Final Examination
Spring Session 2008
WUCT121
Discrete Mathematics

This exam represents 60% of the total subject marks

Reading Time: 5 minutes
Time allowed: 3 Hours

DIRECTIONS TO CANDIDATES
Refer to Page 2
Question 1

(a) Using the 'quick method', prove that the following statement is a tautology:

\[(\neg p \land \neg q) \implies (p \lor q)\]

<table>
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<th>((\neg p \land \neg q))</th>
<th>(\implies)</th>
<th>((p \lor q))</th>
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Step 1: \(F\)

Step 2: \(T\)  \(T\)

Step 3: \(T\)  \(T\)

Step 4: \(F\)  \(F\)  \(F\)

Using the “quick method”, we can assign truth-values to every statement variable and connective. However, we require connective 4 to be false in step 4 but also to be true in step 2. This is impossible. Therefore, as this statement cannot be false, it must be a tautology. \([6\text{ marks}]\)

(b) Using full truth tables, determine whether the following statement is a tautology, a contradiction, or is contingent:

\[\neg ((q \lor \neg p) \land (p \lor \neg q)) \iff (\neg p \iff \neg q)\]

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<th>((\sim p \lor q))</th>
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Therefore, this statement is a tautology. \([12\text{ marks} – \text{allow } 2\text{ marks for correct order of connectives, }1\text{ mark for each correct row of table, }2\text{ marks for conclusion}]\)

(c) Write in predicate calculus notation using quantifiers and variables:

Some students cannot correctly answer all questions in this exam.

Symbolic format (long)

\[\exists s (s \text{ is a student } \land \exists q (q \text{ is a question in this exam } \land s \text{ cannot solve } q \text{ correctly})]\]

Symbolic format (short)

\[\exists s \text{ a student, } \exists q \text{ a question in this exam, } s \text{ cannot solve } q \text{ correctly}\]

[8 marks for either form – allow 4 marks for correct quantifiers and 4 marks for correct predicate joining variables \(s\) and \(q\)]

(d) Write in simple English without using quantifiers or variables:

\[\forall x \in \mathbb{R}^+ \exists y \in \mathbb{R}^+ (y < x), \text{ where } \mathbb{R}^+ \text{ is the set of positive Real numbers.}\]

There is no smallest positive Real number. \([4\text{ marks}]\)
(c) Write down, then simplify, the negations of each of the statements in parts (c) and (d) using

(i) predicate calculus notation using quantifiers and variables

(ii) simple English without using quantifiers and variables.

(c)(i) Symbolic format (long)
\~(∃ i (s is a student ∧ ∃ q (q is a question in this exam ∧ s cannot solve q correctly)))
≡ (∀ s (s is a student ⇒ ∀ q (q is a question in this exam ⇒ s can solve q correctly)))

Symbolic format (short)
\~(∃ a student s, ∃ a question q in this exam, s cannot solve q correctly)
≡ ∀ student s, ∀ question q in this exam, s can solve q correctly))

(c)(ii) Every student can solve every question in this exam correctly.

(d)(i) \~( ∀ x ∈ R⁺, ∃ y ∈ R⁺, (y < x)), where R⁺ is the set of positive Real numbers
≡ ∃ x ∈ R⁺, ∀ y ∈ R⁺, \~(y < x), where R⁺ is the set of positive Real numbers
≡ ∃ x ∈ R⁺, ∀ y ∈ R⁺, (y ≥ x), where R⁺ is the set of positive Real numbers

(d)(ii) There is a smallest positive Real number.
[12 marks – 6 marks for part (c) either form, 6 marks for part (d)]

(f) For the following sentence:
The statement that all statements are sentences but not all sentences are statements is false.

