WUCT121

Discrete Mathematics

Logic

Tutorial Exercises

1. Logic
2. Predicate Logic
3. Proofs
4. Set Theory
5. Relations and Functions
Section 1: Logic

Question 1  For each of the following collections of words:

(a) Determine if it is a statement.
(b) If it is a statement, determine if it is true or false.
(c) Where possible, translate the statement into symbols, using the connectives presented in lectures.

(i)  If \( x = 3 \), then \( x < 2 \).
(ii)  If \( x = 0 \) or \( x = 1 \), then \( x^2 = x \).
(iii) There exists a natural number \( x \) for which \( x^2 = -2x \).
(iv)  If \( x \in \mathbb{N} \) and \( x > 0 \), then if \( \sqrt{x} > 1 \) then \( x > 1 \).
(v)  \( xy = 5 \) \( \Rightarrow \) \( x = 1 \) and \( y = 5 \) or \( x = 5 \) and \( y = 1 \).
(vi)  \( xy = 0 \) \( \Rightarrow \) \( x = 0 \) or \( y = 0 \).
(vii) If \( x \) and \( y \) are real numbers, \( xy = yx \).
(viii) There is a unique even prime number.

Question 2  Translate into symbols the following compound statements and give the form of the compound statement. In each case, list the statements \( p, q, r \ldots \)

(a) If \( x \) is odd and \( y \) is odd then \( x+y \) is even.
(b) It is not both raining and hot.
(c) It is neither raining nor hot.
(d) It is raining but it is hot.
(e) \( -1 \leq x \leq 2 \).

Question 3  Let \( P \) be the statement “Mathematics is easy” and \( Q \) be the statement “I do not need to study”. Write down in words the following statements, and simplify if possible:

(a) \( P \lor Q \)
(b) \( P \land Q \)
(c) \( \sim Q \)
(d) \( \sim P \)
(e) \( \sim P \land Q \)
(f) \( P \Rightarrow Q \)

Question 4  Let \( p \) and \( q \) be statements.

(a) Write down the truth tables for \( \sim (p \lor q) \land q \) and \( \sim (p \land q) \lor q \).
What do you notice about the truth tables?
Based on this result, a creative student concludes that you can always interchange \( \lor \) and \( \land \) without changing the truth table. Is the student right?

(b) Write down the truth tables for \( \sim (p \lor q) \land p \) and \( \sim (p \land q) \lor p \).
What do you think of the rule formulated by the student in 4(a)?
Question 5

(a) Construct truth tables for the compound statements \( p \lor \sim p \) and \( p \land \sim p \).
(b) What do you notice about each of the statements in part (a)?
(c) Determine the truth-value of the compound statements \((p \lor \sim p) \lor q\) and \((p \land \sim p) \land q\). What do you notice?

Question 6

(a) Construct truth tables for the compound statements \((p \lor \sim p) \land (q \lor r)\) and \(q \lor r\). What do you notice?
(b) Construct truth tables for the compound statements \((p \land \sim p) \lor (q \land r)\) and \(q \land r\). What do you notice?

Question 7

Determine which of the following statements are tautologies using the quick method where possible.
(a) \((p \Rightarrow q) \lor (p \Rightarrow \sim q)\)
(b) \((\sim (p \Rightarrow q) \lor (q \Rightarrow p)\)
(c) \((p \land q) \Rightarrow (\sim q \lor (p \Rightarrow q))\)

Question 8

Using Logical Equivalences and Substitution of Equivalence, write the following expressions using only \( \lor, \land \) and \( \sim \). Further, write the expression in the simplest form.
(a) \((p \land q) \Rightarrow r\)
(b) \(p \Rightarrow (p \lor q)\)

Question 9

Let \( p, q \) and \( r \) be statements. Using Logical Equivalences, Substitution and Substitution of Equivalence, prove the following.

(g) \((p \Rightarrow q) \equiv (p \land \sim q)\)
(h) \(((p \land \sim q) \Rightarrow r) \equiv (p \Rightarrow (q \lor r))\)

Question 10

In each case, decide whether the proposition is True or False. Give some reasons.
(a) If \( x \) is a positive integer and \( x^2 \leq 3 \) then \( x = 1 \).
(b) \((\sim (x > 1) \lor \sim (y \leq 0)) \iff (x \leq 1) \land (y > 0))\).

