Asymptotic solitons for a higher-order modified Korteweg–de Vries equation

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(Received 25 June 2002; published 29 October 2002)

Solitary wave interaction for a higher-order modified Korteweg–de Vries (mKdV) equation is examined. The higher-order mKdV equation can be asymptotically transformed to the mKdV equation, if the higher-order coefficients satisfy a certain algebraic relationship. The transformation is used to derive the higher-order two-soliton solution and it is shown that the interaction is asymptotically elastic. Moreover, the higher-order phase shifts are derived using the asymptotic theory. Numerical simulations of the interaction of two higher-order solitary waves are also performed. Two examples are considered, one satisfies the algebraic relationship derived from the asymptotic theory, and the other does not. For the example which satisfies the algebraic relationship the numerical results confirm that the collision is elastic. The numerical and theoretical predictions for the higher-order phase shifts are also in strong agreement. For the example which does not satisfy the algebraic relationship, the numerical results show that the collision is inelastic; an oscillatory wavetrain is produced by the interacting solitary waves. Also, the higher-order phase shifts for this inelastic example are tabulated, for a range of solitary wave amplitudes. An asymptotic mass-conservation law is derived and used to test the finite-difference scheme for the numerical solutions. It is shown that, in general, mass is not conserved by the higher-order mKdV equation, but varies during the interaction of the solitary waves.

DOI: 10.1103/PhysRevE.66.046623 PACS number(s): 05.45.Yv, 02.30.Jr

I. INTRODUCTION

The Korteweg–de Vries (KdV) equation is the generic model for the study of weakly nonlinear long waves. It arises in physical systems which involve a balance between nonlinearity and dispersion at leading order. For example, it describes surface waves of long wavelength and small amplitude on shallow water and internal waves in a shallow density-stratified fluid. Many other applications for the KdV equation also exist, such as plasma waves, Rossby waves, and magma flow. Also, the KdV equation is integrable. This means that a collision between KdV solitary waves is elastic; after the collision the solitons retain their original shape with the only memory of the collision being a phase shift. The explicit solution for interacting KdV solitons was developed using the bilinear transformation method, by Hirota [1].

The modified Korteweg–de Vries (mKdV) equation (in which the quadratic nonlinearity $\eta\eta_x$ of the KdV equation is replaced by the cubic nonlinearity $\eta^2\eta_x$) has many physical applications also. These include electrodynamics, electromagnetic waves in size-quantised films, internal waves for certain special density stratifications, elastic media, and traffic flow. The mKdV equation occurs in two versions; the cubic nonlinearity term can have either positive or negative sign. Hirota [2] found the *N*-soliton solution for the version of the mKdV equation with positive sign. Both Perelman *et al.* [3] and Ono [4] developed the *N*-soliton solution for the case with negative sign. Of particular interest in the negative case is the elastic interaction of a soliton with a dissipationless shock wave.

A higher-order mKdV equation with positive sign will be examined,

$$\eta_t+24\eta^2\eta_x+\eta_{3x}+\alpha c_1\eta^4\eta_x+\alpha c_2\eta_x^3+\alpha c_3\eta^2\eta_{3x}+\alpha c_4\eta\eta_x\eta_{xx}+\alpha c_5\eta_{5x}=0,\quad \alpha\ll 1. \tag{1}$$

When the higher-order coefficients are given by

$$(c_1,c_2,c_3,c_4,c_5)=(1,\tfrac{1}{12},\tfrac{1}{12},\tfrac{1}{3},\tfrac{1}{480}), \tag{2}$$

then Eq. (1) is a member of the mKdV integrable hierarchy of equations, see, Matsuno [5]. Hence Eq. (1) is a generalization of the integrable higher-order mKdV equation (2).

Asymptotic transformations have been developed for the higher-order KdV equation

$$\eta_t+6\eta\eta_x+\eta_{3x}+\alpha c_1\eta^2\eta_x+\alpha c_2\eta_x\eta_{xx}+\alpha c_3\eta\eta_{3x}+\alpha c_4\eta_{5x}=0,\quad \alpha\ll 1, \tag{3}$$

by Kodama [6]. More recently, Fokas and Liu [7], showed that asymptotic transformations can be developed by considering the master symmetries of the associated integrable equation. Marchant [8] asymptotically transformed the higher-order KdV equation (3) to the KdV equation. The transformation used included a nonlocal term and allowed the higher-order two-soliton solution for the higher-order KdV equation (3) to be constructed. The higher-order solitary waves were found to be asymptotic solitons as the collision was elastic to $O(\alpha)$. Also, the higher-order corrections to the phase shifts of the waves after collision were found.

The aim of this paper is to examine the nature of the interaction of solitary waves governed by the higher-order mKdV equation (1). In §2 the asymptotic theory is developed and the theoretical results presented. The asymptotic transformation used is nonlocal and is similar to the asymptotic transformation applicable for the higher-order KdV equation (3). A single higher-order solitary wave is derived from the mKdV one-soliton solution using the transformation. Then a special case, when the higher-order coefficients satisfy a given algebraic relationship, is identified, for which the higher-order collisions are asymptotically elastic. For this special case the mKdV two-soliton solution is used to derive the two-soliton solution for the higher-order mKdV equation (1), and the higher-order phase shifts are found. In the case of arbitrary higher-order coefficients, the asymptotic transformation is used to derive a formula for the higher-order phase shifts in terms of the phase shifts, found numerically in §3, for an inelastic example.

