Barycentric coordinates

This is a brief introduction to barycentric coordinates in the plane. We begin with a result not as widely known as one might expect.

Let $P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3)$ be points defining a triangle in the plane. We will assume that the points are labelled in counter-clockwise order, for reasons that will become clear. Then the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = A$$

is twice the area of the triangle $P_1P_2P_3$.

Perhaps the easiest way to see this is to note that

$$\begin{vmatrix} 1 & 0 & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 \\ y_1 & y_2 - y_1 & y_3 - y_1 \end{vmatrix} = A$$

Let

$$x_2 = x_1 + a * cos(\theta),$$
$$y_2 = y_1 + a * sin(\theta)$$
$$x_3 = x_1 + b * cos(\phi)$$
$$y_3 = y_1 + b * sin(\phi)$$

Then

$$\begin{aligned} A &= a * b * (cos(\theta) * sin(\phi) - sin(\theta) * cos(\phi)) \\ &= 2 * (\frac{1}{2} * a * b * sin(\phi - \theta)) \qquad (1) \end{aligned}$$

where the second term is easily recognisable as the area of the triangle in the form $\frac{1}{2}a * b * sin(C)$ It also explains the reason for numbering the vertices of the triangle in counterclockwise order, so that $sin(\phi - \theta)$ will be positive.

Next we turn to the definition of barycentric coordinates (b_1, b_2, b_3) for a point $P(x, y)$ in the plane, in relation to the triangle $P_1P_2P_3$. They are defined by the relation

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

Some properties that flow easily from the definition are as follows:

1. The b_i are linear functions of x and y.
2. $$b_1 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}$$

   shows $b_1 = \text{Area}(PP_2P_3) \ / \ \text{Area}(P_1P_2P_3)$
3. Similar results hold for b_2 and b_3.
4. For points $P(x, y)$ inside the triangle, $0 \le b_i \le 1$
5. For all points $P(x, y)$ in the plane, $b_1 + b_2 + b_3 = 1$.

These coordinates are used to define **bivariate splines** of degree d, Bezier-Bernstein polynomials, which are polynomials in x and y at each point $P(x, y)$ in the plane.

$$B^d_{i,j,k}(b_1, b_2, b_3) = \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k, \quad i + j + k = d$$

For $d = 0$, $B^0_{0,0,0} = 1$

For $d = 1$, $B^1_{1,0,0} = b_1, B^1_{0,1,0} = b_2, B^1_{0,0,1} = b_3$

These span the same space as $1,\ x,\ y$

For $d = 2,\ B^2_{2,0,0}, B^2_{1,1,0}, B^2_{1,0,1}, B^2_{0,2,0}, B^2_{0,1,1}, B^2_{0.0.2}$

which are, respectively, $b_1^2,\ 2b_1b_2,\ 2b_1b_3,\ b_2^2,\ 2b_2b_3,\ b_3^2,$

span the same space as $1,\ x,\ y,\ x^2,\ xy,\ y^2$.

So, for example, any polynomial of degree 2 in x and y in the triangle, or indeed the plane, can be written

$$p(v(x,y)) = \sum_{i+j+k=2} c_{i,j,k} B^2_{i,j,k}(b_1, b_2, b_3)$$

where the $b_1,\ b_2\ , b_3$ are evaluated at the point $v(x,y)$.

If we wanted to fit the surface p(x,y) to a set of experimental observations (x_i, y_i, f_i) using the usual least squares criterion, we would simply minimise with respect to the **c** vector, the function

$$R(\mathbf{c}) = \frac{1}{2}\sum_i (p(x_i, y_i) - f_i)^2$$

This is a relatively simple problem if we have only one triangle. Complications arise as soon as the plane is subdivided into a set of touching triangles, and we want to approximate a function over the whole region.

If we performed the calculation, triangle-by-triangle in isolation, we would get an optimum fit to the points in each triangle, but the overall surface generated by this approach would be discontinuous along the edges between adjacent triangles. To obtain a smoother surface we need to impose continuity relations between the functions on different triangles, which amount to linear relationships between the **c** vectors on adjacent triangles.

We can illustrate the procedure in the more familiar basis setting for $d = 1$. Suppose on triangle 1:

$$p^1(v(x,y)) = c^1_0\ + c^1_1 * x\ + c^1_2 * y$$

and on triangle 2:

$$p^2(v(x,y)) = c_0^2\ + c_1^2 * x\ + c_2^2 * y$$

and that the equation of the edge connecting the triangles is

$$y = \alpha\ *\ x\ +\ \beta$$

So, along the edge

$$p^1(v) = c_0^1 + c_1^1 * x + c_2^1 * (\alpha + \beta * x)$$

is continuous. ie

$$d_1 * c_1^1 + d_2 * c_2^1 = d_1 * c_1^2 + d_2 * c_2^2$$

For the direction along the edge $(1, \alpha)$ this condition is already satisfied, so if we choose any other **d**, it will be true for any linear combination of directions, and hence for any direction.

Similar arguments can be applied when the functions $p(x, y)$ on neighbouring triangles are expressed in the Bezier spline basis. I will simply quote the general result derived by Lai and Schumaker (2007). If the first triangle has vertices (v_1, v_2, v_3), in clockwise order, with a common edge (v_2, v_3) with a second triangle (v_4, v_3, v_2), in clockwise order, the polynomial functions of degree d, and their derivatives up to order r, are continuous on the edge if

$$C_{n,j,k} = \sum_{f+g+h=n} c_{f,k+g,j+h} B^n_{f,g,h}(v_4)$$

for $j + k = d - n$, and $n = 0 \ldots r$.

$C_{i,j,k}$ refers to the coefficients on the second triangle, and $c_{i,j,k}$ refers to the coefficients on the first triangle.

Again you can see that these are linear relations between the coefficients on different triangles.

If we return now to the problem of fitting the splines to a set of measurements (x_i, y_i, f_i), spread across a collection of neighbouring triangles, we now have to minimise the least squares function

$$R(\mathbf{c}) = \frac{1}{2}\sum(p(v_i) - f_i)^2$$

which we could solve as a series of $A^T A\mathbf{c} = A^T\mathbf{f}$ problems on individual triangles, but which is now subject to a set of linear constraints $E\mathbf{c} = \mathbf{0}$ spread across all the triangles. Firstly, we can consolidate the individual problems of minimising the least squares function in each triangle into one large minimisation problem of the function

$$R(\mathbf{c}) = \frac{1}{2}(A\mathbf{c} - \mathbf{f})^T(A\mathbf{c} - \mathbf{f})$$

subject to

$$E\mathbf{c} = \mathbf{0}$$

We can introduce a vector of Lagrange multipliers λ and solve the matrix problem

$$\begin{bmatrix} A^T A & E^T \\ E & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \lambda \end{bmatrix} = \begin{bmatrix} A^T\mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

which we can solve using the standard LAPACK routines. Inspection of the Lagrange multipliers λ can tell us the importance of insisting on a particular order of continuity across the edges of the triangles.

References:

Lai M-J., and Schumaker, L.L. (2007) Spline Functions on Triangulations. Cambridge University Press