

High-Order Interaction of Solitary Waves on Shallow Water

By T. R. Marchant

The interaction of solitary waves on shallow water is examined to fourth order. At first order the interaction is governed by the Korteweg–de Vries (KdV) equation, and it is shown that the unidirectional assumption, of right-moving waves only, is incompatible with mass conservation at third order. To resolve this, a mass conserving system of KdV equations, involving both right- and left-moving waves, is derived to third order. A fourth-order interaction term, in which the right- and left-moving waves are coupled, is also derived as this term is crucial in determining the fourth-order change in solitary wave amplitude. The form of the unidirectional KdV equation is also discussed with nonlocal terms derived at fourth order. The solitary wave interaction is examined using the inverse scattering method for perturbed KdV equations. Central to the analysis at fourth order is the left-moving wave, for which the solution, in integral form, is derived. A symmetry property for the left-moving wave is found, which is used to show that no change in solitary wave amplitude occurs to fourth order. Hence it is concluded that, for surface waves on shallow water, the change in solitary wave amplitude is of fifth order.

Address for correspondence: T. R. Marchant, School of Mathematics and Applied Statistics, The University of Wollongong, Wollongong, NSW 2522, Australia.

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1. Introduction

The Korteweg–de Vries (KdV) equation is the generic model for the study of weakly nonlinear long waves, incorporating leading-order nonlinearity and dispersion. For example, it describes surface waves of long wavelength and small amplitude on shallow water (Whitham [1]) and internal waves in a shallow density-stratified fluid (Benney [2]). The solitary wave solution of the KdV equation, thus named because it consists of a single humped wave, has a number of special properties. Zabusky and Kruskal [3] numerically examined the nonlinear interaction of a large solitary wave overtaking a smaller one. It was found that, after interaction, the solitary waves retained their original shapes, the only memory of the collision being a phase shift. Due to this special property, amongst others, the solitary wave solution of the KdV equation is termed a soliton. Gardner et al. [4] developed the so-called inverse scattering transform for solving the KdV equation and showed that the KdV solitary wave collision is elastic. The explicit solution for interacting KdV solitons was developed using inverse scattering by Hirota [5].

If higher order nonlinear and dispersive effects are of interest, then the third-order KdV equation

$$\begin{aligned} \eta_t + 6\eta\eta_x + \eta_{3x} + \alpha(c_1\eta^2\eta_x + c_2\eta_x\eta_{xx} + c_3\eta\eta_{3x} + c_4\eta_{5x}) \\ + \alpha^2(d_1\eta^3\eta_x + d_2\eta^2\eta_{3x} + d_3\eta_x^3 + d_4\eta_{xx}\eta_{3x} + d_5\eta_x\eta_{4x} \\ + d_6\eta\eta_{5x} + d_7\eta\eta_x\eta_{xx} + d_8\eta_{7x}) = 0, \quad \alpha \ll 1, \end{aligned} \quad (1)$$

is obtained, where α is a nondimensional measure of the (small) wave amplitude. Note that (1) is scaled with respect to space and time, hence third-order terms appear in (1) at $O(\alpha^2)$. This equation describes the evolution of steeper waves of shorter wavelength than does the KdV equation. The coefficients c_i and d_i depend on the physical context. In the special case of surface water waves the coefficients at second order are

$$c_1 = -\frac{3}{2}, \quad c_2 = \frac{23}{4}, \quad c_3 = \frac{5}{2}, \quad c_4 = \frac{19}{40}. \quad (2)$$

The second-order KdV equation ((1) to $O(\alpha)$) has been derived with the coefficients (2) by Marchant and Smyth [6] to enable resonant flow over topography to be modeled more accurately and by Byatt-Smith [7] to examine second-order solitary wave interactions.

The nature of the solitary wave interaction for higher order KdV models has been considered by a number of authors, both theoretically and numerically. Kodama [8] described a method for the asymptotic transformation of the second-order KdV equation to the second-order member of the KdV integrable hierarchy. Hence the second-order KdV equation is nearly integrable and the solitary waves are asymptotic solitons. Marchant and Smyth [9] used a

local variant of the transformation of Kodama [8] to construct the asymptotic two-soliton solution of the second-order KdV equation. The $O(\alpha)$ corrections to the phase shifts of the solitary waves after collision were also found. Numerical solutions showed that a dispersive wavetrain of very small amplitude was generated by the collision. The inelastic effect was shown to be beyond the second-order asymptotic validity of the model equation; hence the existence of asymptotic second-order solitons was confirmed.

Zou and Su [10] considered higher order interactions of solitary waves on shallow water by using a perturbation expansion of the Euler water wave equations. Using the KdV two-soliton solution as the first-order solution gives partial differential equations describing the solitary wave collision at second and third order. At second order the solitary wave collision was found to be elastic. At third order numerical solutions showed no change in solitary wave amplitude due to the collision. However, a dispersive wavetrain is generated after collision, indicating that the third-order terms cause an inelastic effect beyond third order.

Kodama [11] considered solitary wave interaction for the BBM equation. This was transformed to a member of the KdV integrable hierarchy plus inelastic terms at third order. The inverse scattering perturbation method was then used to show that the inelastic terms at third order generate a fifth-order change in solitary wave amplitude. Marchant [12] considered solitary wave interaction for an extended BBM (eBBM) equation numerically. The eBBM equation was preferred to the asymptotically equivalent second-order KdV equation, due to the superior stability properties of its numerical scheme. It was numerically verified that the inelastic effect of the collision was indeed of fifth order, as predicted theoretically by Kodama [11] for the BBM and eBBM equations.