In §3 numerical solutions are presented for two examples. For the example of an elastic collision, the theoretical predictions of §2 are confirmed. In particular, the theoretical and numerical values for the higher-order phase shifts are in close agreement. For the example of an inelastic collision the higher-order phase shifts are tabulated for a range of wave amplitudes. The Appendix A has details of the finite-difference scheme used to calculate numerical solutions while the Appendix B describes the application of asymptotic theory to an alternative version of the higher-order mKdV equation.

II. THE ASYMPTOTIC THEORY

Consider the transformation

$$\eta = u + \frac{\alpha}{240}(3c_4 - c_1 - c_2 - c_3 - 160c_5)u_{xx} + \frac{\alpha}{120}(12c_3 - 3c_1 - 8c_2 + 4c_4)u^3,$$

$$\tau = t + \alpha\frac{1}{3}c_5 x, \xi = x + \alpha\frac{1}{3}(32c_5 - c_3)\int_{-\infty}^{x} u^2(p,t)dp, \alpha \ll 1, \tag{4}$$

where $u(x,t)\to 0$ as $x\to\pm\infty$. This form of transformation is appropriate for solutions which approach zero far up and downstream, such as the solitary wave solutions considered here. For other forms of solution, such as periodic solutions, the nonlocal term in the transformation (4) needs to be modified slightly. If Eq. (4) is substituted into Eq. (1), and terms of $O(\alpha^2)$ are neglected, then $u(\xi,\tau)$ is a solution of the higher-order mKdV equation

$$u_\tau + 24u^2 u_\xi + u_{3\xi} + \alpha' 4u^4 u_\xi - \alpha' u_\xi^3 + \alpha' u u_\xi u_{\xi\xi} = 0,$$

$$\text{where} \quad \alpha' = \frac{\alpha}{20}(4c_3 - c_1 - 16c_2 + 8c_4 - 320c_5) \ll 1. \tag{5}$$

The particular transformation (4) is chosen because the higher-order mKdV equation (5) has exact solitary wave solutions. Even though they have the same form as the mKdV soliton, the solitary wave solutions of Eq. (5) are not solitons. The fact that the solitary wave solutions have no $O(\alpha)$ amplitude or velocity corrections makes Eq. (5) an ideal example for the numerical simulations in §3. The asymptotic theory and numerical results of §3 are linked together in §2.3.

Also, note that if the algebraic relationship, between the higher-order coefficients,

$$c_1 + 16c_2 - 4c_3 - 8c_4 + 320c_5 = 0, \tag{6}$$

is satisfied then $\alpha' = 0$ and Eq. (5) becomes the mKdV equation.

A. A single higher-order solitary wave

The mKdV soliton solution is

$$u = A\,\text{sech}\,\theta, \quad \theta = 2A(\xi - s - V\tau), \tag{7}$$

where A is the amplitude, the velocity $V = 4A^2$ and the soliton is at $\xi = s$ at $\tau = 0$. The exact solitary wave solution of Eq. (5) is Eq. (7). Using Eq. (7) in the transformation Eq. (4) gives

$$\eta = A\,\text{sech}\,\theta + \alpha\,\tfrac{1}{60}(3c_4 - c_1 - c_2 - c_3 - 160c_5)A^3\,\text{sech}\,\theta + \alpha\,\tfrac{1}{120}(c_1 - 4c_2 + 16c_3 - 8c_4 + 640c_5)A^3\,\text{sech}^3\theta + \cdots,$$

$$\theta = 2A[x(1 - \tfrac{4}{3}c_5A^2) - s + \alpha A\,\tfrac{1}{6}(32c_5 - c_3) \times (\tanh\theta + 1) - Vt]. \tag{8}$$

The phase θ must be made explicit by expanding it in a Taylor series. Also the amplitude is rescaled by

$$A = A^*(1 + \alpha\,\tfrac{4}{3}c_5 A^{*2}), \tag{9}$$

which gives (after dropping the stars)

$$\eta = A\,\text{sech}\,\theta + \alpha a_3 A^3\,\text{sech}\,\theta + \alpha a_4 A^3\,\text{sech}^3\theta + \cdots,$$

$$\theta = 2A[x - s(1 + \alpha\tfrac{4}{3}c_5A^2) + \alpha A\,\tfrac{1}{6}(32c_5 - c_3) - Vt],$$

$$\text{where} \quad V = 4A^2 + \alpha 16 c_5 A^4,$$

$$a_3 = \tfrac{1}{60}(-c_1 - c_2 + 19c_3 + 3c_4 - 720c_5),$$

$$a_4 = \tfrac{1}{120}(c_1 - 4c_2 - 24c_3 - 8c_4 + 1920c_5), \tag{10}$$

for a single solitary wave of the higher-order mKdV equation (1). Expression (10) is the same as that derived directly. Moreover, the higher-order solitary wave has been shifted from $\xi = s$ at $\tau = 0$ to $x = s(1 + \alpha\tfrac{4}{3}c_5A^2) - \alpha A\tfrac{1}{6}(32c_5 - c_3)$ at $t = 0$.