Question 11

Using Logical Equivalences and Substitution of Equivalence, write the following logical expressions using \( \lor \) and \( \land \) only (even without \( \sim \)).
(a) \((\sim (x > 1) \lor \sim (y \leq 0))\)
(b) \((y \leq 0) \implies (x > 1)\).

Question 12

Simplify the expression \(\sim (\sim (p \lor q) \land \sim q)\), using Logical Equivalences.
Section 2 :Predicate Logic

Question1 Write each of the following statements in words. Write down whether you think the statement is true or false.
(a) \( \forall x \in \mathbb{R}, (x \neq 0 \Rightarrow (x > 0 \lor x < 0)) \)
(b) \( \forall x \in \mathbb{N}, \sqrt{x} \in \mathbb{N} \)
(c) \( \forall \text{ students } s \text{ in WUCT121}, \exists \text{ an assigned problem } p, s \text{ can correctly solve } p. \)

Question2 Write each of the following statements using logical quantifiers and variables. Write down whether you think the statement is true or false.
(a) If the product of two numbers is 0, then both of the numbers are 0.
(b) Each real number is less than or equal to some integer.
(c) There is a student in WUCT121 who has never laughed at any lecturer’s jokes.

Question3 Translate each of the following statements into the notation of predicate logic and simplify the negation of each statement. Which statements do you think are true?
(a) \( P: \text{ Someone loves everybody.} \)
(b) \( P: \text{ Everybody loves everybody.} \)
(c) \( P: \text{ Somebody loves somebody.} \)
(d) \( P: \text{ Everybody loves somebody.} \)
(e) \( P: \text{ All rational numbers are integers.} \)
(f) \( P: \text{ Not all natural numbers are even.} \)
(g) \( P: \text{ There exists a natural number that is not prime.} \)
(h) \( P: \text{ Every triangle is a right triangle.} \)

Question4 Are the following statements true or false? Give brief reasons why.
(a) \( \forall x \in \mathbb{R}, (x > 1 \Rightarrow x > 0) \)
(b) \( \forall x \in \mathbb{R}, (x > 1 \Rightarrow x > 2) \)
(c) \( \exists x \in \mathbb{R}, (x > 1 \Rightarrow x^2 > x) \)
(d) \( \exists x \in \mathbb{R}, \left( x > 1 \Rightarrow \frac{x}{x^2 + 1} < \frac{1}{3} \right) \)
(e) \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 = 9 \)
(f) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 < y + 1 \)
(g) \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 \geq 0 \)
(h) \( \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, (x < y \Rightarrow x^2 < y^2) \)

Question5 For each of the following statements,
(a) Write down the negation of the statement,
(b) Write down whether the statement or its negation is false, and
(c) THINK about how you would disprove it.
(i) \( \forall \xi > 0, \exists x \neq 0, |x| < \xi \)

(ii) \( \exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y < x^2 \)

(iii) \( \forall y \in \mathbb{R}, \forall x \in \mathbb{R}, x < y \Rightarrow x < \frac{x+y}{2} < y \)

**Question 6** Write down the negations of the following statements.

(a) \( P : \exists x \in \mathbb{Q}, x^2 = 2 \)

(b) \( Q : \forall x \in \mathbb{R}, x^2 + 1 \geq 2x \)

**Question 7** Write the following statements using quantifiers. Find their negations and determine in each case whether the statement or its negation is true, giving a brief reason.

(a) \( P : \) For each real number, there is a smaller real number

(b) \( Q : \) Every real number is either positive or negative

**Question 8** Write down the negations of the following statements. In each case decide whether the statement or its negation is true.