(i) Determine if it is a statement;
The sentence “The statement that all statements are sentences but not all sentences are statements is false” is a statement because it makes a claim that can be tested for truth or falsity. [4 marks]

(ii) If it is a statement, determine whether it is true or false, giving reasons for your answer.
The given sentence has the form: Statement P is false, where P is the statement that: “All statements are sentences but not all sentences are statements.” Since statement P is true, the given sentence, which declares that P is false, must, itself, be false. [4 marks]

Question 2
(50 marks)

(a) Using a direct proof, prove the following statement:

“For any real number x, 20(x – 1) ≤ 4x² + 5”

A known property of the set of real numbers is ∀ x ∈ R (x² ≥ 0), hence ∀ x ∈ R, (2x – 5)² ≥ 0 (as 2x – 5 ∈ R, by closure of multiplication and addition on R).
Now
(2x – 5)² ≥ 0
⇒ 4x² – 20x + 25 ≥ 0
⇒ 4x² + 5 – 20(x – 1) ≥ 0
⇒ 4x² + 5 ≥ 20(x – 1)
Therefore, ∀ x ∈ R, 20(x – 1) ≤ 4x² + 5 must also be true. [4 marks]
(b) Briefly explain how the following tautologies may be used in the method of proof by contradiction:

(i) \((\neg p \Rightarrow (q \land \neg q)) \Rightarrow p\)

To prove a statement \(p\) is true, we may instead attempt to prove that \(\neg p\) is false, that is, if assuming that \(\neg p\) is true leads to a contradiction namely \(q\) and \(\neg q\), then \(\neg p\) must be false and so \(p\) must be true. [4 marks]

(ii) \((p \Rightarrow q) \iff (\neg q \Rightarrow \neg p)\)

To prove a conditional statement of the form \(p \Rightarrow q\), we may instead attempt to prove the equivalent conditional statement \(\neg q \Rightarrow \neg p\). [4 marks]

(c) Using proof by contradiction or otherwise, prove the following statements:

(i) “For all integers \(n\), if \(n^3\) is even then \(n\) is even”.

This is a conditional statement of the form \(\forall n \in \mathbb{Z} \ (P \Rightarrow Q)\) where \(P\): \(n^3\) is even, and \(Q\): \(n\) is even

To prove this statement, we may instead prove the equivalent statement \(\forall n \in \mathbb{Z} \ (\neg Q \Rightarrow \neg P)\), that is, for all integers \(n\), if \(n\) is odd then \(n^3\) is odd.

Assume that \(n\) is an odd integer, that is, \(n = 2k + 1\), for some \(k \in \mathbb{Z}\).

\[n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 = 2l + 1\]

where \(l = 4k^3 + 6k^2 + 3k \in \mathbb{Z}\)

Therefore, \(n^3\) is also an odd integer. Therefore, \(\neg Q \Rightarrow \neg P\) is true for an arbitrary odd integer \(n\) and, using proof by contradiction, so is \(P \Rightarrow Q\). Therefore, the original conditional statement must also be true.

[8 marks – allow 1 mark for statement of tautology, 2 marks for restatement of problem, 1 mark for definition of even, 2 marks for algebraic manipulation of \(n^3\), 2 marks for appropriate conclusion]

(ii) “There is no largest even integer”.

Let \(P\) be the statement that “There is no largest even integer”.

Then the statement \(\neg P\) is: There is a largest even integer. If \(\neg P\) is true, let the largest even integer be \(x\) and then form the integer \(x + 2\). Clearly, \(x + 2\) is larger than \(x\) and is an even integer (because \(x = 2k\) for some integer \(k\) so that \(x + 2 = 2(k + 1)\) i.e \(x + 2 = 2j\), where \(j = k + 1\) is also an integer as the integers are closed under addition). Hence, assuming that \(\neg P\) is true leads to a contradiction viz. that \(x + 2\) is an even integer that is larger than \(x\), which is the largest even integer. Consequently, (using the tautology \((\neg p \Rightarrow (q \land \neg q)) \Rightarrow p)\) \(\neg P\) must be false and \(P\) true. i.e. There is no largest even integer is a true statement. [8 marks]
(d) Consider following statements:

- \( P: \ 2x + 1 = 3 \)
- \( Q: \ 2x = 2 \)
- \( R: \ x = 1 \)

If we assume that \( P \) is a true statement, explain how these statements, together with Modus Ponens and the Law of Syllogism, can be used to prove that \( R \) is also a true statement.