Kodama [13] considered solitary wave interaction for a fourth-order unidirectional KdV equation with arbitrary higher order coefficients. The higher order terms in the equation are found via a recursive formula and include a nonlocal term at fourth order. It was shown, using the method of Kodama [11], that the change in amplitude is again of fifth order, due to the dispersive radiation generated by the inelastic third-order terms. No leading-order change in amplitude resulted from the fourth-order terms.

Prasad and Akylas [14] considered the generation of shelves by long nonlinear internal waves in a stratified fluid. The main wave disturbance is described by an integro-differential equation, and is typically generated by resonant flow over topography. However, the solution is nonuniform far downstream and mass is not conserved. A solution for the downstream flow was derived by introducing a long length-scale, which was found to resolve the mass conservation inconsistencies. For each longwave mode the downstream solution consists of left- and right-moving shelves, relative to the background flow. Also, transient effects in the main wave disturbance were shown to be

responsible for generating the shelves. In the weakly nonlinear formulation the evolution equation was shown to be a higher order KdV equation; left- and right-moving shelves were predicted at fourth order in the wave amplitude.

Equation (1) does not, in general, conserve mass at third order. This is due to the fact that the unidirectional assumption, which allows for right-moving waves only, breaks down. Hence the derivation of a correct higher order KdV model requires the introduction of a left-moving wave at third order. This is analogous to Prasad and Akylas [14], who showed that mass conservation implies the existence of left-moving internal waves.

The aim of this article is to derive the appropriate system of KdV equations, relevant for surface water waves, to third order (a crucial fourth-order interaction term is also calculated) and then examine the nature of the solitary wave interaction to fourth order. Central to the analysis at fourth order is the left-moving wave, associated with the solitary wave interaction, which is found as an integral.

The fourth-order interaction term, in which the right- and left-moving waves are coupled, is different from the fourth-order nonlocal terms (B.1), obtained from a formal derivation using the unidirectional assumption. Note also that Kodama's fourth-order unidirectional equation contains only the first of the nonlocal terms (B.1). These three model equations are different at fourth order, which potentially, could lead to different predictions for the change of solitary wave amplitude at this order.

In Section 2 the KdV system of equations, appropriate for surface water waves, is derived to third order. This includes left-moving waves, which resolve mass conservation inconsistencies in the unidirectional model. A fourth-order term associated with the interaction of the right- and left-moving waves is also found. In Section 3 the inverse scattering perturbation method is used to show theoretically that no change in solitary wave amplitude occurs at third order and that the change in amplitude, due to third-order terms, is of fifth order. In Section 4 the left-moving wave is described and a relevant symmetry property is derived. It is then shown theoretically, for the KdV system of equations, that the fourth-order change in solitary wave amplitude is zero. Appendix A contains the third-order Boussinesq equations, which are used to derive the third-order KdV model. Appendix B presents the fourth-order nonlocal terms which occur using the unidirectional assumption. It is also shown that no change in solitary wave amplitude occurs as a result of these nonlocal terms.

2. The KdV system of equations

The Euler water wave equations describe waves propagating on the surface of an incompressible, inviscid, irrotational fluid of undisturbed constant depth h . The perturbation expansion assumes that the wave amplitude is small and the

wavelength is large, therefore the two small parameters are

$$\alpha = \frac{a}{h}, \quad \beta = \left(\frac{h}{\ell}\right)^2, \quad (3)$$

where a is a typical wave amplitude and ℓ is a typical wavelength. Expanding the Euler water wave equations and retaining terms up to third order gives the Boussinesq equations (A.1) and (A.2), which describe both left- and right-moving waves. The KdV equation is obtained from the Boussinesq equations by using the unidirectional assumption, and only describes right-moving waves. As will be seen later, the unidirectional assumption causes mass conservation to break down at third order. To overcome this difficulty a left-moving wave is introduced at third order and a system of KdV equations is derived. In a similar manner to Mattioli [15] the horizontal velocity and free surface elevation are written as

$$\omega = \omega^+ + \alpha^3 \omega^-, \quad \eta = \eta^+ + \alpha^3 \eta^-, \quad (4)$$

where both quantities are now split into two components. The superscript $+$ refers to right-moving waves while the superscript $-$ refers to left-moving waves. The right- and left-moving horizontal velocities are chosen to be

$$\begin{aligned} \omega^+ = & \eta^+ - \frac{1}{4}\alpha\eta^{+2} + \frac{1}{3}\alpha\eta_{xx}^+ + \frac{1}{8}\alpha^2\eta^{+3} + \frac{3}{16}\alpha^2\eta_x^{+2} \\ & + \frac{1}{2}\alpha^2\eta^+\eta_{xx}^+ + \frac{1}{10}\alpha^2\eta_{4x}^+ - \frac{5}{64}\alpha^3\eta^{+4} + \frac{61}{1890}\alpha^3\eta_{6x}^+ \\ & + \frac{163}{360}\alpha^3\eta_{xx}^{+2} + \frac{319}{144}\alpha^3\eta_x^+\eta_{3x}^+ + \frac{1}{8}\alpha^3\eta^{+2}\eta_{xx}^+ \\ & + \frac{9}{32}\alpha^3\eta^+\eta_x^{+2} + \frac{7}{20}\alpha^3\eta^+\eta_{4x}^+ + a\alpha^3 \int_{-\infty}^x \eta^+\eta_u^+\eta_{uu}^+ du, \\ \omega^- = & -\eta^- + O(\alpha). \end{aligned} \quad (5)$$