(a) \( \forall x \in \mathbb{R}, x \geq 0 \)

(b) \( \exists z \in \mathbb{Z}, (z \text{ is odd}) \lor (z \text{ is even}) \)

(c) \( \exists n \in \mathbb{N}, (n \text{ is even} \land \sqrt{n} \text{ is prime}) \)

(d) \( \forall y \in \mathbb{R}, \left( y \neq 0 \Rightarrow \frac{y+1}{y} < 1 \right) \)

(e) \( \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy = 1 \)

(f) \( \forall n \in \mathbb{N}, \exists p \in \mathbb{N}, n = 2p \)

(g) \( \forall \epsilon \in \mathbb{R}, \forall x \in \mathbb{Z}, \exists y \in \mathbb{Q}, (\epsilon > 0 \Rightarrow |x - y| < \epsilon) \)

**Question 9** Write down the negations of the following statements. In each case decide whether the statement or its negation is false, giving a brief reason where possible.

(a) \( \forall y \in \mathbb{R}, (y > -1 \Rightarrow y^2 > 1) \)

(b) \( \exists x \in \mathbb{R}, x^2 + 1 = 0 \)

(c) \( \forall x, y, z \in \mathbb{R}, (y - z) \neq (x - y) - z \)

(d) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 0 \)

**Question 10** Write the following statements using quantifiers. Find their negations and determine in each case whether the statement or its negation is false, giving brief reason where possible.

(a) \( P : \) For each natural number there is a smaller natural number.

(b) \( P : \) The square of any real number is non-negative.

(c) \( P : \) Some dogs are vegetarians.

(d) \( P : \) There is a real number that is rational.

(e) \( P : \) Every student likes at least one Mathematics subject.
Section 3: Proofs

Question 1 Each of the following demonstrates the Rule of Modus Ponens, Modus Tollens or the Law of Syllogism. In each case, answer the question or complete the sentence and indicate which of the logical rules is being demonstrated.

(a) If Peter is unsure of an address, then he will phone. Peter is unsure of John’s address. What does Peter do?
(b) If $x^2 - 3x + 2 = 0$, then $(x-2)(x-1) = 0$. If $(x-2)(x-1) = 0$, then $x-2 = 0$ or $x-1 = 0$. If $x-2 = 0$ or $x-1 = 0$, then $x = 2$ or $x = 1$. Therefore, if $x^2 - 3x + 2 = 0$, then ...
(c) We know that if $x$ is a real number, then its square is positive or zero. If $y^2 = -1$, what do we know about $y$?

Question 2 Prove or disprove the following statements

(a) For all $n \in \mathbb{N}$, the expression $n^2 + n + 29$ is prime.
(b) $\exists x \in \mathbb{Q}, \forall y \in \mathbb{Q}, xy \neq 1$
(c) $\forall a, b \in \mathbb{R}, (a+b)^2 = a^2 + b^2$
(d) The average of any two odd integers is odd

Question 3 Find the mistakes in the following “proofs”.

(a) Result: $\forall k \in \mathbb{Z}, k > 0 \Rightarrow k^2 + 2k + 1$ is not prime.
   Proof: For $k = 2$, $k^2 + 2k + 1 = 9$, which is not prime. Therefore the result is true.
(b) Result: The difference between any odd integer and any even integer is odd.
   Proof: Let $n$ be any odd integer and $m$ be any even integer. By definition of odd $n = 2k + 1$, $k \in \mathbb{Z}$, and by definition of even $m = 2k$, $k \in \mathbb{Z}$. Then $n - m = (2k + 1) - 2k = 1$. But 1 is odd. Therefore the result holds.

Question 4 Prove each of the following results using a direct proof:

(a) For $x \in \mathbb{R}, x^2 + 1 \geq 2x$
(b) For $n \in \mathbb{N}$, if $n$ is odd, $n^2$ is odd.
(c) The sum of any two odd integers is even.
(d) If the sum of two angles of a triangle is equal to the third angle, then the triangle is a right angled triangle

Question 5 Prove that if $x$ is a negative real number, then $(x - 2)^2 > 4$.

Question 6 Prove that there is an integer $n > 5$, such that $2^n - 1$ is prime.

Question 7 Prove that for each integer $n$ such that $1 \leq n \leq 10$, $n^2 - n + 41$ is a prime number.
**Question 8**  Prove that if $n$ is an odd integer, then $(-1)^n = -1$.