If \( 2x + 1 = 3 \) then \( 2x = 3 - 1 = 2 \), and
If \( 2x = 2 \) then \( x = 1 \)

i.e. \( P \implies Q \) and \( Q \implies R \), so by the Law of Syllogism \( P \implies R \).

Therefore, by Modus Ponens, if both \( P \) and \( P \implies R \) are tautologies, so is \( R \), and we may conclude that, if \( 2x + 1 = 3 \) is a true statement, then \( x = 1 \) must also be a true statement.

[10 marks – allow partial marks as appropriate]

(e)

(i) Complete the following tautology, which is used in the method of proof by cases:

\[ \begin{align*}
\text{\( (p \lor q) \implies r \text{ if and only if } (p \implies r) \land (q \implies r) \text{ is a tautology.} \)}
\end{align*} \]

(ii) Using proof by cases or otherwise, prove the following statement:

For all integers \( n \), \( 2n^2 - 1 \) is odd.

This statement may be written as \( \forall n \in \mathbb{Z} \) \( (2n^2 - 1 \) is odd).

Using the Quotient-Remainder Theorem it can be shown that every integer is either even or odd.

Consequently, there are two cases to consider:

Case 1: \( n \) is odd \( \Rightarrow \exists k \in \mathbb{Z}, n = 2k + 1 \)

\[ 2n^2 - 1 = 2(2k + 1)^2 - 1 = 2(4k^2 + 4k + 1) - 1 = 2m + 1 \]

where \( m = 4k^2 + 4k - 1 \in \mathbb{Z} \), hence \( n \) is odd \( \Rightarrow 2n^2 - 1 \) is odd.

Case 2: \( n \) is even \( \Rightarrow \exists k \in \mathbb{Z}, n = 2k \)

\[ 2n^2 - 1 = 2(2k)^2 - 1 = 2(4k^2) - 1 = 2m + 1 \]

where \( m = 4k^2 - 1 \in \mathbb{Z} \), hence \( n \) is even \( \Rightarrow 2n^2 - 1 \) is odd.

Therefore, \( (\text{n is even} \lor \text{n is odd}) \Rightarrow 2n^2 - 1 \text{ and so } \forall n \in \mathbb{Z}, (2n^2 - 1 \text{ is odd}). \] [10 marks]
Question 3

(a) Use mathematical induction to prove that $5^n - 1$ is divisible by 4 for all $n \in \mathbb{N}$.

Let Claim($n$) be $4 \mid 5^n - 1$ for all $n \in \mathbb{N}$, i.e. $\exists m \in \mathbb{N} \ (5^n - 1 = 4m)$

Claim(1) is $\exists m \in \mathbb{N} \ (5^1 - 1 = 4m)$ where, $m = 1 \in \mathbb{N}$, and so Claim(1) is true.

Assume Claim($k$), $k \in \mathbb{N}$, that is,

Claim (k): $\exists p \in \mathbb{N} \ (5^k - 1 = 4p)$ \quad (*)

Show Claim (k+1), that is,

Claim (k + 1): $\exists q \in \mathbb{N} \ (5^{k+1} - 1 = 4q)$

$LHS = 5^{k+1} - 1$

$= 5(5^k) - 1$

$= 4(5^k) + 5^k - 1$

$= 4(5^k) + 4p \quad$ by Assumption (*)

$= 4(5^k + p)$

$= 4q$

$= RHS$

where $q = 5^k + p \in \mathbb{N}$

Therefore, by the Principle of Mathematical Induction, Claim($n$) must be true for all natural numbers. \[6 \text{ marks}\]

(b)

(i) Use the Euclidean Algorithm to find $gcd(126, -39)$.

