Course Number: AMAT.2110
Course Name: Discrete Dynamics

Question 5. (20 marks)

(a) The logistic difference equation is

\[ x_{n+1} = rx_n (1 - x_n) . \]

(i) Show that the fixed points of the logistic difference equation are \( x^* = 0 \) and \( x^* = (r - 1)/r \). \[2 \text{ marks}\]

(ii) Determine the stability of the fixed points as a function of \( r \) and explain the biological implications of your answer. \[6 \text{ marks}\]

(iii) How does the solution to the logistic equation change as the parameter \( r \) is increased from 2.9 to 3.1? \[1 \text{ mark}\]

(b) Consider the difference equation

\[ x_{n+1} = 0.1 + x_n^2. \]

(i) Find the fixed points of this map and determine their stability. \[3 \text{ marks}\]

(ii) If \( x_0 = 0.8 \) explain what happens as \( n \to \infty \). \[2 \text{ mark}\]

(iii) If \( x_0 = 0.9 \) explain what happens as \( n \to \infty \). \[2 \text{ mark}\]

(c) Consider the difference equation

\[ x_{n+1} = f (x_n) , \]

(i) What do we mean when we say that the pair \( x_0^* \) and \( x_1^* \) is a period-2 orbit of the function \( f \)? \[1 \text{ mark}\]

(ii) Write down the conditions for the pair \( x_0^* \) and \( x_1^* \) to be stable and unstable. \( \text{(Remember to carefully define all the terms that appear in your solution).} \) \[3 \text{ marks}\]
Question 5. (13 marks)

Consider the logistic equation with fixed harvesting

\[ x_{n+1} = rx_n (1 - x_n) - h, \]

where the positive quantities \( r \) and \( h \) are the static birth rate and the number of animals that are harvested respectively.

(a) Show that the fixed points of the above difference equation are given by

\[ x^* = \frac{(r - 1) \pm \sqrt{(1 - r)^2 - 4rh}}{2r}. \]

[2 marks]

(b) Hence show that there is a critical value of the harvesting parameter, \( h = h_{\sigma} \), such that if \( h > h_{\sigma} \) harvesting is not sustainable. Identify this value \( h_{\sigma} \).

[3 marks]

(c) Suppose that \( r = 2 \) and \( h = 0.1 \).

(i) Sketch the graph \( y = rx_n (1 - x_n) - h \).

[1 mark]

(ii) Using your graph explain why the population will become extinct if the initial value \( (x_0) \) is either ‘too small’ or ‘too large’.

[2 marks]

(iii) Locate the values ‘too small’ and ‘too large’ on your graph.

[1 mark]

(d) Suppose that \( x \) represents the population of scallops in a particular location and that \( r = 2 \). A company decides to harvest the scallops using fixed harvesting. The profit \( P \) that the company makes is given by the formulae

\[ P = h - 0.06. \]

The company asks you to recommend a value for its annual harvest. What would you suggest to the company? (Give reasons for your suggestions.)

[4 marks]