B. The higher-order two-soliton solution

Here the two-soliton solution of the higher-order mKdV equation (1) is found for the case when the higher-order coefficients satisfy Eq. (6). The two-soliton solution of the mKdV equation [(5) with $\alpha' = 0$] is

$$u = \phi_\xi, \quad \tan\phi = \frac{g}{f}, \quad \text{where} \quad g = h_1 + h_2,$$

$$f = 1 + m h_1 h_2, \quad h_i = \exp 2A_i(\xi - 4A_i^2\tau - p_i), \quad i = 1,2,$$

$$m = -\left(\frac{A_2 - A_1}{A_2 + A_1}\right)^2, \tag{11}$$

see Hirota [2]. The velocity of the ith soliton is $4A_i^2$, while p_i is an arbitrary phase constant. Well before interaction (as $\tau\to-\infty$) the two-soliton solution (11) is

$$u=A_1 \operatorname{sech}\theta_1+A_2 \operatorname{sech}\theta_2, \quad \theta_i=2A_i(\xi-s_i-4A_i^2\tau), \quad i=1,2. \tag{12}$$

Expression (12) is just the sum of two solitons; it satisfies the mKdV equation [Eq. (5) with $\alpha'=0$] because the solitons are a long distance apart [Eq. (12) is obtained from Eq. (11) by following each soliton in turn and letting $\tau\to-\infty$]. Also we choose $A_2>A_1$ and $s_2<s_1$; this means that the larger soliton is behind the smaller one initially. The larger soliton travels faster; hence it will interact with the smaller soliton as it overtakes it. To see the result of the collision the solution is considered well after interaction (as $\tau\to\infty$). After interaction the solution is again Eq. (12), but with the phase shifts

$$\frac{1}{A_2}\ln\left(\frac{A_2+A_1}{A_2-A_1}\right), \quad -\frac{1}{A_1}\ln\left(\frac{A_2+A_1}{A_2-A_1}\right), \tag{13}$$

for the larger and smaller solitons, respectively. Hence the collision is elastic, with no change in the solitons' shapes (hence mass and energy are conserved for each soliton) and no dispersive radiation is produced as a result of the interaction. The only memory of the collision is a phase shift forwards for the larger soliton and a phase shift backwards for the smaller soliton.

The higher-order two-soliton solution, to $O(\alpha)$, describing the interaction of two higher-order solitary waves governed by Eq. (1) is just the mKdV two-soliton solution (11) transformed by using Eq. (4). Due to the complicated form of Eq. (11) the explicit higher-order two-soliton solution will not be calculated; the nature of the collision can be found by considering the solution well before and after interaction. Expression (12) describes the mKdV two-soliton solution (11) before and after interaction; substituting Eq. (12) into the transformation (4) gives

$$\eta=A_1 \operatorname{sech}\theta_1+A_2 \operatorname{sech}\theta_2+\alpha a_3A_1^3 \operatorname{sech}\theta_1+\alpha a_3A_2^3 \operatorname{sech}\theta_2 +\alpha a_4A_1^3 \operatorname{sech}^3\theta_1+\alpha a_4A_2^3 \operatorname{sech}^3\theta_2+\cdots,$$

$$\theta_i=2A_i[x-s_i(1+\alpha\tfrac{4}{3}c_5A_i^2)-s_i'-V_it], \quad i=1,2,$$

$$V_i=4A_i^2+\alpha 16c_5A_i^4t, \tag{14}$$

which is just two single higher-order solitary waves as required. The solitary waves are unchanged in shape after the collision and no dispersive radiation is generated, hence the collision is elastic.

The transformation modifies the phase shifts (13), which occur after interaction, of the mKdV solitons. Before and after interaction the phase constants s_i' of each solitary wave are

$$s_1'=-\alpha\tfrac{1}{6}(32c_5-c_3)(A_1+2A_2),$$

$$s_2'=-\alpha\tfrac{1}{6}(32c_5-c_3)A_2, \quad \tau\to-\infty,$$

$$s_1'=-\alpha\tfrac{1}{6}(32c_5-c_3)A_1,$$

$$s_2'=-\alpha\tfrac{1}{6}(32c_5-c_3)(2A_1+A_2), \quad \tau\to\infty. \tag{15}$$

As the nonlocal term in the transformation of the phase θ is an integral from far behind the solitary wave to the current position, an extra term, due to the integration over the trailing solitary wave, appears in the phase of the leading solitary wave. The extra term appears in the phase of the smaller wave before collision and the larger wave after collision. There are two contributions to the higher-order phase shifts, the integration terms (15) and the amplitude scaling (9). Combining these terms gives the phase shifts as

$$\frac{1}{A_2}\ln\left(\frac{A_2+A_1}{A_2-A_1}\right)-\alpha\frac{1}{3}(40c_5-c_3)A_1,$$

$$-\frac{1}{A_1}\ln\left(\frac{A_2+A_1}{A_2-A_1}\right)+\alpha\frac{1}{3}(40c_5-c_3)A_2, \tag{16}$$

for the larger and smaller higher-order solitary waves, respectively. In the case of higher-order coefficients which satisfy Eq. (6) and $c_3=40c_5$ the phase shifts are unchanged from the mKdV case. This is true of the integrable version of the higher-order mKdV equation [Eq. (1) with the coefficients Eq. (2)], hence the transformation correctly predicts that there is no higher-order phase shift in that case.

