Lambda w \neq \emptyset$,
(b) for every $v \in T$ and $1 \leq i \leq k$ there exists $\lambda \in v\Lambda^{e_i}$ such that $s(\lambda) \in T$, and
(c) for $w \in T$ and $v \in \Lambda^0$ with $v\Lambda w \neq \emptyset$ we have $v \in T$.

In [5], Davidson and Yang comprehensively analyse aperiodicity for single-vertex 2-graphs. The following results substantially generalise this analysis, but the fundamental idea behind them is due to Davidson-Yang.

Let Λ be a row-finite k-graph with no sources such that Λ^0 is a maximal tail. We define a relation on Λ by

$$\mu \sim \nu \quad \text{if and only if} \quad s(\mu) = s(\nu) \text{ and } \mu x = \nu x \text{ for all } x \in s(\mu)\Lambda^\infty.$$

This is an equivalence relation on Λ which respects range, source and composition. Thus $\Lambda/{\sim}$ is a category with respect to $r([\lambda]) = [r(\lambda)]$, $s([\lambda]) = [s(\lambda)]$ and $[\lambda][\mu] = [\lambda\mu]$. Define $\operatorname{Per}(\Lambda) \subseteq \mathbb{Z}^k$ by

$$\operatorname{Per}(\Lambda) := \{d(\mu) - d(\nu) : \mu, \nu \in \Lambda \text{ and } \mu \sim \nu\},$$

and define

$$H_{\rm Per} := \big\{v \in \Lambda^0 : \text{ for all } \lambda \in v\Lambda \text{ and } m \in \mathbb{N}^k \text{ such that } d(\lambda) - m \in \operatorname{Per}(\Lambda), \\ \text{there exists } \mu \in v\Lambda^m \text{ such that } \lambda \sim \mu\big\}. \tag{2}$$

For the next result recall that if Λ is a k-graph then a subset H of Λ^0 is *hereditary* if $s(H\Lambda) \subseteq H$.

Theorem 4.2. *Let Λ be a row-finite k-graph with no sources such that Λ^0 is a maximal tail.*

(1) *The set $\operatorname{Per}(\Lambda)$ is a subgroup of $\mathbb{Z}^k$.*
(2) *The set $H_{\rm Per}$ is a nonempty hereditary subset of Λ^0, and for all $p, q \in \mathbb{N}^k$ such that $p - q \in \operatorname{Per}(\Lambda)$ and all $x \in H_{\rm Per}\Lambda^\infty$, we have $\sigma^p(x) = \sigma^q(x)$.*
(3) *If $r(\lambda) \in H_{\rm Per}$ and $d(\lambda) - m \in \operatorname{Per}(\Lambda)$, then there is a unique $\mu \in r(\lambda)\Lambda^m$ such that $\mu \sim \lambda$; in particular, if $\lambda \sim \mu$ and $d(\lambda) = d(\mu)$, then $\lambda = \mu$.*
(4) *If $q_{\rm Per} : \mathbb{Z}^k \to \mathbb{Z}^k/\operatorname{Per}(\Lambda)$ is the quotient map, then the set $\Gamma := (H_{\rm Per}\Lambda)/{\sim}$ is a $q_{\rm Per}(\mathbb{N}^k)$-graph with degree map $\widetilde{d} := q_{\rm Per} \circ d$.*
(5) *The assignment $\lambda \mapsto ([\lambda], d(\lambda))$ is an isomorphism of $H_{\rm Per}\Lambda$ onto the pullback $q_{\rm Per}^*\Gamma$.*

Proof of Theorem 4.2(1). Since $\lambda \sim \lambda$ for all $\lambda \in \Lambda$, we have $0 \in \operatorname{Per}(\Lambda)$. That $\sim$ is symmetric shows that $\operatorname{Per}(\Lambda)$ is closed under inverses in $\mathbb{Z}^k$. To see that it is closed under addition, suppose that $m, n \in \operatorname{Per}(\Lambda)$, and fix $\mu \sim \nu$ and $\eta \sim \zeta$ such that $d(\mu) - d(\nu) = m$

and $d(\eta) - d(\zeta) = n$. Since Λ^0 is a maximal tail, there exist $\alpha \in s(\mu)\Lambda$ and $\beta \in s(\zeta)\Lambda$ such that $s(\alpha) = s(\beta) = v$. We have $\mu\alpha y = \nu\alpha y$ for all $y \in v\Lambda^\infty$. Hence

$$\sigma^{d(\mu)}(y) = \sigma^{d(\mu)+d(\nu\alpha)}(\nu\alpha y) = \sigma^{d(\nu)+d(\mu\alpha)}(\mu\alpha y) = \sigma^{d(\nu)}(y)$$

for all $y \in v\Lambda^\infty$; and similarly, $\sigma^{d(\eta)}(y) = \sigma^{d(\zeta)}(y)$ for all $y \in v\Lambda^\infty$. Now

$$\begin{aligned}\sigma^{d(\mu)+d(\eta)}(y) = \sigma^{d(\mu)}(\sigma^{d(\eta)}(y)) &= \sigma^{d(\mu)}(\sigma^{d(\zeta)}(y))\\ &= \sigma^{d(\zeta)}(\sigma^{d(\mu)}(y)) = \sigma^{d(\zeta)}(\sigma^{d(\nu)}(y)) = \sigma^{d(\nu)+d(\zeta)}(y)\end{aligned}$$

for all $y \in v\Lambda^\infty$. Fix $\xi \in v\Lambda^{d(\mu)+d(\eta)}$ and $\xi' \in s(\xi)\Lambda^{d(\nu)+d(\zeta)}$. Factorise $\xi\xi' = \tau\tau'$ with $d(\tau) = d(\xi')$ and $d(\tau') = d(\xi)$. For $z \in s(\xi')\Lambda^\infty$, we have $\xi\xi' z \in v\Lambda^\infty$, and hence

$$\xi' z = \sigma^{d(\xi)}(\xi\xi' z) = \sigma^{d(\mu)+d(\eta)}(\xi\xi' z) = \sigma^{d(\nu)+d(\zeta)}(\tau\tau' z) = \tau' z.$$

It follows that $\xi' \sim \tau'$ and hence that $m + n = d(\tau') - d(\xi')$ belongs to $\mathrm{Per}(\Lambda)$. □

To prove Theorem 4.2(2), we need some technical results.

