Pythagoras tells us that in a right angled triangle with sides a, b, and hypotenuse c that

$$a^2 + b^2 = c^2$$

We are all familiar with the integer triples $(3, 4, 5)$ and $(5, 12, 13)$. It might be of interest to know how many other integer Pythagorean triples there are.

The first observation that we can make, from similar triangles, is that each triple we know about is a member of a family.

Thus $(3, 4, 5)$ is the first member of the family $(3, 4, 5)$, $(6, 8, 10)$, $(9, 12, 15)$, $(12, 16, 20)$, $(15, 20, 25)$, $(18, 24, 30)$ ..... and so on. So there is at least one Pythagorean triple whenever the smallest side a is a multiple of 3.

Similarly, $(5, 12, 13)$ is the first member of the family $(5, 12, 13)$, $(10, 24, 26)$, $(15, 36, 39)$, $(20, 48, 52)$ .... and so on.

We notice that there are two distinct triples with a smallest side of 15, because there are two prime factors of 15.

So far we have established that at least one triple exists, with a smallest side respectively, of $3, 5, 6, 9, 10, 12, 15, 18, 20, 21, \ldots$. Maybe now we should try to fill in some more holes.

You could keep yourself awake at night playing with different combinations, if you are good at mental arithmetic, or if you have a calculator handy. But looking at the two triples that we akready know, we can see that if the smallest side is a prime, the other two sides are separated by 1. So let's make an inspired guess (or Ansatz, a word mathematicians like to use), that this holds true for any prime.

So we assume that $(p, q, q + 1)$ will be a Pythagorean triple for any prime p.

$$p^2 + q^2 = (q + 1)^2$$

$$2q = p^2 - 1$$

We should always be able to find a q for any odd prime. Thus $(7, 24, 25)$, and $(11, 60, 61)$ are also Pythagorean triples, each of which will give rise to a whole family. Fiddling around with a calculator might have let us find $(7, 24, 25)$, but beyond that might have been a bit of a stretch.

Looking now at the families of triples we have found so far, we have one for each of the respective smallest sides: $3, 5, 6, 7, 9, 10, 11, 12, 13, 17, 15, 17, 18, \ldots$. The only holes that we have left are the powers of 2.

So, choose $a = 2^k$. If we try our previous Ansatz, we would have

$$2^{2k} + q^2 = (q + 1)^2$$

$$2^{2k} = 2q + 1$$

which must fail because one side is even and the other is odd. So, in the spirit of guessing, suppose we try $(2^k, q, q+2)$

$$2^{2k} + q^2 = (q+2)^2$$

$$4q + 4 = 2^{2k}$$

This looks more promising, becauee 2^{2k} will always be divisible by 4.

$k = 3$ yields the family beginning $(8, 15, 17)$.

$k = 4$ yields the family beginning $(16, 63, 65)$, as well as $(16, 30, 34)$ from the $(8, 15, 17)$ family.

You might wonder why I started with 8 and not 4. The above approach does yield the result $(4, 3, 5)$, but we have already decided to characterise our families by beginning with the smallest side, so 4 misses out.

That completes the whole zoo. There are many more families than I expected to uncover when I first wondered. In fact it is only the primes and 8 that are associated with just one family.