D Taylor series

D.1 Taylor series expansions

Values of polynomials, such as $p(x) = x^3 + 2x^2 - 3x + 1$, may be readily calculated for any value of $x$ whereas many other functions, such as $f(x) = \sin x$, cannot be evaluated, for most values of $x$, without the aid of a calculator.

In section D.2 we show how to approximate a function $f$ near a given point $a$ by a polynomial.

D.2 Taylor series expansion

The Taylor series of degree $n$ that approximates a function $f$ about the point $x = a$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k,$$

where $f^{(0)}(x) = f(x)$. 

This equation

\[ f(x + a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k. \]

provides an extremely accurate polynomial approximation for a large class of functions. With reference to figure D.1, close to \( x = x_1 \) we can approximate the curve \( f(x) \) with the line tangent to the curve at \( x = x_1 \). The approximation is not so good further away from \( x = x_1 \).

It should be noted that when finding the \( n \)th Taylor polynomial, the ‘\( n \)’ refers to the degree of the highest term, and not to the number of terms.

Figure D.1: Diagram showing the first-order Taylor expansion in the vicinity of the point \( x_1 \).
Example

1. Suppose \( f(x) = \ln(x) \) with \( a = 1 \).

\[
f(1) = \ln(1) = 0
\]

\[
f'(x) = \quad f'(1) = \_
\]

\[
f''(x) = \quad f''(1) = \_
\]

\[
f'''(x) = \quad f'''(1) = \_
\]

Thus we can approximate the function \( f(x) = \ln(x) \) near the point \( x = 1 \) by the following sequence of functions.

\[
p_1(x) = \quad \_
\]

\[
p_2(x) = \quad \_
\]

\[
p_3(x) = \quad \_
\]

\[
0 + (x - 1)
\]

\[
0 + (x - 1) - \frac{1}{2!} (x - 1)^2
\]

\[
0 + (x - 1) - \frac{1}{2!} (x - 1)^2 + \frac{2}{3!} (x - 1)^3
\]
Each successive approximation is better than the previous one.

Suppose that we want to approximate the value of \( \ln(1.1) \). The values of the three polynomials given above evaluated at \( x = 1.1 \) are

\[
\begin{align*}
p_1(1.1) &= 0.1 \\
p_2(1.1) &= 0.095 \\
p_3(1.1) &= 0.095333
\end{align*}
\]

respectively whereas the exact value, correct to six decimal places, is 0.095308.
2. Find the first two terms in the Taylor series approximation to \( f(x) = \sin(x) \) near the point \( a = 0 \).

\[
\sin x = \sin 0 + x \sin' 0 + \frac{x^2}{2} \sin'' 0 + \frac{x^3}{6} \sin''' 0 + \ldots
\]

\[
= \boxed{\sin 0 + x \cos 0 - \frac{x^2}{2} \sin 0 - \frac{x^3}{6} \cos 0 + \ldots}
\]

\[
x - \frac{x^3}{6} + \ldots
\]