Note that the horizontal velocity for the right-moving wave contains a nonlocal term at third order with an undetermined coefficient a . Substituting (4) and (5) into the third-order Boussinesq equations (A.1) and (A.2) and rearranging gives the equations for the right- and left-moving waves as

$$\begin{aligned} \eta_t^+ + \eta_x^+ + \alpha \left(\frac{3}{2}\eta^+\eta_x^+ + \frac{1}{6}\eta_{3x}^+ \right) + \alpha^2 \left(\frac{23}{24}\eta_x^+\eta_{xx}^+ + \frac{5}{12}\eta^+\eta_{3x}^+ \right. \\ + \frac{19}{360}\eta_{5x}^+ - \frac{3}{8}\eta^{+2}\eta_x^+ \left. \right) + \alpha^3 \left(\frac{3}{16}\eta^{+3}\eta_x^+ + \frac{5}{16}\eta^{+2}\eta_{3x}^+ + \frac{19}{32}\eta_x^{+3} \right. \\ + \frac{317}{288}\eta_{xx}^+\eta_{3x}^+ + \frac{1079}{1440}\eta_x^+\eta_{4x}^+ + \frac{19}{80}\eta^+\eta_{5x}^+ \\ \left. + \frac{23}{16}\eta^+\eta_x^+\eta_{xx}^+ + \frac{55}{3024}\eta_{7x}^+ \right) \end{aligned} \quad (6)$$

and

$$\eta_t^- - \eta_x^- + \left(\frac{3}{8} + a\right)\eta^+\eta_x^+\eta_{xx}^+ = 0. \quad (7)$$

Mass conservation is now used to determine the coefficient, a , of the nonlocal term. Putting (6) and (7) in mass conservation form gives

$$\frac{d}{dt} \int_{-\infty}^{\infty} \eta^+ dx = \frac{3}{8}\alpha^3 I, \quad \frac{d}{dt} \int_{-\infty}^{\infty} \eta^- dx = -\left(\frac{3}{8} + a\right)I, \quad (8)$$

where the mass-flux integral is

$$I = \int_{-\infty}^{\infty} \eta^+\eta_x^+\eta_{xx}^+ dx = -\frac{1}{2} \int_{-\infty}^{\infty} \eta_x^{+3} dx. \quad (9)$$

The first of (8) illustrates that mass is not conserved by the right-moving wave to third order. As the total mass $\eta = \eta^+ + \alpha^3\eta^-$ must be conserved (8) implies that the coefficient $a = 0$. Hence when $a = 0$ the mass-loss from the right-moving waves is balanced by a corresponding mass-gain by the left-moving waves. The integral I is proportional to the mass-flux between the right- and left-moving waves. If the unidirectional assumption is used (where $u^- = 0$) then the coefficient $a = -\frac{3}{8}$ and mass conservation breaks down. Hence the mass conservation at third order implies the existence of a left-moving wave, η^- . The mass-flux integral (9) has the same form as integral (7.10) in Prasad and Akylas[14], indicating that a similar connection occurs between mass conservation and left-moving waves for the case of internal waves in a stratified fluid.

Retaining appropriate terms in the Boussinesq equations and from the substitution of (4) gives the interaction term

$$-\frac{1}{2} \frac{\partial(\eta^+\eta^-)}{x}, \quad (10)$$

in the equation for the right-moving wave (6), at fourth order. The interaction term (10) is different in form from the nonlocal fourth-order terms (B.1) which occur in the unidirectional equation.

The system of KdV equations can be uncoupled as the equation for the right-moving wave (6) only involves η^- at fourth order. The left-moving wave η^- can be found to leading order by solving (7) with η^+ being the solution of the KdV equation. This solution for η^- can then be used in (6), uncoupling the system of equations.

3. The solitary wave interaction at third order

In this section the interaction between two solitary waves governed by the third-order KdV equation is examined by using the inverse scattering

perturbation method. Firstly, (6) is rescaled, which gives the third-order KdV equation, in its more usual form, as

$$\begin{aligned} \eta_t + 6\eta\eta_x + \eta_{3x} + \alpha \left(\frac{23}{4}\eta_x\eta_{xx} + \frac{5}{2}\eta\eta_{3x} + \frac{19}{40}\eta_{5x} - \frac{3}{2}\eta^2\eta_x \right) \\ + \alpha^2 \left(\frac{3}{4}\eta^3\eta_x + \frac{15}{8}\eta^2\eta_{3x} + \frac{57}{16}\eta_x^3 + \frac{317}{32}\eta_{xx}\eta_{3x} \right. \\ \left. + \frac{1079}{160}\eta_x\eta_{4x} + \frac{171}{80}\eta\eta_{5x} + \frac{69}{8}\eta\eta_x\eta_{xx} + \frac{55}{224}\eta_{7x} \right) = 0. \end{aligned} \quad (11)$$

Note that the third-order terms are now rescaled to $O(\alpha^2)$ in (11) and the superscripts on the right-moving wave have been dropped for convenience. Equation (11) was checked by comparing its solitary wave solution with that found directly from the Euler water wave equations (see Fenton [16]).

Before applying the inverse scattering perturbation method the method of Kodama [11] is used to put the third-order KdV equation in normal form. After application of an asymptotic transformation, the third-order KdV equation becomes

$$\eta_t + 6\eta\eta_x + \eta_{3x} + \alpha\mu_1 X_1(\eta) + \alpha^2\mu_2 X_2(\eta) + \alpha^2 R_2(\eta) = 0, \quad (12)$$

where $X_1(\eta)$ and $X_2(\eta)$ are members of the KdV hierarchy of integrable equations and terms of $O(\alpha^3)$ are neglected. Equation (12) is a member of the family of integrable KdV equations, plus inelastic terms, given by $R_2(\eta)$, at $O(\alpha^2)$. The terms in $R_2(\eta)$ are just those at $O(\alpha^2)$ in (11), but with different coefficients.