Lemma 4.3. *Let Λ be a row-finite k-graph with no sources such that Λ^0 is a maximal tail. For $v \in \Lambda^0$, let $\Sigma_v := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : \sigma^p(x) = \sigma^q(x)$ for all $x \in v\Lambda^\infty\}$.*

(1) *For each $\lambda \in \Lambda$, we have $\Sigma_{r(\lambda)} \subseteq \Sigma_{s(\lambda)}$.*
(2) *The relation Σ_v is an equivalence relation on $\mathbb{N}^k$ for each $v \in \Lambda^0$.*
(3) *The set Σ_v is a sub-monoid of $\mathbb{N}^k \times \mathbb{N}^k$ for each $v \in \Lambda^0$.*

Proof. (1) Fix $\lambda \in \Lambda$ and $(p, q) \in \Sigma_{r(\lambda)}$. For $x \in s(\lambda)\Lambda^\infty$, we have

$$\sigma^p(x) = \sigma^{d(\lambda)}(\sigma^p(\lambda x)) = \sigma^{d(\lambda)}(\sigma^q(\lambda x)) = \sigma^q(x),$$

so $(p, q) \in \Sigma_{s(\lambda)}$.

(2) It is routine to check that Σ_v is reflexive, symmetric and transitive.

(3) Fix $v \in \Lambda^0$. Part (2) implies that $(0, 0) \in \Sigma_v$. Suppose that $(p, q), (p', q') \in \Sigma_v$, and fix $x \in v\Lambda^\infty$. Then

$$\sigma^{p+p'}(x) = \sigma^p(\sigma^{p'}(x)) = \sigma^p(\sigma^{q'}(x)).$$

Part (1) implies that $(p, q) \in \Sigma_{s(x(0,q'))} = \Sigma_{r(\sigma^{q'}(x))}$, and hence $\sigma^p(\sigma^{q'}(x)) = \sigma^q(\sigma^{q'}(x)) = \sigma^{q+q'}(x)$. Hence $(p + p', q + q') \in \Sigma_v$. □

Proposition 4.4. *Let Λ be a row-finite k-graph with no sources such that Λ^0 is a maximal tail. Let $\Sigma_\Lambda = \bigcup_{v\in\Lambda^0} \Sigma_v$. Then Σ_Λ is an equivalence relation on $\mathbb{N}^k$ and a submonoid of $\mathbb{N}^k \times \mathbb{N}^k$. The set $\Sigma_\Lambda^{\min}$ of minimal elements of $\Sigma_\Lambda \setminus \{0\}$ (under the usual ordering on $\mathbb{N}^k \times \mathbb{N}^k$) is finite and generates Σ_Λ as a monoid; and $(\Sigma_\Lambda - \Sigma_\Lambda) \cap (\mathbb{N}^k \times \mathbb{N}^k) = \Sigma_\Lambda$.*

Proof. That Σ_Λ is reflexive and symmetric follows from the same properties of the Σ_v. If $(p, q), (q, r) \in \Sigma_\Lambda$ then there exist v, w such that $(p, q) \in \Sigma_v$ and $(q, r) \in \Sigma_w$. Since Λ^0 is a maximal tail, there exists $u \in \Lambda^0$ such that $v\Lambda u$ and $w\Lambda u$ are nonempty. Lemma 4.3(1) then implies that $(p, q), (q, r) \in \Sigma_u$. Lemma 4.3(2) therefore implies that $(p, r) \in \Sigma_u \subseteq \Sigma_\Lambda$.

We show that Σ_Λ is a submonoid of $\mathbb{N}^k \times \mathbb{N}^k$. That each Σ_v is a monoid gives $(0, 0) \in \Sigma_\Lambda$, so it suffices to show that Σ_Λ is closed under addition. Suppose that (p, q) and (r, s) belong to Σ_Λ. As above, there exists $u \in \Lambda^0$ such that $(p, q), (r, s) \in \Sigma_u$. Lemma 4.3(3) then implies that $(p, q) + (r, s) \in \Sigma_u \subseteq \Sigma_\Lambda$.

Let $G := \{p - q : p, q \in \Sigma_\Lambda \ \subseteq \mathbb{Z}^k \times \mathbb{Z}^k$. We must show that $G \cap (\mathbb{N}^k \times \mathbb{N}^k) = \Sigma_\Lambda$. Since Σ_Λ is a monoid, the containment $\supseteq$ is clear. For the reverse containment, suppose that $(p, q), (r, s) \in \Sigma_\Lambda$ and that $p-r$ and $q-s$ belong to $\mathbb{N}^k$; we must show that $(p, q)-(r, s) \in \Sigma_\Lambda$. As above, there exists $u \in \Lambda^0$ such that (p, q) and (r, s) both belong to Σ_u. Fix

$\lambda \in u\Lambda^{r+s}$, and let $w = s(\lambda)$. Lemma 4.3(2) implies that (s, r) belongs to Σ_u and then that $(p + s, q + r) \in \Sigma_u$. So for $x \in w\Lambda^\infty$, we have

$$\sigma^{p-r}(x) = \sigma^{p-r}(\sigma^{r+s}(\lambda x)) = \sigma^{p+s}(\lambda x) = \sigma^{q+r}(\lambda x) = \sigma^{q-s}(\sigma^{r+s}(\lambda x)) = \sigma^{q-s}(x).$$

Thus $(p, q) - (r, s) \in \Sigma_w \subseteq \Sigma_\Lambda$.