**Question 9**  Prove, by contraposition, that if $n^2$ is even, then $n$ is even.

**Question 10**  Prove by cases, if $m$ is an integer, then $m^2 + m + 1$ is always odd.

**Question 11**  Disprove the statement: $\forall a, b \in \mathbb{Z}, a \neq 0, b \neq 0, \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$. Are there any values for $a, b$ that make the statement true? Explain.

**Question 12**  Prove or disprove this statement: For all integers, $a, b$ if $a < b$, then $a^2 < b^2$.

**Question 13**  Prove if $n^2$ is odd, then $n$ is odd.

**Question 14**  Prove there is no smallest positive real number.

**Question 15**  Prove each of the following using proof by cases
(a) If $x = 4, 5, or 6$, then $x^2 - 3x + 21 \neq x$.
(b) $\forall x \in \mathbb{Z}, x \neq 0 \Rightarrow 2^x + 3 \neq 4$

**Question 16**  Prove there is a perfect square that can be written as the sum of two other perfect squares. (Note an integer $n$ is a perfect square if and only if $\exists k \in \mathbb{Z}, n = k^2$)

**Question 17**  Prove that the product of two odd integers is also an odd integer

**Question 18**  Prove or disprove the following statements:
(a) The difference between any two odd integers is also an odd integer
(b) For any integer $n$, $3 \mid n(6n + 3)$.
(c) The cube of any odd integer is an odd integer.
(d) For any integers $a, b, c$, if $a \mid c$, then $ab \mid c$.
(e) There is no largest even integer.
(f) For all integers $a, b, c$, if $a \mid bc$, then $a \mid b$.
(g) For all integers $n$, $4(n^2 + n + 1) - 3n^2$ is a perfect square.
(h) For any integers $a, b$, if $a \mid b$ then $a^2 \mid b^2$.
(i) For all integers $n$, $n^2 - n + 41$ is prime.
(j) For all integers $n$ and $m$, if $n-m$ is even, then $n^3 - m^3$ is even.

**Question 19**  Prove that the product of any four consecutive numbers, increased by one, is a perfect square?
Section 4: Set Theory

Question 1 Let \( U = \mathbb{R} \).
Let \( A = \{1\} \), \( B = (0,1) = \{ x \in \mathbb{R} : 0 < x < 1 \} \) and \( C = [0,1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \).
Write down the following sets:

(a) \( A \cup B \)
(b) \( A \cap B \)
(c) \( B \cap C \)
(d) \( A \cup C \)
(e) \( A \cap C \)
(f) \( \overline{A} \)
(g) \( \overline{C} \)
(h) \( C - A \)
(i) \( C - B \)
(j) \( A - C \)

Are any of the pairs of sets \( A, B \) and \( C \) disjoint?

Question 2 Let \( U = \mathbb{N} \).
Let \( A = \{ x \in \mathbb{N} : x \text{ is odd} \} \), \( B = \{ x \in \mathbb{N} : x \text{ is even} \} \) and \( P = \{ x \in \mathbb{N} : x \text{ is prime} \} \).
Write down the following sets:

(a) \( A \cup B \)
(b) \( A \cap B \)
(c) \( B \cap P \)
(d) \( A \cup P \)
(e) \( A \cap P \)
(f) \( \overline{A} \)
(g) \( \overline{P} \)
(h) \( P - A \)
(i) \( B - P \)
(j) \( A - B \)

Are \( A \) and \( B \) disjoint? Is \( P \subseteq A \)?

Question 3 Let \( X = \{1,2,3,4,5\} \).
(a) Write down the set \( \mathcal{P}(X) \) by listing its elements.
(b) How many elements in \( \mathcal{P}(X) \)?
(c) Is \( \emptyset \in \mathcal{P}(X) \)?
(d) Is \( \{\emptyset\} \subseteq \mathcal{P}(X) \)?

Question 4 Write down the set \( \mathcal{P}(\emptyset) \) by listing its elements.
How many elements in \( \mathcal{P}(\emptyset) \)?

Question 5 Let \( X = \{1,2,3,\ldots,n\} \), that is, \( X \) is a finite set with \( n \) elements.
How many elements does \( \mathcal{P}(X) \) have?

Question 6 Let \( X = \{1,2,3\} \). For each of the following statements, write down whether it is True or False. Give reasons.