The normal form implies that (12) has the exact sech^2 solitary wave solution of the higher order integrable equation (when $R_2(\eta) = 0$). This means that when the inverse scattering perturbation method is applied to the transformed equation (12) no spurious radiation, due to the propagation of a single solitary wave, is generated (see Kodama [11]). The coefficients of the terms in $R_2(\eta)$ are not needed for this work however; they are only important if the magnitude of the radiation, which is of $O(\alpha^5)$, needs to be found.

The inverse scattering perturbation method of Kaup and Newell [17] is now used to find the effect of collision on the solitary wave amplitudes. In the method, ordinary differential equations describing the perturbation to the eigenvalues, and hence the amplitudes of the solitary waves, are obtained. To find the change in solitary wave amplitude these equations must be integrated from an initial time, well before the solitary waves interact, to a final time, well after interaction. See Byatt-Smith [7] or Myint and Grimshaw [18] for previous applications.

The leading-order change to the eigenvalues is given by

$$\frac{d\beta_i^2}{dt} = -\alpha^2 \int_{-\infty}^{\infty} \omega_i^2(x, t) R_2(\eta) dx, \quad i = 1, 2, \quad (13)$$

where β_i are the eigenvalues and ω_i are the eigenfunctions. The amplitude of the solitary waves is given by $a_i = 2\beta_i^2$. Only the perturbation term $R_2(\eta)$ is included in the integral (13); the terms $X_1(\eta)$ and $X_2(\eta)$ make no contribution as they represent the KdV integrable hierarchy. The eigenfunctions and two-soliton solution of the second-order KdV equation ((12) to $O(\alpha)$) are given by

$$\begin{aligned} \omega_i &= (2\beta_i E_i)^{\frac{1}{2}} (1 + \beta_{ij} E_j) E^{-1}, \quad i = 1, 2, \quad i \neq j, \\ E_i &= \exp \left[-2\beta_i \left(x - \left(4\beta_i^2 + \alpha\mu_1 \frac{8}{15} \beta_i^4 \right) t - x_i \right) \right], \\ \beta_{ij} &= \frac{\beta_i - \beta_j}{\beta_i + \beta_j}, \quad E = 1 + E_1 + E_2 + \beta_{12}^2 E_1 E_2, \\ \eta &= 4\beta_1 \omega_1^2 + 4\beta_2 \omega_2^2, \end{aligned} \quad (14)$$

where x_i are the phase shifts of the solitons. The eigenvalues and the phase shifts are chosen as

$$\gamma\beta_1 = \beta_2 = \frac{1}{\sqrt{2}}, \quad \gamma > 1, \quad \beta_1 x_1 = \beta_2 x_2 = \frac{\ln(\beta_{12})}{2}, \quad (15)$$

hence the collision represents a wave of amplitude unity overtaking a smaller wave of amplitude γ^{-2} . The center of the interaction is at time $t = 0$, hence the effect of the collision is found by integrating from times $t = -\infty$ to ∞ .

To find the leading-order change in amplitude the integral on the right-hand side of (13) is examined at times $t = \pm a$. This integral represents the perturbation to the eigenvalues. Firstly, it is noted that $\omega_i^2(x, t) = \omega_i^2(-x, -t)$ and hence $\eta(x, t) = \eta(-x, -t)$ for the two-soliton solution (14). Then the integral at time $t = -a$ becomes

$$\int_{-\infty}^{\infty} \omega_i^2(x, -a) R_2(x, -a) dx = - \int_{-\infty}^{\infty} \omega_i^2(x, a) R_2(x, a) dx, \quad (16)$$

where the fact that all the terms in $R_2(\eta)$ (see (11) at $O(\alpha^2)$) contain an odd number of derivatives is used. Hence the perturbations to the eigenvalues at times $t = \pm a$ are equal in magnitude and opposite in sign; hence the full integration implies that no change in amplitude occurs at third order.

4. The solitary wave interaction at fourth order

The terms at fourth order in the equation for the right-moving wave consist of the interaction term (10) plus purely local terms which are only a function of η^+ . Hence any fourth-order change in solitary wave amplitude, due to the collision, will only be due to the interaction term (10) as the result in Section 3 indicates that the other fourth-order terms cause no change in solitary wave amplitude.

4.1. The left-moving wave

To investigate the effect of the interaction term (10) on the solitary wave interaction, the left-moving wave η^- must first be found. Firstly, it is noted that the two-soliton solution can be written in the form

$$\eta^+ = \eta^+(x - t, \alpha t), \quad (17)$$

where the frame of reference now moves with the linear phase velocity and it can be clearly seen that the soliton interaction occurs on the slow timescale $T = \alpha t$. Equation (7), with $a = 0$, is a hyperbolic equation describing the left-moving wave at third order. This equation can be written in the form

$$\eta_t^- - \eta_x^- = f^+(x - t, \alpha t), \quad f^+ = -\frac{3}{8}\eta^+\eta_x^+\eta_{xx}^+, \quad (18)$$

where the forcing term f^+ depends on the right wave η^+ . Solving the above equation using the method of characteristics gives

$$\frac{d\eta^-}{dx} = -f^+ \quad \text{on } x = -t + \tau, \quad (19)$$

where the characteristics all have a slope of negative one and τ is the constant of integration which identifies a particular characteristic. The left wave η^- can then be written as the integral

$$\eta^-(x, t) = -\frac{1}{2} \int_{\infty}^{x-t} f^+\left(u, \frac{\alpha}{2}(x + t - u)\right) du. \quad (20)$$

Well before interaction, as $t \rightarrow -\infty$, following a particular characteristic implies $x \rightarrow \infty$. Hence the integration constant in (20) is taken so that the left wave η^- is zero as $x - t \rightarrow \infty$.