The set $\Sigma_\Lambda^{\min}$ is completely unarranged in the sense that no two distinct elements of $\Sigma_\Lambda^{\min}$ are comparable under $\leq$. Hence $\Sigma_\Lambda^{\min}$ is finite by Dickson's Theorem (see [22, Theorem 5.1]). An induction (see, for example, [22, Proposition 8.1]) shows that it generates Σ_Λ as a monoid. □

Lemma 4.5. *Let Λ be a row-finite k-graph with no sources and suppose that Λ^0 is a maximal tail. Suppose that $p, q \in \mathbb{N}^k$ and $n \in \mathbb{Z}^k$ satisfy $(p - n, q - n) \in \Sigma_\Lambda$. Then $(p, q) \in \Sigma_\Lambda$.*

Proof. Let $n_+ = n \vee 0$ and $n_- = (-n) \vee 0$. Then $n_+, n_- \in \mathbb{N}^k$ and $n = n_+ - n_-$. Putting $p' := p - n$ and $q' := q - n$, we have

$$(p, q) = ((p', q') + (n_+, n_+)) - (n_-, n_-) \in (\Sigma_\Lambda - \Sigma_\Lambda) \cap (\mathbb{N}^k \times \mathbb{N}^k).$$

Proposition 4.4 therefore implies that $(p, q) \in \Sigma_\Lambda$. □

Lemma 4.6. *Let Λ be a row-finite k-graph with no sources such that Λ^0 is a maximal tail. For $p, q \in \mathbb{N}^k$, we have $(p, q) \in \Sigma_\Lambda$ if and only if $p - q \in \mathrm{Per}(\Lambda)$.*

Proof. Suppose that $p, q \in \mathbb{N}^k$ satisfy $p - q \in \mathrm{Per}(\Lambda)$. Fix $\mu \sim \nu$ in Λ such that $p - q = d(\mu) - d(\nu)$. Let $p' = d(\mu)$ and $q' = d(\nu)$. Since $\mu \sim \nu$, for $x \in s(\mu)\Lambda^\infty$ we have

$$\sigma^{p'}(x) = \sigma^{p'+q'}(\nu x) = \sigma^{p'+q'}(\mu x) = \sigma^{q'}(x).$$

So $(p', q') \in \Sigma_{s(\mu)} \subseteq \Sigma_\Lambda$. We have $p - q = p' - q'$, and hence $p - p' = q - q'$. Thus $n := p - p' \in \mathbb{Z}^k$ and $p, q \in \mathbb{N}^k$ satisfy $(p - n, q - n) \in \Sigma_\Lambda$. Thus Lemma 4.5 implies that $(p, q) \in \Sigma_\Lambda$.

Now suppose that $(p, q) \in \Sigma_\Lambda$. Fix v such that $(p, q) \in \Sigma_v$. Fix $\lambda \in v\Lambda^{p+q}$. Factorise $\lambda = \mu\alpha = \nu\beta$ with $d(\mu) = d(\beta) = p$ and $d(\nu) = d(\alpha) = q$. For $x \in s(\alpha)\Lambda^\infty$, we have

$$\alpha x = \sigma^p(\mu\alpha x) = \sigma^q(\nu\beta x) = \beta x.$$

Hence $\alpha \sim \beta$ and so $p - q = d(\beta) - d(\alpha) \in \mathrm{Per}(\Lambda)$. □

Proof of Theorem 4.2(2). We begin by showing that H_{Per} is nonempty. By Proposition 4.4 the finite set $\Sigma_\Lambda^{\min}$ generates Σ_Λ as a monoid. For each $(p, q) \in \Sigma_\Lambda^{\min}$ there exists $w \in \Lambda^0$ such that $(p, q) \in \Sigma_w$. Let $|\Sigma_\Lambda^{\min}|$ be the cardinality of $\Sigma_\Lambda^{\min}$. By $|\Sigma_\Lambda^{\min}|$ applications of condition (a) for the maximal tail Λ^0, there exists $w \in \Lambda^0$ such that $\Sigma_\Lambda^{\min} \subseteq \Sigma_w$. Since Σ_w is a monoid and $\Sigma_w \subseteq \Sigma_\Lambda$, it follows that $\Sigma_w = \Sigma_\Lambda$. Let $N := \bigvee\{p \vee q : (p, q) \in \Sigma_\Lambda^{\min}\}$, and fix $\eta \in w\Lambda^N$. We claim that $v := s(\eta)$ belongs to H_{Per}. To see this, fix $\lambda \in v\Lambda$ and $m \in \mathbb{N}^k$ such that $d(\lambda) - m \in \mathrm{Per}(\Lambda)$. Lemma 4.6 implies that $(d(\lambda), m) \in \Sigma_\Lambda$. So Proposition 4.4 implies that there exist elements $(p_1, q_1), \dots, (p_l, q_l) \in \Sigma_\Lambda^{\min}$ such that

$$(d(\lambda), m) = \sum_{i=1}^{l} (p_i, q_i).$$

We will argue by induction that for $0 \leq j \leq l$ there exists $\lambda_j \in v\Lambda$ such that $\lambda_j \sim \lambda$ and $d(\lambda_j) = d(\lambda) + \sum_{i=1}^{j}(q_i - p_i)$. Putting $\lambda_0 := \lambda$ establishes a base case. Now suppose that we have λ_{j-1} with the desired properties. Recall that $r(\lambda) = s(\eta)$ where $\eta \in \Lambda^N$ and $\Sigma_{r(\eta)} = \Sigma_\Lambda$. Let $\mu := \eta(N - q_j, N)$. By construction of λ_{j-1} we have

$$d(\mu\lambda_{j-1}) = q_j + d(\lambda) + \textstyle\sum_{i=1}^{j-1}(q_i - p_i) \geq p_j,$$

so $\lambda_j := (\mu\lambda_{j-1})(p_j, d(\mu\lambda_{j-1}))$ satisfies

$$d(\lambda_j) = d(\mu\lambda_{j-1}) - p_j = d(\lambda) + \textstyle\sum_{i=1}^{j}(q_i - p_i).$$