In summary, when Eq. (6) is satisfied, the higher-order mKdV solitary wave collision is asymptotically elastic with phase shifts given by Eq. (16).

C. The higher-order phase shifts for arbitrary higher-order coefficients

What is the nature of the higher-order mKdV solitary wave interaction when the algebraic relation (6) is not satisfied? Inverse scattering for perturbed mKdV equations indicates that there will be no amplitude change at $O(\alpha)$ in these cases. However the collision is not asymptotically elastic, as radiation, of magnitude $O(\alpha^2)$, will be shed. There is no simple analytical way to calculate the higher-order phase shifts in this case. However, the asymptotic transformation can be used to relate the phase shifts for the special case (5) of the higher-order mKdV equation, to the phase shifts of the higher-order mKdV equation (1) with arbitrary coefficients.

After an interaction governed by Eq. (5), the two higher-order solitary waves are given by Eq. (14), except for the unknown $O(\alpha)$ phase shifts and the radiation shed, of $O(\alpha^2)$. Applying the transformation, and ignoring the radiation of $O(\alpha^2)$, gives the $O(\alpha)$ phase shifts, for Eq. (1) with arbitrary higher-order coefficients, as

$$-\alpha\tfrac{1}{3}(40c_5-c_3)A_1+\alpha' s_l, \quad \alpha\tfrac{1}{3}(40c_5-c_3)A_2+\alpha' s_s, \tag{17}$$

for the larger and smaller waves, respectively. s_l and s_s are the higher-order phase shifts for the special case (5). If the algebraic relationship (6) is satisfied then $\alpha'=0$ and Eq. (17) reduces to the prediction of the asymptotic theory. The usefulness of Eq. (17) is that numerical estimates of the higher-order phase shifts need only be found for Eq. (5), as Eq. (17) then allows the calculation of the higher-order phase shifts in the case of arbitrary coefficients.

III. NUMERICAL RESULTS

In this section the interaction of higher-order mKdV solitary waves is examined numerically. This allows the theoretical results, which apply when Eq. (6) is satisfied, to be verified and allows the nature of the collision to be determined for an example not covered by the asymptotic theory.

The numerical solution of higher-order KdV-type equations is problematic. For example, Marchant and Smyth [9] solved the higher-order KdV equation numerically, using a hybrid Runge-Kutta method for time and centred finite differences for the spatial coordinate. They reported that the numerical scheme was only stable for small values of α, large Δx and small Δt. The lack of spatial resolution in the numerical solution made the determination of any small inelastic effects and small higher-order phase shifts extremely difficult.

To overcome these shortcomings, the numerical method used here (see, the Appendix A for details) is based on a perturbation technique, similar to that used by Zou and Su [10]. They considered the interaction of higher-order KdV solitary waves numerically, at second and third orders, by using a perturbation expansion. The lowest-order solution, about which the perturbation was based, was the KdV two-soliton solution. Their finite-difference scheme is unconditionally stable so is ideal for accurately resolving inelastic effects and higher-order phase shifts.

The higher-order mKdV equation (1) is expanded using

$$\eta=v+\alpha p, \tag{18}$$

where v is the two-soliton solution (11) of the mKdV equation and p is the $O(\alpha)$ correction term of interest here. The expansion (18) is substituted into the higher-order mKdV equation (1), which leads to a linear KdV-type equation with forcing terms

$$p_t+24\frac{\partial}{\partial x}(v^2p)+p_{3x}=-f, \quad \text{where}$$

$$f=c_1v^4v_x+c_2v_x^3+c_3v^2v_{3x}+c_4vv_xv_{xx}+c_5v_{5x}. \tag{19}$$

Before considering the interaction of higher-order mKdV solitary waves the contribution to the perturbation p, from a single higher-order mKdV solitary wave (10) needs to be considered. The parameter choices

$$(c_1,c_2,c_3,c_4,c_5)=(20,-1,1,0,0), \tag{20}$$

$$(c_1,c_2,c_3,c_4,c_5)=(4,-1,0,1,0), \tag{21}$$

are used for the numerical examples as they have no $O(\alpha)$ amplitude or velocity corrections. The special case (20) satisfies (6), hence is asymptotically elastic and can be used to verify the theoretical results. The special case (21) does not satisfy (6), hence is not covered by the asymptotic theory.

For these special cases, the perturbation p after interaction only represents the higher-order phase shifts and any inelastic effects. The higher-order solitary wave solution for the special cases (20) and (21) is the mKdV soliton

$$\eta=A\,\text{sech}\,\theta, \quad \theta=2A(x-\alpha s-4A^2t), \tag{22}$$

where the soliton is located at $x=\alpha s$ at $t=0$. Expanding Eq. (22) in a Taylor series expansion gives

$$\eta=A\,\text{sech}\,\theta+\alpha 2sA^2\,\text{sech}\,\theta\tanh\theta, \tag{23}$$

hence if an $O(\alpha)$ phase shift of s is applied to the solitary wave then the perturbation is given by the antisymmetric function $p=2sA^2\,\text{sech}\,\theta\tanh\theta$. Hence the higher-order phase shift s can be deduced by measuring the amplitude of the antisymmetric $\text{sech}\,\theta\tanh\theta$ function.