To first order the right wave η^+ is a solution of the KdV equation; the corresponding left wave η^- can be calculated to leading order by using (20). The soliton solution of the KdV equation is

$$\eta^+ = aS^2, \quad S = \text{sech } k(x - Vt), \quad V = 1 + \alpha\frac{a}{2}, \quad k = \left(\frac{3}{4}a\right)^{\frac{1}{2}}, \quad (21)$$

where in this section the scaling used corresponds to that of the KdV system

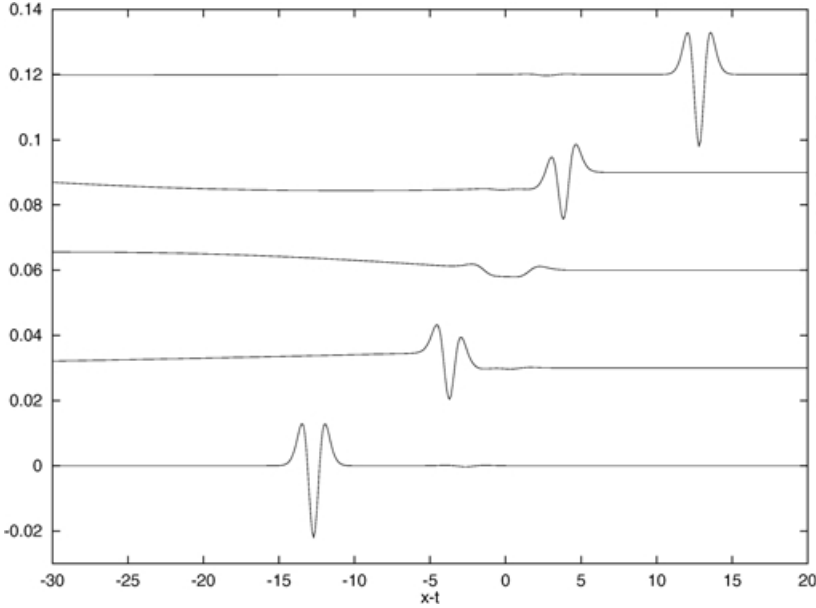


Figure 1. The left wave η^- during the interaction.

(6)–(7). Substituting (21) into (20) gives the left wave as

$$\eta^- = \frac{3}{16}a^4 \left(S^6 - \frac{9}{8}S^8 \right). \quad (22)$$

The left wave corresponding to a right-moving soliton is a right-moving forcing. The left wave η^- is symmetric, with a minimum, where $\eta^- = -\frac{3}{128}a^4$, at its center. It is flanked by two peaks smaller in magnitude than the central depression (see Figure 1).

When the right wave η^+ is given by the two-soliton solution (14) rescaled appropriately, the left wave is not purely a right-moving forcing, as the interaction causes a genuine left-moving wave to be generated. In the two-soliton case an explicit expression for the left wave cannot be obtained, as it could for the one-soliton case. However, the integral form (20) allows certain properties of the left-moving portion of η^- to be deduced. These will be complemented by numerical evaluations of (20), calculated using Simpson's rule.

The left wave η^- is considered at long time, as $t \rightarrow \infty$. This allows the solitons to complete their interaction with the right-moving forcings of the form (22) to moving to $x \rightarrow \infty$ and the left-moving portion moving to $x \rightarrow -\infty$. Hence, to find the left-moving portion let $x - t \rightarrow -\infty$ and note that $x + t$ is constant on a characteristic. This gives the left-moving portion of (20) as

$$l^- = \frac{1}{2} \int_{-\infty}^{\infty} f^+ \left(u, \frac{1}{2}(X + T - \alpha u) \right) du, \quad (23)$$

where T is the slow time and $X = \alpha x$ is a stretched length coordinate. As $l^- = l^-(X + T)$, it moves to the left at speed unity and varies on a much longer length-scale than the right-moving solitary waves. Also, using the fact that η^+ satisfies the KdV equation, (23) can be written in the alternative form

$$l^- = \frac{3}{16} \frac{\partial}{\partial T} \int_{-\infty}^{\infty} \eta^{+3} \left(u, \frac{1}{2}(X + T - \alpha u) \right) du. \quad (24)$$

It can now be seen that left-moving waves are only generated when the flow is transient. For the one-soliton solution, which moves without change of shape, the integral in (24) is constant, hence a left-moving wave does not exist. This agrees with Prasad and Akylas [14] who concluded that transient effects in stratified flows caused the generation of left-moving internal shelves. They also found that the internal shelves vary on a long length-scale.