$126 = 3 \times 39 + 9$

$39 = 4 \times 9 + 3$

$9 = 3 \times 3 + 0$

$\therefore \ gcd(126, -39) = 3.$ \[4 \text{ marks}\]

(ii) Find $m, n \in \mathbb{Z}$ such that $126m - 39n = gcd(126, -39)$

Rewrite the equations from (i) with the remainder as the subject:

$3 = 39 - 4 \times 9 \quad$ (1)

$9 = 126 - 3 \times 39 \quad$ (2)

$3 = 39 - 4 \times (126 - 3 \times 39)$

$= 13 \times 39 - 4 \times 126$

$= 126 \times (-4) - 39 \times (-13)$

$\therefore m = -4, n = -13$

[6 marks – allow 3 marks for rewriting equations in terms of remainders and 3 marks for back-substitution to find $m$ and $n$; allow partial marks as appropriate]
Showing all steps clearly, use the Sieve of Eratosthenes to find all the prime numbers between 2 and 52. Write down all the twin primes between 2 and 52.

Write down the primes from 2 to $\sqrt{52}$ that is, 2, 3, 5, and 7

Write down all of the integers between 1 and 52.

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

First eliminate all integers that are multiples of 2.

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

Then eliminate all integers that are multiples of 3.

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

Then eliminate all integers that are multiples of 5.

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

Then eliminate all integers that are multiples of 7.
The remaining integers (excluding 1) are the set of primes between 2 and 52, that is, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

[8 marks – allow 2 marks for determination of primes to use in sieve; 4 marks for use of sieve; and 2 mark for appropriate conclusion]

(ii) Write down all the twin primes between 2 and 52.
From the primes found in (i) the twin primes between 2 and 52 are seen to be (3, 5), (5, 7), (7, 11), (11, 13), (17, 19), (29, 31), (41, 43). [2 marks]

(d) Use the Generalized Pigeonhole Principle to justify your answer to the following question:
In a group of 2000 students, must at least five have the same day of the year as their birthday?
In any given year, there are 365 (ignoring leap years) different birthdays. Treating the birthdays as pigeonholes ($k$) and the students as pigeons ($n$), the largest possible value of $m$ in the inequality $n > mk$ is sought. Since $5 \times 365 = 1825 < 2000$ and $6 \times 365 = 2190 > 2000$, it follows from the Generalised Pigeonhole Principle that at least $5 + 1$ i.e. 6 students must have the same birthday. [4 marks]

(e) Find a value of $x$ such that $x = 5^7 \pmod{17}$ and $0 \leq x < 17$.
\[5^2 \equiv 25 \equiv 8 \pmod{17} \]
\[5^4 \equiv 8^2 \equiv 13 \pmod{17} \]
\[5^6 \equiv 8 \times 8^2 \equiv 8 \times 13 \equiv 2 \pmod{17} \]
\[5^7 \equiv 5 \times 2 \equiv 10 \pmod{17} \] [4 marks]

(f) Write down the complete set of residue classes modulo 11, $\mathbb{Z}_{11}$.
$\mathbb{Z}_{11} = \{[0], [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]\}$ [3 marks]

(g) Express each of the elements $[-3], [7], [44] \in \mathbb{Z}_{11}$ as $[x] \in \mathbb{Z}_{11}, 0 \leq x < 11$.
$[-3] = [8]; [7] = [7]; [44] = [0]$ [3 marks]

(h) Let $S \subseteq \mathbb{Z}_{11}$ be given by $S = \{[1], [4], [7]\}$.

(iii) Construct the multiplication table for $S$.