Fix $x \in s(\lambda_j)\Lambda^\infty$. We have

$$\lambda_j x = (\mu\lambda_{j-1})(p_j, d(\mu\lambda_{j-1}))x = \sigma^{p_j}(\mu\lambda_{j-1}x).$$

Lemma 4.3(1) implies that $\Sigma_\Lambda = \Sigma_{r(\eta)} \subseteq \Sigma_{r(\nu)} \subseteq \Sigma_\Lambda$, giving equality throughout. Thus $(p_j, q_j) \in \Sigma_{r(\nu)}$, and so

$$\sigma^{p_j}(\mu\lambda_{j-1}x) = \sigma^{q_j}(\mu\lambda_{j-1}x) = \lambda_{j-1}x.$$

So $\lambda_j \sim \lambda_{j-1}$. Since $\lambda_{j-1} \sim \lambda$ by the inductive hypothesis, $\lambda_j \sim \lambda$.

Now $\mu := \lambda_l$ satisfies $\mu \in v\Lambda^m$ and $\lambda \sim \mu$. Hence $v \in H_{\mathrm{Per}}$ as claimed.

We show that H_{Per} is hereditary. Suppose that $r(\alpha) \in H_{\mathrm{Per}}$. Fix $\lambda \in s(\alpha)\Lambda$ and $m \in \mathbb{N}^k$ such that $d(\lambda) - m \in \mathrm{Per}(\Lambda)$. Then $d(\alpha\lambda) - (m + d(\alpha)) \in \mathrm{Per}(\Lambda)$. Hence there exists $\zeta \in r(\alpha)\Lambda$ such that $d(\zeta) = m + d(\alpha)$ and $\zeta \sim \alpha\lambda$. Since $\zeta x = \alpha\lambda x$ for all $x \in s(\lambda)\Lambda^\infty$, and since and $d(\zeta) > d(\alpha)$, we have $\zeta(0, d(\alpha)) = \alpha$. Thus $\zeta = \alpha\mu$ for some $\mu \in s(\alpha)\Lambda^m$. For $x \in s(\lambda)\Lambda$,

$$\lambda x = \sigma^{d(\alpha)}(\alpha\lambda x) = \sigma^{d(\alpha)}(\zeta x) = \sigma^{d(\alpha)}(\alpha\mu x) = \mu x.$$

So $\lambda \sim \mu$, whence $s(\alpha) \in H_{\mathrm{Per}}$.

To see that $\sigma^p(x) = \sigma^q(x)$ whenever $p - q \in \mathrm{Per}(\Lambda)$ and $x \in H_{\mathrm{Per}}\Lambda^\infty$, fix such p, q, x. Let $\lambda = x(0, p)$. Then $x = \lambda\sigma^p(x)$. Since $r(x) \in H_{\mathrm{Per}}$, there exists $\mu \in r(x)\Lambda^q$ such that $\mu \sim \lambda$. We then have

$$\sigma^p(x) = \sigma^q(\mu\sigma^p(x)) = \sigma^q(\lambda\sigma^p(x)) = \sigma^q(x). \qquad \square$$

Proof of Theorem 4.2(3). Fix $\lambda \in H_{\mathrm{Per}}\Lambda$ and $m \in \mathbb{N}^k$ such that $d(\lambda) - m \in \mathrm{Per}(\Lambda)$. Suppose that $\mu, \nu \in \Lambda^m$ satisfy $\mu \sim \lambda \sim \nu$. Since $\sim$ is an equivalence relation, we have $\mu \sim \nu$. So it suffices to show that if $\mu \sim \nu$ and $d(\mu) = d(\nu)$, then $\mu = \nu$. Fix $x \in s(\mu)\Lambda^\infty$ and observe that $\mu \sim \nu$ implies $\mu x = \nu x$ and so $\mu = (\mu x)(0, m) = (\nu x)(0, m) = \nu$. $\square$

Proof of Theorem 4.2(4). We showed that Γ is a category at the beginning of this section. Since d is a functor, $\widetilde{d} := q_{\mathrm{Per}} \circ d$ is also a functor. We must show that $(\Gamma, \widetilde{d})$ has the unique factorisation property.

Let $q = q_{\mathrm{Per}(\Lambda)} : \mathbb{Z}^k \to \mathbb{Z}^k/\,\mathrm{Per}(\Lambda)$. Fix $\lambda \in \Lambda$ with $r(\lambda) \in H_{\mathrm{Per}}$ and $m, n \in \mathbb{N}^k$ such that $\widetilde{d}([\lambda]) = q(m) + q(n)$. We must show that there is a unique pair $\alpha, \beta \in \Gamma$ such that $\widetilde{d}(\alpha) = q(m)$, $\widetilde{d}(\beta) = q(n)$ and $\alpha\beta = [\lambda]$. We begin by showing that such a pair exists. Since $d(\lambda)-(m+n) \in \mathrm{Per}(\Lambda)$ and $r(\lambda) \in H_{\mathrm{Per}}$, there exists $\mu \in r(\lambda)\Lambda^{m+n}$ such that $\mu \sim \lambda$. Let $\alpha := [\mu(0, m)]$ and $\beta := [\mu(m, m + n)]$. Then $\alpha\beta = [\mu(0, m)\mu(m, m + n)] = [\mu] = [\lambda]$ because $\mu \sim \lambda$; and $d(\alpha) = q(m)$ and $d(\beta) = q(n)$ by construction.