The higher-order mKdV equation (1) has the asymptotic mass-conservation law

$$\int_{-\infty}^{\infty}\eta+\alpha\frac{1}{6}a_5\eta^3dx, \quad a_5=c_4-2c_2-2c_3, \tag{24}$$

hence mass is only conserved if $a_5=0$. Any physically applicable model equation would need to conserve mass, hence $a_5=0$ would be a requirement. However, it is also of interest to note that the algebraic relation (6), for which asymptotic solitons exist, includes both mass-conserving and nonmass-conserving cases. The conservation law (24) can be evaluated as

$$\pi+\alpha\pi\tfrac{1}{24}(a_5+12a_3+6a_4)(A_1^2+A_2^2), \tag{25}$$

by considering the interaction well before or after the collision, when the solitary waves are well separated. If the mass of the system is written as $\pi+\alpha M_p$, then the equation for the mass of the perturbation p is

$$M_p=\int_{-\infty}^{\infty}p\,dx=\frac{\pi}{4}(2a_3+a_4)(A_1^2+A_2^2)+\frac{a_5}{24}\left(\pi(A_1^2+A_2^2)-4\int_{-\infty}^{\infty}v^3dx\right), \tag{26}$$

where v is the two-soliton solution (11) of the mKdV equation. The $O(\alpha)$ mass has been split into two parts. The first part, associated with the $O(\alpha)$ corrections to the solitary wave's profile, is constant (and equal to zero for the special cases considered here) throughout the collision. The second part, associated with the cubic term in the conservation law (24), varies through the interaction. The expression (26), for the mass of the perturbation p, shall be used as an additional check on the accuracy of the numerical results.

Example 1 is of the higher-order mKdV equation with coefficients (20), hence it satisfies Eq. (6) and is asymptoti-

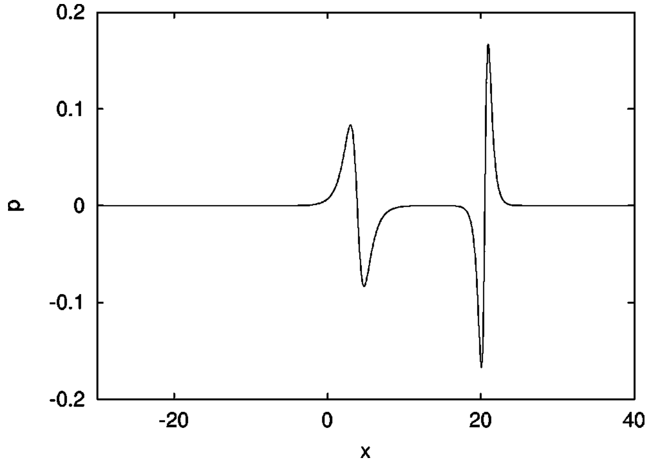

FIG. 1. The perturbation p at $t=5$ for example 1.

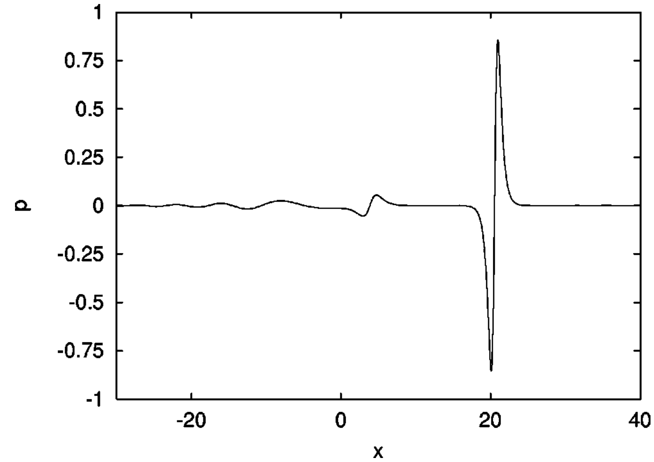

FIG. 2. The perturbation p at $t=5$ for example 2.

cally elastic. Initially the perturbation $p=0$ and time $t=-5$, which corresponds to a time, well before interaction of the solitary waves. The forced linear KdV-type equation (19) is solved numerically, from $t=-5$ up until $t=5$, which represents a time, well after interaction. The spatial and temporal gridspacings used are $\Delta x=2.5\times10^{-3}$ and $\Delta t=5\times10^{-4}$.

Figure 1 shows the perturbation p for example 1 after interaction, at $t=5$. The amplitudes of the two solitons are $A_2=1$ and $A_1=0.5$. It shows that p comprises two, well separated, antisymmetric $\operatorname{sech}\theta\tanh\theta$ functions, which represent the higher-order phase shifts of the two solitary waves. First, note that the leading antisymmetric wave represents a phase shift forwards for the large solitary wave, while the trailing antisymmetric wave (as the position of its positive and negative peaks have been reversed) represents a phase shift backwards for the smaller solitary wave.