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<th>[1]</th>
<th>[4]</th>
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<tr>
<td>[1]</td>
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<td>[28] = [6]</td>
<td>[49] = [5]</td>
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</table>

[6 marks]

(iv) Using this table, or otherwise, determine the identity for multiplication and the multiplicative inverses of all the elements in $S$ which have them.
The identity for multiplication is [1]. The multiplicative inverse of [1] is [1], and [1] is the only element of $S$ that has a multiplicative inverse.
[4 marks – 1 mark each for: finding the multiplicative identity, the inverse of [1], noting that [4] lacks a multiplicative inverse, noting that [7] lacks a multiplicative inverse].
Question 4 (50 marks)

(a) Let \( U = \{d, i, s, r, e, t, e, m, a, t, h, e, m, a, t, i, c, s\} \) be the Universal set.
Let \( S = \{x \in U \mid x \in \{s, e, c, r, e, t, i, c, s\}\} \), \( T = \{x \in U \mid x \in \{t, h, e, m, e, s\}\} \) and
\( C = \{x \in U \mid x \in \{t, a, c, t, i, c, s\}\} \) be subsets of the Universal set.

(i) Draw a Venn diagram showing \( S, T \) and \( C \) and \( U \).

(ii) Write down the following sets:
   
   (a) \( S \cup T = \{c, e, h, m, r, s, t\} \)
   
   (b) \( T - S = \{h, m\} \)
   
   (c) \( S - C = \{e, r\} \)
   
   (d) \( C \cap T = \{d, r\} \)

(iii) Which of the following are false? Give reasons.

   (a) \( c \subseteq C \)
   
   (b) \( \{c\} \subseteq C \)
   
   (c) \( \emptyset \in \mathcal{P}(S) \)
   
   (d) \( \emptyset \subseteq \mathcal{P}(S) \)

   (a) \( c \subseteq C \) is false as \( c \) is not a set and so cannot be a subset of \( C \).
   
   (b) \( \{c\} \subseteq C \) is true since \( c \) is an element of \( C \).
   
   (c) \( \emptyset \in \mathcal{P}(S) \) is true as the empty set is a subset of every set and so is an element of every power set.
   
   (d) \( \emptyset \subseteq \mathcal{P}(S) \) is true as the empty set is a subset of every set.

(b) Briefly explain what is meant by the term Singleton set.

A singleton set is a set that contains exactly one element.
Let $A = \{1\}$ and $B = \{4, 3, 2\}$. Write down the following sets:

(i) \((A \times B)\)

\((A \times B) = \{(1, 4), (1, 3), (1, 2)\}\) [4 marks]

(ii) \(\mathcal{P}(A \times B)\)

\(\mathcal{P}(A \times B) = \{\emptyset, \{(1, 4)\}, \{(1, 3)\}, \{(1, 2)\}, \{(1, 4), (1, 3)\}, \{(1, 4), (1, 2)\}, \{(1, 3), (1, 2)\}\) [6 marks]

(d) State the Axioms of Specification and the Axiom of Extent.

Axiom of Specification:
Given a set universe $\mathcal{U}$ and $P(x)$ any predicate statement involving $x$, then there exists a set $A$ such that $\forall x \ (x \in A \iff (x \in \mathcal{U} \land P(x)))$.

We may write this set as $A = \{x \mid x \in \mathcal{U} \land P(x)\}$, or $A = \{x \in \mathcal{U} \mid P(x)\}$.

Axiom of Extent:
If $A$ and $B$ are sets, then $A = B \iff \forall x \ (x \in A \iff x \in B)$ [4 marks – allow 2 marks for each axiom]

(e) Using (d), and the definitions of Intersection, Difference, and Complement, prove the following statement using a typical element argument:

\[X - Y = X \cap \overline{Y}\]

To prove $X - Y = X \cap \overline{Y}$, we must prove $\forall x (x \in X - Y \iff x \in X \cap \overline{Y})$.

Need to prove $\forall x (x \in X - Y \Rightarrow x \in X \cap \overline{Y})$ and $\forall x (x \in X \cap \overline{Y} \Rightarrow x \in X - Y)$ for an arbitrary element $x$.