For uniqueness, suppose that $\mu_1, \nu_1, \mu_2, \nu_2 \in H_{\mathrm{Per}}\Lambda$, satisfy $r(\nu_i) = s(\mu_i)$, $q(d(\mu_i)) = q(m)$ and $q(d(\nu_i)) = q(n)$. Suppose further that $\mu_1\nu_1 \sim \lambda \sim \mu_2\nu_2$. We must show that $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$. Fix $x \in s(\nu_1)\Lambda^\infty$. Then

$$\nu_1 x = \sigma^{d(\mu_1)}(\mu_1\nu_1 x) = \sigma^{d(\mu_1)}(\mu_2\nu_2 x).$$

We have $q(d(\mu_1)) = q(m) = q(d(\mu_2))$, and so $d(\mu_1) - d(\mu_2) \in \mathrm{Per}(\Lambda)$. Hence Theorem 4.2(2) implies that $\sigma^{d(\mu_1)}(y) = \sigma^{d(\mu_2)}(y)$ for all $y \in r(\mu_2)\Lambda^\infty$, and in particular

$$\sigma^{d(\mu_1)}(\mu_2\nu_2 x) = \sigma^{d(\mu_2)}(\mu_2\nu_2 x) = \nu_2 x.$$

Thus $\nu_1 \sim \nu_2$. Since $d(\mu_1) - d(\mu_2) \in \mathrm{Per}(\Lambda)$ and since $r(\mu_1) \in H_{\mathrm{Per}}$, there exists a unique $\zeta \in r(\mu_1)\Lambda^{d(\mu_2)}$ such that $\mu_1 \sim \zeta$. Fix $x \in s(\lambda)\Lambda^\infty$. We have

$$\zeta = (\zeta\nu_1 x)(0, d(\mu_2)) = (\mu_1\nu_1 x)(0, d(\mu_2)) = (\mu_2\nu_2 x)(0, d(\mu_2)) = \mu_2. \qquad \square$$

Proof of Theorem 4.2(5). It is routine to check that $\lambda \mapsto ([\lambda], d(\lambda))$ is a k-graph morphism from $H_{\mathrm{Per}}\Lambda$ to $q^*_{\mathrm{Per}(\Lambda)}(\Gamma)$. This k-graph morphism is injective by part (3). Fix $\xi \in H_{\mathrm{Per}}\Lambda$ and $m \in \mathbb{N}^k$ such that $m - d(\xi) \in \mathrm{Per}(\Lambda)$. Then there exists $\eta \in r(\xi)\Lambda^m$ such that $\eta \sim \xi$. Thus $([\eta], d(\eta)) = ([\xi], m)$. So $\lambda \mapsto ([\lambda], d(\lambda))$ is surjective and hence an isomorphism. □

5. The primitive ideals of a k-graph algebra

In this section we prove our main theorem, giving a complete listing of the primitive ideals in the C^*-algebra of a row-finite k-graph with no sources.

Lemma 5.1. *Let Λ be a row-finite k-graph such that Λ^0 is a maximal tail. Let $H = H_{\mathrm{Per}}$ be the hereditary set (2), and let $\Gamma = H\Lambda/\sim$. Let $q = q_{\mathrm{Per}} : \mathbb{Z}^k \to \mathbb{Z}^k/\mathrm{Per}(\Lambda)$ be the quotient map, and let $P := q(\mathbb{N}^k)$. Then*

1. *there is a unique $*$-homomorphism $\phi : C^*(q^*\Gamma) \to C^*(\Lambda)$ such that $\phi(s_{([\lambda],d(\lambda))}) = s_\lambda$ for $\lambda \in H\Lambda$;*
2. *the series $\sum_{v\in H} s_v$ converges strictly to a projection P_H in $\mathcal{M}C^*(\Lambda)$, and ϕ is an isomorphism of $C^*(q^*\Gamma)$ onto $P_H C^*(\Lambda)P_H$; and*
3. *the corner $P_H C^*(\Lambda)P_H$ is a full corner of the ideal $I_H = \overline{\mathrm{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda H\}$.*

Proof. Theorem 4.2(5) implies that $\lambda \mapsto ([\lambda], d(\lambda))$ is a k-graph isomorphism between $H\Lambda$ and $f^*\Gamma$. Statement (1) then follows from the universal property of $C^*(q^*\Gamma)$.

The argument of [3, Lemma 1.1] shows that $\sum_{v\in H} s_v$ converges strictly to a multiplier projection P_H such that $P_H s_\mu s_\nu^* = \chi_H(r(\mu)) s_\mu s_\nu^*$. We have $P_H C^*(\Lambda)P_H = \overline{\mathrm{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in H\Lambda\}$, so the image of ϕ is $P_H C^*(\Lambda)P_H$. Proposition 2.7 implies that ϕ is injective, giving (2).

For (3), observe that I_H is the ideal generated by $\{s_v : v \in H\}$ and so is generated as an ideal of $C^*(\Lambda)$ by P_H. Hence P_H determines a full corner in I_H. □

Lemma 5.2. *Let Λ be a row-finite k-graph with no sources and let T be a maximal tail of Λ. Then there is an infinite path $x \in (\Lambda T)^\infty$ which is cofinal in T in the sense that for each $v \in T$, there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n) \neq \emptyset$.*

Proof. Let $\{v_j\}_{j=0}^\infty$ be a listing of T. Condition (a) for a maximal tail implies that there exists $\alpha_1 \in v_0 T$ such that $v_1\Lambda s(\alpha_1) \neq \emptyset$. Applying the same condition again gives $\alpha_2 \in s(\alpha_1)T$ such that $v_2\Lambda s(\alpha_2) \neq \emptyset$. Inductively, we construct $\{\alpha_j\}_{j=1}^\infty$ such that $r(\alpha_j) = s(\alpha_{j-1})$ and $v_j\Lambda s(\alpha_j) \neq \emptyset$ for all j. Define $\beta_0 = v_0$ and $\beta_i = \beta_{i-1}\alpha_i$ for $i \geq 1$. Then $v_i\Lambda s(\beta_j) \neq \emptyset$ whenever $i \leq j$, and $d(\beta_j) \to (\infty, \dots, \infty)$ as $j \to \infty$. The argument of [12, Remarks 2.2] shows that there is a unique $x \in (\Lambda T)^\infty$ such that $x(0, d(\beta_i)) = \beta_i$ for all i. This x is cofinal by construction. □

In the following, given an infinite path x in a k-graph Λ, we write $[x]$ for the shift-tail equivalence class $\{\lambda\sigma^m(x) : m \in \mathbb{N}^k, \lambda \in \Lambda x(m)\}$.