The numerical estimates of the phase shifts are obtained by measuring the peak amplitude of the antisymmetric waves. The peak amplitudes are also measured using values of Δx and Δt double the values quoted above. Richardson extrapolation is then used to obtain a converged estimate of the phase shifts. The numerically obtained phase shifts are 0.16 628 for the larger wave and $-0.33\,333$ for the smaller wave. The theoretical predictions (16) for the higher-order phase shifts in this example are $\frac{1}{6}$ for the large wave and $-\frac{1}{3}$ for the small wave. Hence the errors in the numerical phase shifts are 0.2% and 1×10^{-5}%, for the large and small solitary waves, respectively.

Examination of the free surface behind the antisymmetric waves after interaction shows that it essentially flat, with no dispersive wave train produced. The amplitude of the largest oscillation behind the antisymmetric waves is $O(10^{-6})$. Richardson extrapolation indicates the converged amplitude of this oscillation is $O(10^{-7})$, confirming that it is merely a result of the discretisation. Hence, no dispersive wave train occurs as a result of the collision, so it is elastic to $O(\alpha)$.

Example 1 is mass conserving as $a_5=0$. Hence, as $a_3=a_4=0$ also, Eq. (26) indicates that the mass of the perturbation, M_p, is zero throughout the interaction of the higher-order solitary waves. The mass of the perturbation is calculated numerically from the finite-difference solution at each time step, by using Simpson's method. The numerical results indicate, that for the spatial and time-discretisations used, that the magnitude of the perturbation mass M_p is never greater than 1.9×10^{-6}. This represents an extremely small change to the total mass, π, of the interacting solitary waves. Moreover, Richardson extrapolation indicates the converged mass of the perturbation is no greater than $O(10^{-8})$. Hence the numerical scheme conserves the mass of the perturbation extremely accurately throughout the interaction of the higher-order mKdV solitary waves.

In summary, it can be seen that the theoretical predictions have been confirmed for this example. The numerical results show that the collision is asymptotically elastic to $O(\alpha)$ and the numerical estimates for the higher-order phase shifts are very close to the theoretical predictions (16). These results also show that the numerical scheme is an extremely accurate method of determining the higher-order phase shifts, which will now be used for the special case (21), which is not asymptotically elastic.

Example 2 is of the higher-order mKdV equation (1) with coefficients (21), hence it does not satisfy Eq. (6) and the asymptotic results of §2 do not apply. As for example 1, the perturbation $p=0$ and time $t=-5$ initially. The results described here are for $t=12$, longer than example 1, which allows the dispersive wave train to completely separate from the smaller antisymmetric wave. The spatial and temporal gridspacings used here are the same as those in example 1.

Figure 2 shows the perturbation p for example 2 after interaction, at $t=5$. The amplitudes of the two solitons are $A_2=1$ and $A_1=0.5$. The figure shows the wave train at an earlier time ($t=5$ rather than $t=12$) simply for graphical convenience, as at longer times the large antisymmetric wave moves well away from the smaller wave. It shows that p comprises two, well separated, antisymmetric $\operatorname{sech}\theta\tanh\theta$ functions, which represent the higher-order phase shifts of the solitary waves and a dispersive wave train representing the inelastic effects of the collision. Note in this case that the antisymmetric waves represent a phase shift forwards, at higher-order, for both solitary waves. This is in contrast to example 1, where the higher-order phase shift for the large

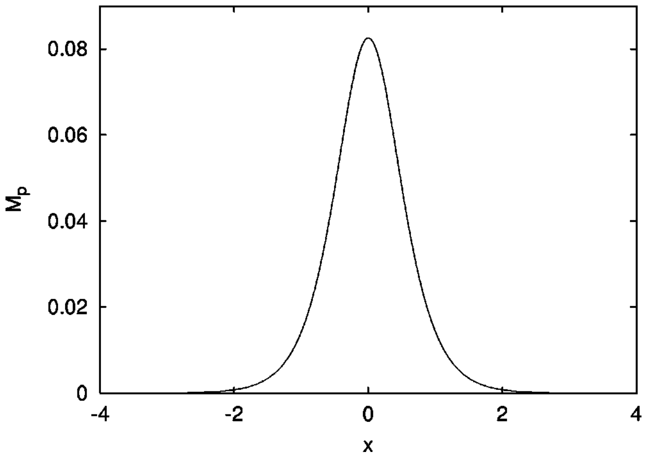

FIG. 3. The change in the mass of the perturbation M_p, during the interaction, for example 2.

solitary wave was forwards and the phase shift for the small solitary wave was backwards.

Examination of the free surface behind the nonsymmetric waves after interaction shows that a dispersive wave train has been produced. The amplitude of the largest wave in the dispersive wave train is 1.8×10^{-2} with Richardson extrapolation indicating that this is very close to the converged amplitude (as $\Delta t,\ \Delta x\rightarrow0$). Hence this dispersive wave train is a result of an inelastic solitary wave interaction and is not due merely to the numerical discretisation. The converged numerical estimates for the higher-order phase shifts are 0.8585 for the larger wave and 0.2270 for the smaller wave. Richardson extrapolation shows that these estimates have converged to at least three (for the larger wave) and four (for the smaller wave) decimal places.