Let $x \in X - Y$

$\Rightarrow x \in X \land x \notin Y$ Definition of set difference

$\Rightarrow x \in X \land x \in \overline{Y}$

$\Rightarrow x \in X \cap \overline{Y}$

$\therefore \forall x (x \in X - Y \Rightarrow x \in X \cap \overline{Y})$

Let $x \in X \cap \overline{Y}$

$\Rightarrow x \in X \land x \notin \overline{Y}$ Definition of union

$\Rightarrow x \in X \land x \notin Y$

$\Rightarrow x \in X - Y$ Definition of set difference

$\therefore \forall x (x \in X \cap \overline{Y} \Rightarrow x \in X - Y)$

$\therefore \forall x (x \in X - Y \iff x \in X \cap \overline{Y})$ and hence $X - Y = X \cap \overline{Y}$ [12 marks]
Question 5

(a) Explain what is meant by an ordered pair.

An ordered pair is an ordered set with two elements, that is, if \((a, b)\) and \((c, d)\) are ordered pairs, then \((a, b) = (c, d)\) if and only if \(a = c\) and \(b = d\). [3 marks]

(b) Explain what is meant by a binary relation.

A binary relation is any set of ordered pairs. [3 marks]

(c) Explain what is meant by a function from \(A\) to \(B\).

A function \(f\) from \(A\) to \(B\) denoted \(f: A \Rightarrow B\) is a binary relation from \(A\) to \(B\) (i.e. \(f \subseteq A \times B\)) such that the following two properties hold:

(1) Existence:
\[\forall x (x \in A \Rightarrow \exists y (y \in B \wedge (x, y) \in f))\]
This is equivalent to \(\text{dom } f = A\) and \(\text{range } f \subseteq B\).

(2) Uniqueness:
\[\forall x \forall y \forall z (((x, y) \in f \wedge (x, z) \in f) \Rightarrow y = z)\] [4 marks]

(d) Explain what is meant by a permutation.

A function \(f\) on \(A\) is a permutation if \(f\) is both one-to-one and onto \(A\). [2 marks]

(e) Let \(T\) be a relation on \(\{1, 2, 3, 4, 5\}\) defined as follows:

\[T = \{(x, y) \mid x > y \wedge x + y\text{ is odd and prime}\}\]

(i) Write down the domain and range of \(T\).

Consider the Cartesian Product of \(\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}\)
\[= \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)\}\]
Eliminating those elements for which \(x > y\) is false leaves the following relation:
\[\{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3), (5, 4)\}\]
Eliminating those elements for which \(x + y\text{ is odd}\) is false leaves the relation:
\[\{(2, 1), (3, 2), (4, 1), (4, 3), (5, 2), (5, 4)\}\]
Eliminating those elements for which \(x + y\text{ is prime}\) is false leaves the relation
\[T = \{(2, 1), (3, 2), (4, 1), (4, 3), (5, 2)\}\]
\(\text{Dom } T = \{2, 3, 4, 5\}\) \(\text{Range } T = \{1, 2, 3\}\) [6 marks]
(ii) Write down the elements and sketch the graph of $T$. 
From 5(e)(i) $T = \{(2, 1), (3, 2), (4, 1), (4, 3), (5, 2)\}$

![Graph of T]

(iii) Write down the elements and sketch the graph of $T^{-1}$. 
$T^{-1} = \{(1, 2), (2, 3), (1, 4), (3, 4), (2, 5)\}$

![Graph of T^{-1}]

(iv) Are either $T$ or $T^{-1}$ functions? Give a brief explanation why or why not.
Neither $T$ nor $T^{-1}$ are functions because they fail one or both of the required existence and uniqueness properties of functions:
(4, 1) and (4, 3) are both elements of $T$ so $T$ fails the “not one-to-many” property. Similarly, (2, 3) and (2, 5) are both elements of $T^{-1}$ so $T^{-1}$ also fails the “not one-to-many” property.
Also, \( T \) is a relation on \( \{1, 2, 3, 4, 5\} \) but 1 is not an element of the domain of \( T \) so \( T \) fails the “existence” property.