Theorem 5.3. *Let Λ be a row-finite k-graph with no sources.*

1. *Let T be a maximal tail of Λ and γ a character of $\mathrm{Per}(\Lambda T)$. Suppose that $z \in \mathbb{T}^k$ satisfies $z^{m-n} = \gamma(m-n)$ for all $m - n \in \mathrm{Per}(\Lambda T)$. Suppose that $x \in (\Lambda T)^\infty$ is cofinal in T. Then there is an irreducible representation $\pi_{[x],z} : C^*(\Lambda) \to \mathcal{B}(\ell^2([x]))$ determined by*

$$\pi_{[x],z}(s_\lambda)\xi_y = \begin{cases} z^{d(\lambda)}\xi_{\lambda y} & \text{if } y(0) = s(\lambda) \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

(2) *Let I be a primitive ideal in $C^*(\Lambda)$. Then $T = \{v \in \Lambda^0 : s_v \notin I\}$ is a maximal tail. Let $H = H_{\mathrm{Per}}(\Lambda T)$. There is a unique character γ of $\mathrm{Per}(\Lambda T)$ such that $s_\mu - \gamma(d(\mu) - d(\nu))s_\nu \in I$ whenever $\mu, \nu \in H\Lambda T$ and $\mu \sim_{\Lambda T} \nu$. Suppose that $x \in (\Lambda T)^\infty$ is cofinal in ΛT and that $z \in \mathbb{T}^k$ satisfies $z^{m-n} = \gamma(m-n)$ for all $m - n \in \mathrm{Per}(\Lambda T)$. Then $I = \ker \pi_{[x],z}$.*

Proof. (1) It is routine to check that the operators $\{\pi_{[x],z}(s_\lambda) : \lambda \in \Lambda\}$ defined by (3) constitute a Cuntz-Krieger Λ-family. The universal property of $C^*(\Lambda)$ then yields a homomorphism $\pi := \pi_{[x],z} : C^*(\Lambda) \to \mathcal{B}(\ell^2([x]))$ extending (3).

Fix $y \in [x]$. For $w \in [x]$ and $n \in \mathbb{N}^k$,

$$\pi(s_{y(0,n)}s^*_{y(0,n)})\delta_w = \begin{cases} \delta_w & \text{if } w(0,n) = y(0,n) \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\lim_{n\to\infty} \pi(s_{y(0,n)}s^*_{y(0,n)})\delta_w = \theta_{\delta_y,\delta_y}\delta_w$. Thus the net $(\pi(s_{y(0,n)}s^*_{y(0,n)}))_{n\in\mathbb{N}^k}$ converges strongly to $\theta_{y,y}$. We claim that the strong closure of the image of π contains $\mathcal{K}(\ell^2([x]))$. Fix $w, y \in [x]$ and fix $p_i, q_i \in \mathbb{N}^k$ such that $\sigma^{p_1}(w) = \sigma^{q_1}(x)$ and $\sigma^{p_2}(y) = \sigma^{q_2}(x)$. Let $q = q_1 \vee q_2$, $\alpha = w(0, p_1 + (q - q_1))$ and $\beta = y(0, p_2 + (q - q_2))$. Then $w = \alpha\sigma^q(x)$ and $y = \beta\sigma^q(x)$. We have $\pi(s_{\alpha x(q,q+n)}s^*_{\beta x(q,q+n)}) = \pi(s_\alpha s^*_\beta)\pi(s_{y(0,d(\beta)+n)}s^*_{y(0,d(\beta)+n)})$ which converges strongly to $\pi(s_\alpha s^*_\beta)\theta_{y,y} = \theta_{w,y}$. Thus, the strong closure of the image of π contains $\mathcal{K}(\ell^2([x]))$ as claimed. Hence π is irreducible.

(2) Since I is a primitive ideal in $C^*(\Lambda)$, there is an irreducible representation $\pi : C^*(\Lambda) \to \mathcal{B}(\mathcal{H})$ such that $\ker \pi = I$. Let $K = \{v \in \Lambda^0 : \pi(s_v) = 0\}$. Then [17, Theorem 5.2(a)] implies that K is hereditary and saturated. So the set

$$T = \{v \in \Lambda^0 : s_v \notin I\} = \{v \in \Lambda^0 : \pi(s_v) \neq 0\} = \Lambda^0 \setminus K.$$

satisfies conditions (b) and (c) of a maximal tail. To establish (a), fix $v, w \in T$ and $h \in p_v\mathcal{H} \setminus \{0\}$. Since π is irreducible, h is cyclic for π. So there exists $a \in C^*(\Lambda)$ such that $\pi(a)h \in \pi(p_w)\mathcal{H} \setminus \{0\}$; that is, $0 \neq \pi(p_w)\pi(a)h = \pi(p_w a p_v)h$. In particular, $p_w a p_v \neq 0$. Since $C^*(\Lambda) = \overline{\mathrm{span}}\{s_\alpha s^*_\beta : s(\alpha) = s(\beta)\}$ we may approximate a by a linear combination of such $s_\alpha s^*_\beta$. Thus there exist α, β with $s(\alpha) = s(\beta)$ such that $p_w s_\alpha s^*_\beta p_v \neq 0$. Hence $u := s(\alpha) = s(\beta)$ satisfies $v\Lambda u, w\Lambda u \neq \emptyset$, and so T satisfies (a).