Figure 1 shows the left wave η^- versus $x - t$, during the interaction of two solitary waves of nondimensional amplitudes 1 and $\frac{1}{3}$, with $\alpha = 0.3$. Shown is the left wave η^- at times $t = -80, -20, 0, 20, 80$; the mean level at $t = -80$ is zero and the mean level at each subsequent time is increased by 0.03. Before interaction at time $t = -80$, the two solitary left waves are given by (22) and $\eta^- \rightarrow 0$ as $x \rightarrow \pm\infty$. The smaller solitary wave is barely visible on the graph as its amplitude is $\frac{1}{81}$ th of the larger wave's amplitude. At time $t = -20$ the solitary waves are drawing together; the trailing peak on the left wave is now larger than the leading peak; attached behind the solitary wave is a decaying tail of positive displacement. At the center of the interaction, at $t = 0$, the KdV two-soliton solution (14) has the form of a single hump, as the waves have merged. The left waves have also merged with a form qualitatively similar to (22), however, the magnitude of the peaks and the central depression are much smaller than in the solitary left wave (22). A tail with positive mass, generated by the interaction, is attached to the rear of the wave. At time $t = 20$ the solitary waves are drawing apart; the leading peak is now larger than the trailing peak on the solitary left wave. The displacement of the tail is initially negative; it returns to positive values before decaying to the mean level. After interaction, at time $t = 80$, the two solitary left waves have regained the form (22). Note that the two left-solitary waves only depend on α via the slow timescale $T = \alpha t$; the variation of l^- , the left-moving portion of η^- , with α is described below.

Figure 2 shows the left-moving wave l^- versus X for the same parameters as Figure 1. The time is $t = 100$, which is chosen so that the solitons have completed their interaction. The left-moving wave l^- has matching regions of positive and negative displacements (with net mass of zero) and propagates in the negative x -direction with the linear wave speed of unity. The main effect of changing the nonlinearity is to change the length-scale of the wave; it gets shorter as α increases. In Figure 2, for which $\alpha = 0.3$, the amplitude of l^- is 5.59×10^{-3} . This changes only slightly, by about 1.8%, as α decreases from 0.3 to 0.15.

Using (17) implies that the mass-flux integral (9) $I = I(\alpha t)$. This indicates that the mass-flux between the right- and left-moving waves (η^+ and η^-)

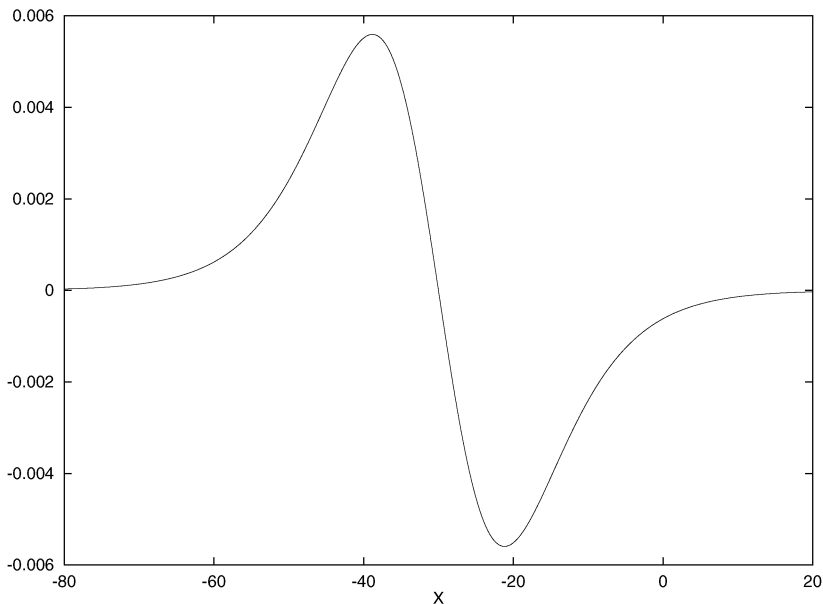


Figure 2. The left-moving wave l^- .

depends only on the slow timescale $T = \alpha t$, as does the soliton evolution. As the magnitude of the mass-flux is constant it is an $O(1)$ (leading-order) quantity. This implies that the left-moving wave, generated by the interaction, has an $O(1)$ amplitude also. This is confirmed above by the numerical evaluations of (23), which show only a small variation of amplitude with α . This small change is due to the forcing term in (18) having a spatial distribution. Therefore the mass-flux, which occurs at any given point over time, varies as the timescale for the soliton interaction changes.

Other qualitative information about the left-moving wave can also be deduced from the mass-flux integral. The integral I is anti-symmetric about $t = 0$ confirming that the net mass of the left-moving wave l^- is zero. At negative times the mass-flux is from η^+ to η^- , hence the leading half of l^- has positive amplitude. For positive times the mass-flux is from η^- to η^+ , which is consistent with the negative amplitude of the trailing half of l^- .

4.2. The change in solitary wave amplitude at fourth order

The change in solitary wave amplitude at fourth order for the system of KdV equations is now investigated. This is done by considering the ordinary differential equations (13) with the perturbation given by (10). At leading order the right wave η^+ corresponds to the KdV two-soliton solution. It obeys the symmetry property

$$\eta^+(x, t) = \eta^+(-x, -t). \quad (25)$$

This property was used in Section 3 to show that there was no change in solitary wave amplitude at third order as the right-hand side of (13), at times $t = \pm a$, are equal in magnitude and opposite in sign. The left wave η^- satisfies

$$\eta^-(x, t) = \eta^-(-x, -t) + l^-(X + T). \quad (26)$$

Now the left-moving wave l^- varies on a much longer length-scale than does the right-moving waves, which are located at $x = t$ to leading order. Hence, to leading order, the product $\eta^+\eta^-$ obeys a similar symmetry property

$$\eta^+(x, t)\eta^-(x, t) = \eta^+(-x, -t)(\eta^-(-x, -t) + l^-(2T)), \quad (27)$$

where $l^-(2T)$ represents a time-dependent adjustment to the mean level. The term involving $l^-(2T)$ in (27) will integrate to zero on the right-hand side of (13); see Appendix B. Hence it can be seen, in a similar manner to Section 3, that the right-hand side of (13) with the perturbation (10) is equal in magnitude and opposite in sign at times $t = \pm a$. This implies that no change in solitary wave amplitude occurs at fourth order.