Further, \( T^{-1} \) is a relation on \( \{1, 2, 3, 4, 5\} \) but neither 4 nor 5 are elements of the domain of \( T^{-1} \) so \( T^{-1} \) also fails the “existence” property. [6 marks – allow 3 marks each for suitable reason]

(v) Are either \( T \) or \( T^{-1} \) onto \( \{1, 2, 3, 4, 5\} \)? Give a brief explanation why or why not.

\( T \) is not onto \( \{1, 2, 3, 4, 5\} \) as the range \( T = \{1, 2, 3\} \) is not equal to \( \{1, 2, 3, 4, 5\} \).

\( T^{-1} \) is not onto \( \{1, 2, 3, 4, 5\} \) as the range \( T^{-1} = \{2, 3, 4, 5\} \) is not equal to \( \{1, 2, 3, 4, 5\} \). [6 marks – allow 3 marks each for suitable reason]

(f) Simplify the following permutations:

(i) \( (1 \ 2) \cdot (4 \ 3) \cdot (2 \ 5) \)

This may be interpreted as follows:
1 goes to 2 and then 2 is unchanged and then 2 goes to 5
2 goes to 1 and then 1 is unchanged
3 is unchanged and then 3 goes to 4 and then 4 is unchanged
4 is unchanged then 4 goes to 3 and then 3 is unchanged
5 is unchanged and again 5 is unchanged then 5 goes to 2
resulting in \( (1 \ 5 \ 2) \cdot (3 \ 4) \) [4 marks]

(ii) \( (\begin{smallmatrix} 1 & 3 & 5 \\ 3 & 1 & 4 \\ 4 & 5 & 2 \end{smallmatrix})^{-1} \cdot (\begin{smallmatrix} 4 & 3 & 5 \\ 5 & 3 & 4 \\ 3 & 4 & 1 \end{smallmatrix})^{-1} \)

\( (\begin{smallmatrix} 1 & 3 & 5 \\ 3 & 1 & 4 \\ 4 & 5 & 2 \end{smallmatrix})^{-1} \cdot (\begin{smallmatrix} 4 & 3 & 5 \\ 5 & 3 & 4 \\ 3 & 4 & 1 \end{smallmatrix})^{-1} = (\begin{smallmatrix} 3 & 4 & 1 \\ 4 & 1 & 3 \\ 5 & 2 & 3 \end{smallmatrix})^{-1} = (\begin{smallmatrix} 3 & 1 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 5 \end{smallmatrix})^{-1} = (\begin{smallmatrix} 3 & 1 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 5 \end{smallmatrix}) \) [4 marks]

Question 6 (50 marks)

(a) Explain what is meant by the following terms:

(i) a graph
A graph consists of a pair of non-empty sets \( V \) and \( E \), where \( V \) is the set of vertices and \( E \) is the set of edges. Each edge in \( E \) is associated with a pair of vertices in \( V \). [2 marks]

(ii) a simple graph
A graph that has neither loops nor parallel edges is a simple graph. [2 marks]

(iii) a simple path
A simple path is a path that has no repeated edges or vertices. [2 marks]
(b) 
(i) Draw a graph that is simple and connected.

![Graph](image1.png)

This is just one of an infinite number of possible answers. [2 marks]

(ii) Draw a graph that is connected but not simple.

![Graph](image2.png)

This is just one of an infinite number of possible answers. [3 marks]

(c) Draw a graph with the specified properties or explain why no such graph exists:

(i) Graph with three vertices of degrees 1, 2, and 3 respectively.

![Graph](image3.png)

[4 marks]
(ii) Graph with four vertices of degrees 1, 2, 3, and 4 respectively.