Let $H = H_{\mathrm{Per}}(\Lambda T)$, and let I_H be the ideal of $C^*(\Lambda)$ generated by $\{s_v : v \in H\}$. Since $\pi|_{I_H}$ is nonzero, Theorem 1.3.4 of [1] shows that $\pi_H := \pi|_{I_H}$ is an irreducible representation. Lemma 5.1(2) implies that $I_H P_H$ is a Morita equivalence between $P_H C^*(\Lambda) P_H$, so $\ker(\pi_H)$ is induced from the primitive ideal $\ker(\pi_H)|_{P_H C^*(\Lambda) P_H}$. Let $\Gamma = (H\Lambda T)/\sim$. Then Lemma 5.1(2) provides an isomorphism $\omega : C^*(q^*\Gamma) \to P_H C^*(\Lambda) P_H$, and then $\pi_H \circ \omega$ is an irreducible representation $C^*(q^*\Gamma)$. Now Theorem 3.5 implies (see Remark 3.6) that there is a unique character γ of $\mathrm{Per}(\Lambda T)$ such that $s_\mu - \gamma(d(\mu) - d(\nu))s_\nu \in I$ whenever $\mu, \nu \in H\Lambda T$ and $\mu \sim_{\Lambda T} \nu$. It is routine to check that if x is cofinal in ΛT and $z^{m-n} = \gamma(m-n)$ for $m - n \in \mathrm{Per}(\Lambda T)$, then $s_\mu - \gamma(d(\mu) - d(\nu))s_\nu \in \ker \pi_{[x],z}$ whenever $\mu, \nu \in H\Lambda T$ and $\mu \sim_{\Lambda T} \nu$.

We claim that $\ker(\pi_H|_{P_H I_H P_H}) = \ker(\pi_{[x],z}|_{P_H I_H P_H})$. Each of $\pi \circ \omega$ and $\pi_{[x],z} \circ \omega$ is a representation of $C^*(q^*\Gamma)$ which vanishes on the generators of the ideal I_z of Lemma 3.4 and is nonzero on every vertex projection $s_{(v,0)} \in C^*(q^*\Gamma)$. So $\pi \circ \omega$ and $\pi_{[x],z} \circ \omega$ descend to representations ρ and $\rho_{[x],z}$ of $C^*(q^*\Gamma)/I_z$. Lemma 3.4 gives an isomorphism $\theta : C^*(\Gamma) \to C^*(q^*\Gamma)/I_z$, and the representations $\rho \circ \theta$ and $\rho_{[x],z} \circ \theta$ of $C^*(\Gamma)$ are nonzero on vertex projections. Hence Corollary 2.8 implies that $\rho \circ \theta$ and $\rho_{[x],z} \circ \theta$ are both injective. Thus ρ and $\rho_{[x],z}$ are both injective, which implies that $\ker(\pi \circ \omega) = I_z = \ker(\pi_{[x],z} \circ \omega)$. Since

P_H is a strict limit of projections in I_H, we have $P_H I_H P_H = P_H C^*(\Lambda) P_H$, and so ω is an isomorphism from $C^*(q^*\Gamma)$ to $P_H I_H P_H$. This proves the claim.

Since $I_H P_H$ is a Morita equivalence, the claim implies that $\ker(\pi_H) = \ker(\pi_{[x],z}|_{I_H})$. Now [19, Proposition 2.72] applied to the adjointable left action of $C^*(\Lambda)$ by left multiplication on the standard Hilbert module $(I_H)_{I_H}$ implies that $\ker \pi = \ker \pi_{[x],z}$. □

Let T be a maximal tail of Λ and γ a character of $\mathrm{Per}(\Lambda T)$. Theorem 5.3 implies that if x, y are both cofinal in T and $w, z \in \mathbb{T}^k$ satisfy $w^{m-n} = \gamma(m-n) = z^{m-n}$ whenever $m - n \in \mathrm{Per}(\Lambda T)$, then $\ker \pi_{[x],z} = \ker \pi_{[y],w}$. We define $I_{T,\gamma} := \ker \pi_{[x],z}$. We write $\mathrm{MT}(\Lambda)$ for the collection of all maximal tails in Λ^0.

Corollary 5.4. *Let Λ be a row finite k-graph with no sources. The assignment $(T, \gamma) \mapsto I_{T,\gamma}$ is a bijection from $\bigcup_{T \in \mathrm{MT}(\Lambda)}(\{T\} \times \widehat{\mathrm{Per}(\Lambda T)})$ to the set of primitive ideals of $C^*(\Lambda)$.*

Proof. This follows immediately from Theorem 5.3 and Lemma 5.2. □

Remark 5.5. It is worthwhile to point out what the above catalogue of primitive ideals says for aperiodic maximal tails T. In this instance, we have $\mathrm{Per}(\Lambda T) = \{0\}$ so the only character is the identity character $\gamma(0) = 1$. The Cuntz-Krieger uniqueness theorem implies that $\pi_{[x],1}$ restricts to a faithful representation of $C^*(H\Lambda T)$. Thus $I_{T,1}$ is precisely the gauge-invariant primitive ideal associated to T described in [11]. In particular, if Λ is strongly aperiodic in the sense of [11], so that every maximal tail T is aperiodic, then our result recovers the listing given in [11].

We close by characterising primitivity of $C^*(\Lambda)$; recall that a C^*-algebra A is primitive if it has a faithful irreducible representation.