(a) \( \forall B \in \mathcal{P}(X), \forall C \in \mathcal{P}(X), (B \subseteq C \lor C \subseteq B) \)
(b) \( \exists B \in \mathcal{P}(X), \forall C \in \mathcal{P}(X), B \subseteq C \).
(c) \( \exists B \in \mathcal{P}(X), \forall C \in \mathcal{P}(X), C \subseteq B \)
(d) The number of proper subsets of \( X \) is \( 2^3 - 1 \).


**Question 7** Let \( X = \{1, 2\} \). Write down the set \( \mathcal{P}(\mathcal{P}(X)) \) by listing its elements. If \( Y = \{1, 2, 3\} \), how many elements would be in the set \( \mathcal{P}(\mathcal{P}(Y)) \)? Write down two elements of \( \mathcal{P}(\mathcal{P}(Y)) \).

**Question 8** Can you write down two elements of \( \mathcal{P}(\mathbb{R}) \)? Can you list the elements of \( \mathcal{P}(\mathbb{R}) \)? What can you say about the set \( \mathcal{P}(\mathbb{R}) \)?

Is \( [-1, 1] = \{x \mid x \in \mathbb{R} \land -1 \leq x \leq 1\} \in \mathcal{P}(\mathbb{R}) \)?

**Question 9** Let \( X = \{1, 2, 3\} \). Draw the Hasse Diagram for \( \mathcal{P}(X) \).

**Question 10** Let \( X = \{1, 2, 3, 4\} \). Try to draw the Hasse Diagram for \( \mathcal{P}(X) \).

**Question 11** Using the Principle of Mathematical Induction, prove that if \( X = \{1, 2, 3, \ldots, n\} \), that is, \( X \) is a finite set with \( n \) elements, then the number of elements in \( \mathcal{P}(X) \) is \( 2^n \). (A procedure for determining the number of sets is sufficient for the inductive step.)

**Question 12** Give examples to demonstrate the following results.

(a) \( \mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y) \) but \( \mathcal{P}(X) \cup \mathcal{P}(Y) \neq \mathcal{P}(X \cup Y) \)

(b) \( \mathcal{P}(X) \cap \mathcal{P}(Y) = \mathcal{P}(X \cap Y) \)

**Question 13** Let \( U \) be the universal set and let \( A, B \) and \( C \) be subsets of \( U \).

Use a typical element argument to prove the following set theoretic results.

(a) \( A \subseteq B \Rightarrow A \cup C \subseteq B \cup C \)

(b) \( (A \cup B) \cap B = B \)

**Question 14** Let \( U \) be the universal set and let \( A, B \) and \( C \) be subsets of \( U \).

Using properties of union, intersection and complement and known set laws, simplify the following:

(a) \( (A \cap \overline{B}) \cap A \)

(b) \( (C \cup B) \cup \overline{C} \)

(c) \( (A \cap \varnothing) \cap U \)

(d) \( (A \cup U) \cup \overline{A} \)

**Question 15** Prove or disprove:

\[ \{0, 1\} = \left\{ n \in \mathbb{Z} : \exists k \in \mathbb{Z}, \left( n = \frac{1 + (-1)^k}{2} \right) \right\} \]

**Question 16** Let \( U = \mathbb{Z} \). Let \( A = \{n \in \mathbb{Z} : \exists k \in \mathbb{Z}, n = 2k - 1\} \) and let \( B = \{m \in \mathbb{Z} : \exists p \in \mathbb{Z}, m = 3p + 2\} \). Write down 4 elements from each set.

Show that \( t \in A \cap B \iff \exists w \in \mathbb{Z}, (w \ is \ odd \land t = 3w + 2) \).

**Question 17** Let \( U \) be the universal set and let \( A, B \) and \( C \) be subsets of \( U \).

Use a typical element argument to prove the following set theoretic results.