Figure 3 shows the perturbation mass M_p versus time t, for the same parameters as Fig. 2. Shown is the solution of Eq. (26). The direct numerical calculation of M_p, from the finite-difference solution of Eq. (19) is not shown as it is the same as Eq. (26) to graphical accuracy. This example is not mass-conserving throughout the collision as $a_5=3$. The mass of the perturbation M_p increases from zero at $t=-5$ to a maximum at the center of the interaction; for positive time M_p decreases back to zero. Hence for this example, the mass of the system increases as the solitary waves approach each other and decreases back to the initial value as they separate. The maximum mass of the perturbation, which occurs at $t=0$, is predicted by Eq. (26) to be 8.25450×10^{-2}. The variation between this prediction for the maximum mass, and that from the numerical scheme, is very small, being only 5.5×10^{-6} or 0.007%. Hence the numerical scheme accurately predicts the mass of the perturbation throughout the collision, for this nonmass-conserving example.

Table I shows the numerically obtained higher-order phase shifts, for Eq. (1) with the coefficients (21). The amplitude of the large solitary wave is $A_2=1$ and the small solitary wave takes a range of different amplitudes. Richardson extrapolation indicates that the phase shift estimates have converged to at least the three significant figures presented.

When the amplitude of the smaller wave $A_1<0.2$, the numerical calculation of the phase shift suffered by small wave, is extremely intensive. This is because the soliton velocity is very small ($V=4A_1^2$) and the antisymmetric wave associated with the phase shift takes a very long time to completely separate from the dispersive wave train. For example, when $A_1=0.1$, the numerical simulation, begun at $t=-15$ must be continued until $t=1135$, for complete separation to occur.

TABLE I. Numerical higher-order phase shifts for Eq. (1) with the coefficients (21).

Amplitude, (A_1) of smaller wave	Phase shift: Larger wave	Phase shift: Smaller wave
0.9	-0.164	0.397
0.8	5.07×10^{-2}	0.409
0.7	0.297	0.382
0.6	0.568	0.321
0.5	0.859	0.227
0.4	1.16	9.91×10^{-2}
0.3	1.48	-5.11×10^{-2}
0.2	1.80	-0.228
0.1	2.13	-0.434

The data shows that the higher-order phase shifts can both be forwards, or one phase shift can be forwards with the other backwards. Moreover, the higher-order phase shifts show no simple functional dependence on the solitary wave amplitude A_1. Table I can be used to determine the phase shifts of Eq. (1) with arbitrary coefficients, by using the expression (17).

IV. CONCLUSION

An asymptotic transformation has been developed for a higher-order mKdV equation. It is found that the higher-order solitary waves are asymptotic solitons when an algebraic relationship (6) involving the higher-order coefficients is satisfied. The higher-order phase shifts, after interaction of the solitary waves, were also found using the asymptotic theory.

Numerical solutions of the higher-order mKdV equation were used to confirm the asymptotic results, namely, the elastic nature of the collision and the value of the phase shift corrections. Moreover, numerical solutions, for an example not satisfying the relation (6) show evidence of an inelastic collision, via the generation of a dispersive wave train. The higher-order phase shifts, for this inelastic example, were also tabulated for a range of wave amplitudes. By using these tabulated values and the asymptotic transformation the higher-order phase shifts can be determined for any choice of the higher-order coefficients.

Only higher-order versions of the mKdV equation with positive sign have been considered in this paper. The methods used here could be easily extended to higher-order versions of the mKdV equation with negative sign. In particular the interaction of a higher-order dissipationless shock wave and solitary wave could be described by applying an asymptotic transformation to the solution of Perelman *et al.* [3].

In summary, the application of asymptotic theory may be useful for analysing other higher-order model equations, which represent small perturbations to an integrable equation. The asymptotic theory allows the straightforward determination of parameter choices, for which the higher-order equation is asymptotically integrable, and of the higher-order phase shifts.

APPENDIX A: THE NUMERICAL SCHEME

The numerical solutions of Eq. (19) are obtained by using an implicit, two level, finite-difference scheme with second-order accuracy. The perturbation at the time t_i is

$$p_{i,j}=p(t=i\Delta t,x=j\Delta x), \quad j=1,\ldots,N, \tag{A1}$$

and the two-soliton solution of the mKdV equation is

$$v_{i,j}=v[t=(i+0.5)\Delta t,x=j\Delta x], \tag{A2}$$

which is evaluated using Eq. (11). Δt and Δx are the time and space steps used in the discretisation. The two-soliton solution is calculated at the half time steps to maintain second-order accuracy (see below). The discretised version of Eq. (19) is

$$\begin{aligned}
&\frac{\Delta t}{4\Delta x^3}(p_{i+1,j+2}-2p_{i+1,j+1}+2p_{i+1,j-1}-p_{i+1,j-2})\\
&\quad+\frac{6\Delta t}{\Delta x}(v_{i,j+1}^2p_{i+1,j+1}-v_{i,j-1}^2p_{i+1,j-1})+p_{i+1,j}\\
&=p_{i-1,j}-\frac{\Delta t}{4\Delta x^3}(p_{i-1,j+2}-2p_{i-1,j+1}+2p_{i-1,j-1}\\
&\quad-p_{i-1,j-2})-\frac{6\Delta t}{\Delta x}(v_{i,j+1}^2p_{i-1,j+1}-v_{i,j-1}^2p_{i-1,j-1})\\
&\quad+\Delta t f_{i,j}, \quad j=2,\cdots,N-2,
\end{aligned} \tag{A3}$$