Corollary 5.6. *Let Λ be a row-finite k-graph with no sources. Then $C^*(\Lambda)$ is primitive if and only if Λ^0 is a maximal tail and Λ is aperiodic.*

Proof. First suppose that Λ^0 is a maximal tail and Λ is aperiodic. Lemma 5.2 implies that there is an infinite path x of Λ which is cofinal in Λ^0. Consider the irreducible representation $\pi_{[x],1}$ of (3). Since Λ is a maximal tail, we have $\pi_{[x],1}(s_v) \neq 0$ for all $v \in \Lambda^0$. Since Λ is aperiodic, the Cuntz-Krieger uniqueness theorem implies that $\pi_{[x],1}$ is faithful. Hence $C^*(\Lambda)$ is primitive.

Now suppose that $C^*(\Lambda)$ is primitive. Then $\{0\}$ is a primitive ideal of $C^*(\Lambda)$. Theorem 5.3(2) implies that $T := \{v \in \Lambda^0 : s_v \notin \{0\}\}$ is a maximal tail, and that there is a character χ of $\mathrm{Per}(T)$ such that $I_{T,\chi} = \{0\}$. Since the generators of $C^*(\Lambda)$ are all nonzero, $T = \Lambda^0$, so Λ^0 is a maximal tail. To see that Λ is aperiodic, we must show that $\mathrm{Per}(\Lambda) = \{0\}$. By Theorem 4.2, it suffices to show that $\mu \sim \nu$ implies $d(\mu) = d(\nu)$. Suppose that $\mu \sim \nu$. Then $s_\mu - \chi(d(\mu) - d(\nu))s_\nu \in I_{T,\chi}$, and hence $s_\mu = \chi(d(\mu) - d(\nu))s_\nu$. For $z \in \mathbb{T}^k$, we then have

$$z^{d(\mu)} s_\mu = \gamma_z(s_\mu) = \gamma_z(\chi(d(\mu) - d(\nu))s_\nu) = z^{d(\nu)}\chi(d(\mu) - d(\nu))s_\nu.$$

So $z^{d(\mu)-d(\nu)} s_\mu = \chi(d(\mu) - d(\nu))s_\nu$ for all $z \in \mathbb{T}^k$. Since the right-hand side is nonzero and independent of z, we deduce that $z^{d(\mu)-d(\nu)}$ is constant with respect to z, forcing $d(\mu) = d(\nu)$. □

References

[1] W. Arveson, An invitation to C^*-algebras, Graduate Texts in Mathematics, No. 39, Springer-Verlag, New York, 1976, x+106.

[2] T. Bates, J.H. Hong, I. Raeburn, and W. Szymański, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois J. Math. **46** (2002), 1159–1176.

[3] T. Bates, D. Pask, I. Raeburn, and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324.

[4] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.

[5] K.R. Davidson and D. Yang, *Periodicity in rank 2 graph algebras*, Canad. J. Math. **61** (2009), 1239–1261.

[6] M. Enomoto and Y. Watatani, *A graph theory for C^*-algebras*, Math. Japon. **25** (1980), 435–442.

[7] R. Exel, *Non-Hausdorff étale groupoids*, Proc. Amer. Math. Soc. **139** (2011), 897–907.

[8] N.J. Fowler, M. Laca, and I. Raeburn, *The C^*-algebras of infinite graphs*, Proc. Amer. Math. Soc. **128** (2000), 2319–2327.

[9] J.H. Hong and W. Szymański, *The primitive ideal space of the C^*-algebras of infinite graphs*, J. Math. Soc. Japan **56** (2004), 45–64.

[10] J.A. Jeong, G.H. Park, and D.Y. Shin, *Stable rank and real rank of graph C^*-algebras*, Pacific J. Math. **200** (2001), 331–343.

[11] S. Kang and D. Pask, *Aperiodicity and the primitive ideal space of a row-finite k-graph C^*-algebra*, preprint 2011 (arXiv:1105.1208 [math.OA]).

[12] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.

[13] A. Kumjian, D. Pask, and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174.

[14] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.

[15] G.J. Murphy, C^*-algebras and Operator Theory, Academic Press Inc., Boston, MA, 1990, x+286.

[16] I. Raeburn, Graph algebras, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005, vi+113.

[17] I. Raeburn, A. Sims, and T. Yeend, *Higher-rank graphs and their C^*-algebras*, Proc. Edinb. Math. Soc. (2) **46** (2003), 99–115.

[18] I. Raeburn and W. Szymański, *Cuntz-Krieger algebras of infinite graphs and matrices*, Trans. Amer. Math. Soc. **356** (2004), 39–59.

[19] I. Raeburn and D.P. Williams, Morita equivalence and continuous-trace C^*-algebras, American Mathematical Society (1998).

[20] J. Renault, A groupoid approach to C^*-algebras, Springer, Berlin, 1980, ii+160.

[21] G. Robertson and T. Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. reine angew. Math. **513** (1999), 115–144.

[22] J.C. Rosales and P.A. García-Sánchez, Finitely Generated Commutative Monoids, Nova Science Publishers Inc., Commack, NY, 1999, xiv+185.

[23] M. Tomforde, *Ext classes and embeddings for C^*-algebras of graphs with sinks*, New York J. Math. **7** (2001), 233–256.

Toke Carlsen, Department of Mathematical Sciences, NTNU, NO-7491, Trondheim, Norway

E-mail address: `Toke.Meier.Carlsen@math.ntnu.no`

Sooran Kang, Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin 9054, New Zealand

E-mail address: `sooran@maths.otago.ac.nz`

Jacob Shotwell, TBLR, Arizona State University, Tempe, AZ 85281, USA

E-mail address: `shotwell@asu.edu`

Aidan Sims, School of Mathematics and Applied Statistics, University of Wollongong, Wollongong 2522, Australia

E-mail address: `asims@uow.edu.au`