(a) \( \overline{A \cup B} = \overline{A} \cap \overline{B} \)

(b) \( A \cap (B - C) = (A \cap B) - C \)
**Question 18** Let $U$ be the universal set and let $A$, $B$ and $C$ be subsets of $U$.
Using properties of union, intersection and complement and known set laws, simplify the following:
(a) $(C \cup U) \cup \overline{C}$  
(b) $(A \cap U) \cup \overline{A}$  
(c) $(C \cup \overline{C}) \cup C$  
(d) $(A \cap B) \cap \overline{A}$

**Question 19** Let $O = \{n \in \mathbb{Z} : n \text{ is odd}\}$ and let $T = \{n \in \mathbb{Z} : n^2 \text{ is odd}\}$.
Prove or disprove:
(a) $T \subseteq O$  
(b) $O \subseteq T$  
(c) $T = O$

**Question 20** Let $T = \{n \in \mathbb{Z} : \exists x, y \in \mathbb{Z}, z = x^2 + y^2\}$ and let $E = \{z \in \mathbb{Z} : z \text{ is even}\}$.
Prove or disprove $T \subseteq E$.

**Question 21** Let $U$ be the universal set and let $A$, $B$ and $C$ be subsets of $U$.
Prove or disprove the following:
(a) $(A \cap B) = A \Rightarrow A \subseteq B$
(b) $(A \cap B) = (A \cap C) \Rightarrow B = C$

**Question 22** Determine if the following statements are true or false:
(a) $A \cap B = \emptyset \Rightarrow A \subseteq \overline{B}$
(b) $(A \subseteq \overline{B} \wedge \overline{A} \subseteq \overline{B}) \Rightarrow B = \emptyset$
(c) $A$ and $B - A$ are disjoint.
Section 5: Relations and Functions

**Question 1** Let \( A = \{1, 2\} \), \( B = \{0, 2, 3\} \) and \( C = \{a, b\} \).

(a) List the elements and sketch the graphs in \( \mathbb{R}^2 \) of:

(i) \( A \times B \)

(ii) \( A \times A \)

(iii) \( B \times A \)

(b) Is \( A \times B \subseteq B \times A \)?

(c) List the elements of \( (A \times B) \times C \) and \( (A \times C) \cup (B \times C) \). What do you notice?

(d) List the elements of \( (A \times B) \times C \) and \( C \times (A \times A) \).

**Question 2** Let \( A = \{1, 2\} \) and \( B = \{a, b\} \). Write down \( D \) \( = \{a, b\} \) \( \times \) \( \). Will this be the same as \( D \times A \)?

**Question 3** Let \( A = \{x \in \mathbb{R} : 0 < x < 1\} \), \( B = \{x \in \mathbb{R} : -1 < x < 3\} \) and \( C = \{x \in \mathbb{R} : 0 \leq x \leq 1\} \).

(a) Sketch the graph of \( A \times B \) in \( \mathbb{R}^2 \).

(b) Sketch the graph of \( C \times C \) in \( \mathbb{R}^2 \). Note: \( C \times C \) is called the until square in \( \mathbb{R}^2 \).

(c) Sketch the graph of \( \mathbb{R} \times C \) in \( \mathbb{R}^2 \).

**Question 4** Let \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_m\} \).

(a) How many elements in \( A \times B \)?

(b) Write out the elements of \( A \times B \).

**Question 5** Let \( A \), \( B \) and \( C \) be elements of \( \mathcal{P}(U) \). Prove that \( (A \cup B) \times C = (A \times C) \cup (B \times C) \).

**Question 6** Sketch the graphs of the following relations in \( \mathbb{R}^2 \).

(a) \( R_1 = \{(x, y) : x = y \wedge x = -y\} \).

(b) \( R_2 = \{(x, y) : x^2 - y = 0\} \).

(c) \( R_3 = \{(x, y) : y^2 = 2 + x\} \).

(d) \( R_4 = \{(x, y) : (x^2 - y)(x - y) = 0\} \).

**Question 7** Sketch the graphs of the following relations in \( \mathbb{R}^2 \).

(a) \( R_1 = \{(x, y) : |x| = |y|\} \).

(b) \( R_2 = \{(x, y) : (x^2 - y)(4x^2 + 9y^2 - 36) = 0\} \).

**Question 8** Let \( A = \{2, 3, 4\} \) and \( B = \{6, 8, 10\} \) and define the relation \( R \) from \( A \) to \( B \) as follows: \( R = \{(x, y) : x \text{ is a factor of } y\} \).