where

$$\begin{aligned}
f_{i,j}=&\frac{c_1}{2\Delta x}v_{i,j}^4(v_{i,j+1}-v_{i,j-1})+\frac{c_2}{8\Delta x^3}(v_{i,j+1}-v_{i,j-1})^3\\
&+\frac{c_3}{2\Delta x^3}v_{i,j}^2(v_{i,j+2}-2v_{i,j+1}+2v_{i,j-1}-v_{i,j-2})\\
&+\frac{c_4}{2\Delta x^3}v_{i,j}(v_{i,j+1}-v_{i,j-1})(v_{i,j+1}-2v_{i,j}+v_{i,j-1})\\
&+\frac{c_5}{8\Delta x^5}(v_{i,j+4}-2v_{i,j+3}-2v_{i,j+2}+6v_{i,j+1}-6v_{i,j-1}\\
&+2v_{i,j-2}+2v_{i,j-3}-v_{i,j+4}).
\end{aligned}$$

The initial and boundary conditions used are

$$p_{i,j}=0, \quad j=1,2,N-1,N, \quad p_{0,j}=0, \tag{A4}$$

which means the perturbation is zero initially and far ahead and behind the location of the solitary waves.

The finite-difference scheme is constructed by using centred differences about the point $(t,x)=[(i+0.5)\Delta t,j\Delta x]$. The accuracy of the numerical method is second-order, $O(\Delta t^2,\Delta x^2)$. Equation (A3) requires the solution of a pentadiagonal matrix at each time step . A fast algorithm for this task is detailed in Conte and deBoor [11].

APPENDIX B: AN ALTERNATIVE HIGHER-ORDER mKdV EQUATION

An alternative higher-order mKdV equation is

$$\begin{aligned}
&\eta_t+24\eta^2\eta_x+\eta_{3x}+\alpha c_1\eta^3\eta_x+\alpha c_2\eta_x\eta_{xx}+\alpha c_3\eta\eta_{3x}\\
&\quad=0, \quad \alpha\ll 1,
\end{aligned} \tag{B1}$$

in which the higher-order nonlinearity, $\eta^3\eta_x$, is quartic, compared to the quintic nonlinearity of Eq. (1). One application of Eq. (B1) is the modelling of internal waves. The mKdV equation arises as a model describing internal waves of small amplitude and long wavelength when the density stratification is such that the leading-order nonlinearity, $\eta\eta_x$, vanishes and a balance occurs between the leading-order dispersion and the higher-order cubic nonlinearity. The higher-order mKdV equation (B1) is obtained by retaining the terms which occur at next order in the expansion; hence it allows the description of steeper and shorter internal waves than does the mKdV equation. Also, a higher-order mKdV equation is used to describe traffic congestion, see Komatsu and Sasa [12].

An asymptotic transformation can be found for Eq. (B1) also. It is

$$\eta=u+\alpha\frac{1}{6}(c_3-c_2)u^2, \quad \tau=t,$$

$$\xi=x-\alpha\frac{1}{3}c_3\int_{-\infty}^{x}u(p,t)dp, \quad \alpha\ll 1, \tag{B2}$$

where $u(x,t)\to 0$ as $x\to\pm\infty$. If Eq. (B2) is substituted into Eq. (B1), and terms of $O(\alpha^2)$ are neglected, then $u(\xi,\tau)$ is a solution of the mKdV equation if the relation between the coefficients of the higher-order terms satisfy

$$c_1+\tfrac{8}{3}c_3-8c_2=0. \tag{B3}$$

This result is a generalization of Alexeyev [13]. He showed that Eq. (B1) with $c_1=0$ and $c_3=3c_2$ [which satisfies Eq. (B3)] was approximately integrable by using a perturbation method based on inverse scattering.

Using the asymptotic theory the phase shifts can be written in the form

$$\frac{1}{A_2}\ln\left(\frac{A_2+A_1}{A_2-A_1}\right)+\alpha c_3\frac{\pi}{6}, \quad -\frac{1}{A_1}\ln\left(\frac{A_2+A_1}{A_2-A_1}\right)-\alpha c_3\frac{\pi}{6}, \tag{B4}$$

for the larger and smaller higher-order solitary waves respectively. In contrast to the higher-order phase shifts for Eq. (1) the higher-order corrections to the phase shifts for Eq. (B1) are constant. They do not depend on the amplitudes of the solitary waves, which is related to the fact that the mass of a mKdV soliton is independent of its amplitude. In the case of higher-order coefficients given by $c_1=8c_2$ and $c_3=0$ the phase shifts are unchanged from the mKdV case.

Hence, when Eq. (B3) is satisfied, the quartic mKdV solitary waves are asymptotic solitons, the only memory of the collision being the phase shifts (B4). It is presumed that when Eq. (B3) is not satisfied, a dispersive wavetrain is produced by the collision, as is the case for Eq. (1) when Eq. (6) is not satisfied.

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