Figure 3 shows the left wave η^- and the product $\eta^+\eta^-$ versus x . The parameters are the same as for Figure 2. Shown are the solutions at times $t = 20$ (solid line) and $t = -20$ (dashed line). The left wave at time $t = -20$ has its mean level adjusted (by $l^-(12) = -5.1 \times 10^{-3}$) and the solutions at time $t = -20$ have been reflected in the line $x = 0$. Also, the product $\eta^+\eta^-$ has 0.03 added to its mean level so that it lies above the left wave η^- in the figure. It can

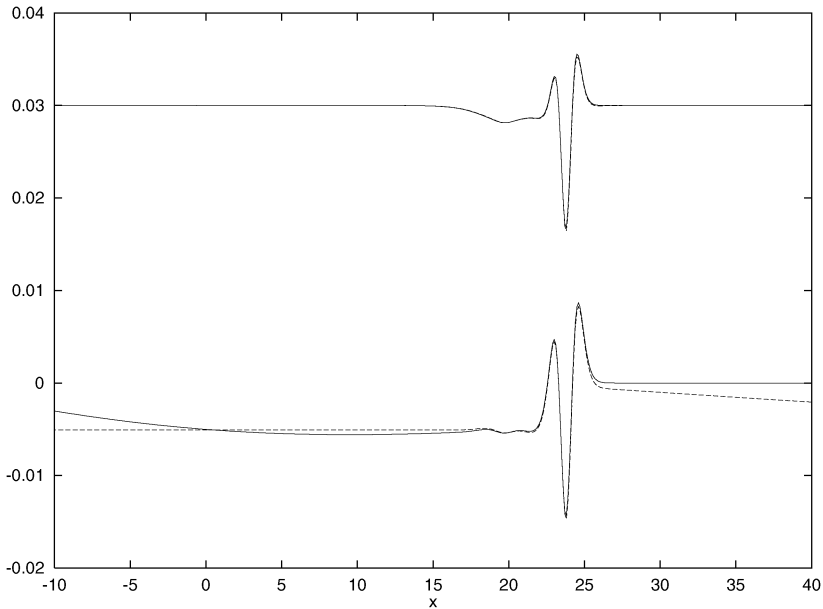


Figure 3. The left wave η^- with the product $\eta^+\eta^-$ above it.

be seen from the figure that the left wave η^- satisfies the symmetry condition ((25) with a change in mean level) near the center of the interaction, but the tails of the left wave are clearly unsymmetric. However, the product $\eta^+\eta^-$ does satisfy the symmetry condition (27). The reason is that the only contribution to the product is where the right-moving solitary waves are located; this is precisely where the left wave is symmetric. The unsymmetric tails of the left wave do not contribute to the product as the right-moving wave is zero there.

5. Conclusion

In this paper a perturbation expansion has been used to derive a system of KdV equations, which allows right- and left-moving waves, to third order. This resolved an inconsistency involving mass conservation, which occurs in the unidirectional model at third order. A fourth-order interaction term, in which the right- and left-moving waves are coupled, was also derived.

The inverse scattering perturbation method was used to investigate the solitary wave collision to fourth order. The left wave, associated with the KdV two-soliton collision, was found and used to show that the fourth-order interaction term caused no change in solitary wave amplitude.

The differences at fourth order between the KdV system of equations, the unidirectional KdV equation (both derived in this paper) and Kodama's fourth-order unidirectional model are illustrated. It was shown in Appendix B that no changes in solitary wave amplitude occur at fourth order for the unidirectional models either.

The conclusion is that interacting solitary waves on shallow water suffer no change in amplitude to fourth order; hence the change in amplitude is of fifth order. A future numerical investigation of solitary wave interaction, using the Euler water wave equations, would complement this theoretical study. The aim of the numerical study would be to confirm the theoretical result that the change in amplitude is of fifth order.

Appendix A. Third-order Boussinesq equations

The third-order Boussinesq equations are

$$\begin{aligned} \omega_t + \eta_x + \alpha \left(\omega \omega_x - \frac{1}{2} \omega_{xxt} \right) + \alpha^2 \left(\frac{1}{2} \omega_x \omega_{xx} - \eta_x \omega_{xt} + \frac{1}{24} \omega_{4xt} - \frac{1}{2} \omega \omega_{3x} \right. \\ \left. - \eta \omega_{xxt} \right) + \alpha^3 \left(\eta_x \omega_x^2 + \eta \omega_x \omega_{xx} - \eta \eta_x \omega_{xt} + \frac{1}{24} \omega \omega_{5x} - \frac{1}{720} \omega_{6xt} - \omega \eta_x \omega_{xx} \right. \\ \left. + \frac{1}{6} \eta_x \omega_{3xt} - \frac{1}{8} \omega_x \omega_{4x} + \frac{1}{6} \eta \omega_{4xt} - \frac{1}{2} \eta^2 \omega_{xxt} - \eta \omega \omega_{3x} + \frac{1}{12} \omega_{2x} \omega_{3x} \right) = 0, \end{aligned} \quad (\text{A.1})$$