(a) List the elements of \( R \).

(b) Graph \( A \times B \) and circle the elements of \( R \).

(c) True or false?

(i) \( 4R6 \)

(ii) \( 4R8 \)

(iii) \( (3, 8) \in R \)

(iv) \( (2, 10) \in R \)

(v) \( (4, 12) \in R \)

**Question 9** Define the relations \( R \) and \( S \) on \( \mathbb{R} \) as follows: \( R = \{(x, y) : y = |x|\} \)

\( S = \{(x, x) : x = 0\} \). Find simple expressions for the relations:

(a) \( R \cup S \) on \( \mathbb{R} \).

(b) \( R \cap S \) on \( \mathbb{R} \).
**Question 10** Write down the domain and range of the relation \( R \) on the given set \( A \).
\[ A = \{ h : h \text{ is a human being} \} , \quad R = \{ (h_1, h_2) : h_1 \text{ is the sister of } h_2 \} \]

**Question 11** Let \( A = \{3, 4, 5\} \) and \( B = \{4, 5, 6\} \) and define the relation \( R \) from \( A \) to \( B \) as follows: \( R = \{ (x, y) : x < y \} \). Write down \( R \) and \( R^{-1} \) by listing their elements.

**Question 12** For the relation \( T \) on \( \mathbb{R} \) given by \( T = \{ (x, y) : \frac{x^2}{4} + \frac{y^2}{9} = 1 \} \), find and expression for the inverse relation \( T^{-1} \). Sketch both \( T \) and \( T^{-1} \).

**Question 13** Determine whether or not the given relation is reflexive, symmetric or transitive. Give a counterexample in each case in which the relation does not satisfy the property.

(a) \( R_1 \) on the set \( A = \{ h : h \text{ is a human being} \} \) given by
\[ R_1 = \{ (h_1, h_2) : h_1 \text{ is the sister of } h_2 \} \]

(b) \( R_2 \) on the set \( A = \{a, b, c, d\} \) given by
\[ R_2 = \{ (a, a), (a, b), (b, a), (b, b), (c, b), (c, c), (d, d) \} \]

**Question 14** Determine whether or not the following relation is an equivalence relation. \( R \) on \( A = \{0, 1, 2, 3\} \) given by \( R = A \times A \).

**Question 15** Show that the relation \( R \) on the set \( A = \{0, 1, 2, 3, 4\} \) given by
\[ R = \{ (0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4) \} \]
is an equivalence relation. Find all the classes of \( R \).

**Question 16** Is the following relation a function? Give brief reason.
\( R \) on \( [-2, 2] = \{ x \in \mathbb{R} : -2 \leq x \leq 2 \} \), where
\[ R = \{ (x, y) : y = \sqrt{1 - (x - 1)^2} \lor y = 1 - \sqrt{1 - (x + 1)^2} \} . \]

**Question 17** Determine whether or not the following functions are:

(a) one-to-one , give brief reasons
(b) onto. Give brief reasons.

(i) Let \( A = \{1, 5, 9\} \) and \( B = \{3, 4, 7\} \). \( F_1 \subseteq A \times B \) and \( F_1 = \{ (1, 7), (5, 3), (9, 4) \} \)

(ii) \( F_2 \) on \( \mathbb{Z} \) and \( F_2 = \{ (x, y) : y = 2x \} \)

**Question 18** Let \( A = \{4, 5, 6\} \) and \( B = \{5, 6, 7\} \) and define the relations \( S \) and \( T \) from \( A \) to \( B \) as follows: \( S = \{ (x, y) : x - y \text{ is even} \} \) and \( T = \{ (4, 6), (6, 5), (6, 7) \} \).

(a) Find expressions for \( S^{-1} \) and \( T^{-1} \)
(b) Which of \( S, T, S^{-1} \) and \( T^{-1} \) are functions?

**Question 19** Simplify the following:

(a) \( (1 \ 3 \ 4)(3 \ 2 \ 4) \)
(b) \( (1 \ 3 \ 4)^{-1} \)
(c) \( (2 \ 5 \ 4)^{-1} \)
(d) \( (3 \ 2)(3 \ 2 \ 4)(3 \ 1)(4 \ 2) \)