(iii) Graph with five vertices of degrees 0, 1, 1, 1, and 5 respectively.

(d) Let $G = \{ V, E \}$ be the graph given below:

(i) Draw the subgraph $H_1$ or explain why no such subgraph exists

Subgraph $H_1 = \{ \{v_1, v_2, v_4\}, \{e_3, e_5, e_6\} \}$.

This subgraph does not exist because it includes edge $e_6$ that links vertices $v_2$ and $v_3$ but $v_3$ is not part of the subgraph $\{ \{v_1, v_2, v_4\}, \{e_3, e_5, e_6\} \}$. [3 marks]
(ii) Starting at $v_1$, write down any circuit of this graph.

A circuit is a path that contains no repeated edges or vertices and has the same starting and ending vertex. One such path starting at $v_1$ is $v_1, e_1, v_1$. Another is $v_1, e_2, v_2, e_6, v_3, e_4$. [3 marks]

(iii) Is this graph an Eulerian graph? Explain.

An Eulerian graph is a graph containing an Eulerian circuit, that is, a circuit in which every edge is included exactly once. A necessary condition for a graph to be Eulerian is that every vertex have even degree. As $v_2$ has degree 5, this graph is not an Eulerian graph. [3 marks]

(iv) Does an Eulerian path exist in $G$? Explain.

An Eulerian path, which is a path that crosses every edge without repeating an edge but starts and ends at different vertices, exists if and only if the graph has exactly two vertices of odd degree. In the graph $G$, there are exactly two such vertices viz. $v_2$ and $v_3$, so $G$ does have an Eulerian path viz. $v_2, e_6, v_3, e_4, v_1, e_1, v_1, e_1, v_1, e_2, v_2, e_3, v_2, e_5, v_4, e_7, v_3$. [3 marks]

(e) For the graph $G$ given in question (d) above, either write down a path with the specified properties, or explain why no such path exists.

(i) A simple path of length 3 from $v_1$ to $v_2$.

$v_1, e_4, v_3, e_7, v_4, e_5, v_2$ [2 marks]

(ii) A path of length 4 from $v_1$ to $v_3$.

$v_1, e_1, v_1, e_1, v_1, e_4, v_3$ [2 marks]

(f) Is the graph $G$ given in question (d) above connected? If so, what is the least number of edges that you must remove to make the graph disconnected? Give an example.

The graph $G$ is connected because it is possible to move from any vertex to any other vertex either directly or indirectly. The minimum number of edges that must be removed to make the resulting subgraph disconnected is two viz. $e_5$ and $e_7$ as removing these edges turns vertex $v_4$ into an isolated vertex. [3 marks]

(g) If the edges in graph $G$ have the following weightings, use Kruskal’s Algorithm to find a minimum spanning tree for $G$: $e_1 = 4, e_2 = 7, e_3 = 9, e_4 = 2, e_5 = 6, e_6 = 3, e_7 = 1$.

To apply Kruskal’s Algorithm, the first step is to place the edges in increasing order of weighting as follows:

$e_7 = 1; e_4 = 2; e_6 = 3; e_1 = 4; e_5 = 6; e_2 = 7; e_3 = 9$

The next step is to select edges until a connected graph is obtained that does not contain any circuits i.e. until a spanning tree is obtained. Clearly choosing the first three edges on the list creates a spanning tree, which then has a total weighting of $1 + 2 + 3 = 6$.

The spanning tree is $\{\{v_1, v_2, v_3, v_4\}, \{e_4, e_6, e_7\}\}$. [4 marks]
(h) Label the vertices and edges of the following graphs in any convenient way and show whether or not they are isomorphic.

These two graphs are isomorphic as can been seen by the labelling above or from the following mapping of vertices and edges: $v_i \rightarrow V_i$ and $e_i \rightarrow E_i$. [4 marks]

END OF EXAM