and

$$\begin{aligned}
& \eta_t + \omega_x + \alpha \left(\omega \eta_x + \eta \omega_x - \frac{2}{3} \omega_{xxt} - \frac{1}{2} \eta_{xxt} \right) + \alpha^2 \left(\frac{2}{15} \omega_{5x} - 4 \eta_x \omega_{xx} \right. \\
& \quad \left. - 2 \eta_x \eta_{xt} - \eta_t \eta_{xx} + \frac{1}{24} \eta_{4xt} - \eta \eta_{xxt} - \frac{1}{2} \omega \eta_{3x} - \frac{5}{2} \omega_x \eta_{xx} - 2 \eta \omega_{3x} \right) \\
& \quad + \alpha^3 \left(-5 \omega_x \eta_x^2 - \eta \omega \eta_{3x} - 2 \eta \eta_x \eta_{xt} - \eta \eta_t \eta_{xx} - 3 \omega \eta_x \eta_{xx} \right. \\
& \quad \left. - 5 \eta \omega_x \eta_{xx} + \frac{1}{24} \omega \eta_{5x} + \frac{1}{6} \eta \eta_{4xt} - \frac{1}{720} \eta_{6xt} - \frac{4}{315} \omega_{7x} \right. \\
& \quad \left. + \frac{1}{6} \eta_t \eta_{4x} + \frac{2}{3} \eta_x \eta_{3xt} + \frac{2}{3} \eta \omega_{5x} - \frac{1}{2} \eta^2 \eta_{xxt} - 2 \eta^2 \omega_{3x} \right. \\
& \quad \left. + \frac{4}{3} \omega_{xx} \eta_{3x} + \frac{2}{3} \eta_{xt} \eta_{3x} + \eta_{xx} \eta_{xxt} - \eta_t \eta_x^2 \right. \\
& \quad \left. + \frac{7}{3} \eta_{xx} \omega_{3x} + 2 \eta_x \omega_{4x} + \frac{3}{8} \omega_x \eta_{3x} - 8 \eta \eta_x \omega_{xx} \right) = 0, \tag{A.2}
\end{aligned}$$

where $\omega = f_x$, which is the lowest order horizontal velocity.

Appendix B. The fourth-order unidirectional equation

If the unidirectional assumption is used, where $u^- = 0$ and the coefficient $a = -\frac{3}{8}$, then the nonlocal term in the horizontal velocity (5) causes nonlocal terms to appear at fourth order in the unidirectional equation. These are

$$-\frac{9}{4} \eta_x \int_{-\infty}^x \eta \eta_u \eta_{uu} du + \frac{27}{32} \int_{-\infty}^x \eta_u \eta_{uu} (6 \eta \eta_u + \eta_{uuu}) du. \tag{B.1}$$

The first of these nonlocal terms is predicted by Kodama [13] (see his (2.3)); however, the second term is not present in his expression.

To find the fourth-order change in solitary wave amplitude the ordinary differential equations (13) are considered, with the perturbation given by the nonlocal terms (B.1). These ordinary differential equations are

$$\begin{aligned}
& \frac{d\beta_i^2}{dt} = I_j(t), \quad j = 1, 2, \\
& I_1(t) = \int_{-\infty}^{\infty} \omega_i^2 \eta_x \int_{-\infty}^x \eta \eta_u \eta_{uu} du dx, \\
& I_2(t) = \int_{-\infty}^{\infty} \omega_i^2 \int_{-\infty}^x \eta_u \eta_{uu} (6 \eta \eta_u + \eta_{uuu}) du dx, \tag{B.2}
\end{aligned}$$

where I_1 and I_2 represent the nonlocal terms in (B.1). Summing the integrals

I_i at times $t = \pm a$ allows the new forms

$$\begin{aligned} I_1(t) &= \int_{-\infty}^{\infty} \omega_i^2 \eta_x dx \int_{-\infty}^{\infty} \eta \eta_x \eta_{xx} dx, \\ I_2(t) &= \int_{-\infty}^{\infty} \eta_x \eta_{xx} (6\eta \eta_x + \eta_{3x}) dx, \end{aligned} \quad (\text{B.3})$$

to be obtained. The total change in amplitude is now found by integrating from time $t = 0$ to ∞ .

Both of the integrals in (B.3) can be shown to give a zero contribution to the change in solitary wave amplitude. Firstly, the square of the eigenvectors satisfies the identity

$$(\omega_i^2)_{3x} + 2(\eta \omega_i^2)_x + 2\eta(\omega_i^2)_x = 4\beta_i^2(\omega_i^2)_x, \quad (\text{B.4})$$

where η is the two-soliton solution (see Kodama [11]). Integrating (B.4) over the real number line and using integration by parts gives $\int_{-\infty}^{\infty} \omega_i^2 \eta_x dx = 0$, which implies $I_1 = 0$. Next, by using the KdV equation and integration by parts the integral I_2 can be written as

$$I_2(t) = \frac{d}{dt} I_2'(t), \quad \text{where} \quad I_2'(t) = -\frac{1}{6} \int_{-\infty}^{\infty} \eta_x^3 dx. \quad (\text{B.5})$$

Integrating the second part of (B.2) gives the change in solitary wave amplitude as $I_2'(\infty) - I_2'(0)$. Now, due to its symmetry, $I_2' = 0$ for a single soliton. As the time becomes large the solution becomes two well separated solitons, therefore $I_2'(\infty) = 0$. At the time $t = 0$ the two solitons are fully merged into a single symmetric profile hence $I_2'(0) = 0$ also. Therefore no change to the solitary wave amplitude results from I_1 or I_2 . This result for the two nonlocal terms (B.1) was confirmed by solving the equations (B.2) numerically. Hence there is no fourth-order change in solitary wave amplitude, after collision, for the unidirectional model either.

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