10 The period-doubling cascade to chaos

10.1 Behaviour in the logistic map when $0 < r < 1 + \sqrt{6}$

In section 4 we found that the logistic model (16) has two fixed points

$$x_0^* = 0, \quad x_1^* = 1 - \frac{1}{r}.$$  \hspace{1cm} (83)

When $0 < r < 1$ the only fixed-point of biological interest is the trivial solution $(x_0^*)$. This is stable for $0 < r < 1$ and unstable for $1 < r \leq 4$. The second fixed point $(x_1^*)$ only makes biological sense for $1 < r \leq 4$. It is stable for $1 < r < 3$ and unstable for $3 < r \leq 4$. When $r > 3$ both fixed points are unstable. In section 8.1 we showed that for $3 < r \leq 4$ there is a period-2 orbit which is stable for $3 < r < 1 + \sqrt{6}$ and unstable for $1 + \sqrt{6} < r \leq 4$.

This behaviour is best appreciated using a diagram on which one plots the fixed points, as functions of $r$, and also indicates their stability. Figure 9 (a) shows such a diagram. Stability is indicated by showing stable orbits with solid lines and unstable ones as dashed lines. Observe that the two points comprising the period-2 orbit ‘sandwich’ the now unstable period-1 point $(x_1^*)$ that generated them. Usually the unstable solution branches are not shown, resulting in a figure similar to figure 9 (b).

When $r = 3$ the eigenvalue of the non-trivial fixed point decreases through the value $\lambda = -1$. This corresponds to a pitchfork bifurcation.

For $3 < r < 1 + \sqrt{6}$ the solutions $x_t$ simply oscillates between the two points which are the intersections of a vertical line through the $r$-value. Numerically the map $f^2(u)$ converges to one of these two points. What happens when $r > 1 + \sqrt{6}$?

10.2 Behaviour in the logistic map when $1 + \sqrt{6} < r < r_\infty (r_\infty \sim 3.57)$ — the period-doubling cascade to chaos

When $r = 3$ the fixed point $x_1^*$ destabilised, producing two points that comprised the period-2 solution. As $r$ increases from $r = 3$, the eigenvalues $\lambda$ at $A$ and $C$ in figure 8 decrease, eventually passing through $\lambda = -1$. At this point $(r = 1 + \sqrt{6})$ the period-2 solutions become unstable. The mechanism that produced the period-2 solution from the period-1 solution is repeated: each of the period-2 points is destabilised, producing two additional solutions. A period-4 solution therefore appears at the point $r = 1 + \sqrt{6}$. Thus if $r_4 < r < r_8$, where $r_8$ is the bifurcation value to a period-8 solution, $x_t$ exhibits a period-4 solution with the values given by the intersection of the curve of equilibrium states with the vertical line through the $r$-value in figure 9 (b). The period-4 points are found by solving the equation $x^* = f^4(x^*)$.

The function $f^4(u)$ tends to one of the four period-4 points unless for some $n$, $f^n(u)$ equals one of the points of period 1 or 2.

The eigenvalue of the period-4 solution decrease as $r$ increases and eventually pass through the value $\lambda = -1$. At this point the stable period-4 solutions become unstable and a stable
Figure 9: Steady-state diagrams for the logistic model. Figure (a) shows the locus of the stable and unstable branches for the period-1 and period-2 solutions. Period doubling at $r = 3$; a period 2 orbit is born as a fixed point becomes unstable. Stable solutions are denoted by solid lines and unstable solutions by dashed lines. (b) Stable solutions for the logistic model as $r$ passes through bifurcation values. At each bifurcation the previous state becomes unstable. The sequence of stable solutions have periods $2, 2^2, 2^3, \ldots$.

The sequence $1 \rightarrow 2 \rightarrow 2^2 \rightarrow 2^3 \rightarrow \cdots$ is known as an infinite cascade of periodic orbits. At the critical value $r_\infty \approx 3.57$ all periodic solution of period $2^n$ are unstable and ‘chaos
Table 1: The values of $r$ at which an orbit of period $2^n$ becomes stable in the logistic map $u_{n+1} = ru_n(1-u_n)$.

<table>
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<th>$n$</th>
<th>Value</th>
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<tbody>
<tr>
<td>2</td>
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</tr>
</tbody>
</table>

Figure 10: Stable solutions for the logistic model (16). This picture is typical of discrete models which exhibit period doubling and eventually chaos and the subsequent path through chaos. Picture downloaded from [www.pha.jhu/~ldb/seminar/logdiffeqn.html](http://www.pha.jhu/~ldb/seminar/logdiffeqn.html).

sets in’. This process in which an orbit of period-$2^n$ successively lose stability to an orbit of period-$2^{n+1}$, ending at a limiting value at which all periodic solutions are unstable is known as the **period doubling route to chaos**.

10.3 Behaviour in the logistic map when $r_\infty < r \leq 4$

Is $r_\infty$ the end of the story? What happens when $r_\infty < r \leq 4$?\(^2\)

As the parameter $r$ is increased from $r_\infty$ there are regions where the solution is not chaotic but is instead periodic. These regions of periodicity are known as ‘windows’ and for $r_\infty < r < 4$ parameter windows of periodicity are interlaced with windows of aperiodicity. The interlacing of periodicity and aperiodicity is apparent from figures 10 & 11.

If the regions of periodicity are blow-up it is seen that each window contains its own period-doubling sequence. For instance, the period-doubling cascade associated with the period-3 window will be $3 \rightarrow 2 \times 3 \rightarrow 2^2 \times 3 \rightarrow 2^m \times 3 \rightarrow r_{3,\infty}$, where $r_{3,\infty}$ is the accumulation point at which the period-3 period-doubling cascade becomes chaotic. In fact for $r > r_\infty$ there is

\(^2\)The material in this section will not be tested. If you find it interesting you can do a project on it.
a periodic window of base period for any \( k \) (with \( k \) odd) and an associated period-doubling cascade \( k \rightarrow 2 \times k \rightarrow 2^2 \times k \rightarrow 2^m \times k \rightarrow r_{k,\infty} \) ending in a chaotic region. This behaviour is illustrated in figures 10 & 11. In particular the sequence of aperiodicity - periodicity - aperiodicity is shown in the enlargements of figure 11 (a). There exist an infinite number of windows with a finite width. The period-3 window around \( r \sim 3.84 \) is the largest window.

Although we have concentrated here on the logistic model (16) the phenomena of period-doubling cascades to chaos and of windows surrounded by regions of aperiodicity is typical of difference equation models with the dynamics like (1) and schematically illustrated in figure 1.