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Difference Spaces and Invariant Linear Forms

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To Sylvia

PREFACE

The ideas in this work have had a long period of gestation, although the fuller development and expression of them has occurred over a relatively short and intense period. The dominant ideas emerged in an endeavour to answer the following question: if f is a function in $L^2(\mathbb{R}^n)$, how can the behaviour of its Fourier transform near the origin of $\mathbb{R}^n$ be described and characterized? The corresponding question for the circle group had been given a satisfactory answer by Gary Meisters and Wolfgang Schmidt in 1972, so the work also can be regarded as arising from an attempt to extend their result from the compact case of the circle group to the non-compact case of $\mathbb{R}^n$. The answers presented to these and related questions have implications for other areas of analysis; the notable ones being the ranges of partial differential operators, and the behaviour of some of the singular integral operators of classical analysis.

I have received a great deal of help and encouragment from many individuals, many of whom are probably not aware of the positive effect they have had upon my carrying out this work. Their input has been too multifarious and diverse for me to record them all by name or to list all of the many ways in which their ideas or suggestions or actions have been of aid to me.

There are, however, some for whom it is incumbent upon me to record their very substantial and specific assistance. One of these was Igor Kluvánek, who sustained an interest in and an encouragment for this work over an extended period – in particular, since late in 1990, when in an extended conversation in a Randwick restaurant, we discussed the general question of the behaviour of the Fourier transform near the origin. As well, to the extent that the present work is considered to emphasise clarity and precision of thought, and is considered to be written in a way which shows respect and consideration for the reader, I would like to think that the writing of it shows his positive influence which extends back over a much longer period. His death in 1994 meant the loss of a close friend who had also been a great intellectual influence upon me. Among my colleagues at Wollongong, I am particularly indebted to Graham Williams and Keith Tognetti. The former has provided me with invaluable opportunities for discussing many of the ideas in this work, which is all the more appreciated because of the time which was freely given, but which often could be ill-afforded. The latter has had an infectious enthusiasm and openness to new ideas, which has also had an effect on my own more sceptical nature, and this has been all to my own good. I am also indebted to Wai Lok Lo, who detected several errors in various earlier versions of the manuscript.

As well, I am aware that my mathematical colleagues at Wollongong have played a very positive, if indirect, rôle in this work, by maintaining an harmonious and intellectually alive environment amongst themselves which has contributed in no small measure to my capacity and opportunity to carry this work through to completion. Also, I have been fortunate in receiving support directly from the Analysis Research

Group at my University, and indirectly from the Graduate Faculty. All of these have sustained me during a period in which changing attitudes towards universities, and within them, have created difficulties of acceptance, understanding and appreciation of those disciplines and researches primarily concerned with the basic understanding of phenomena and abstract concepts.

I have been extremely fortunate with the dedication and attention to detail shown by Carolyn Silveri in what has proved to be a long and demanding task of word-processing. Her many suggestions have greatly added to the presentation of the completed work. I am also indebted to Kerrie Gamble, who originally prepared the draft of Chapter IV.

The work has been completed in the face of many competing tasks, and the tolerance, understanding, encouragement and support of my wife have been immeasurably important. The good humour with which my children have borne both my absences and unavailability for sufficiently many pastimes has also played its part in my completing this work. I am also very grateful to my parents, who have invariably provided me with encouragement in my work over many years.

Notwithstanding the help I have received from so many people, for any errors, omissions or deficiencies in the work, the author should be considered responsible.

Rodney Nillsen,

August, 1994

INTRODUCTION

1. Difference spaces

Let G be a Hausdorff locally compact group and let $M(G)$ denote the space of Radon measures on G. Let $*$ denote the usual convolution of functions, measures and distributions on G. Let $1 \leq p < \infty$ and let X be a subspace of $L^p(G)$ or a space of distribution on G with the property that $M(G) * X \subseteq X$. If $x \in G$ let $\delta_x \in M(G)$ denote the Dirac measure at x. Then if $\emptyset \neq S \subseteq M(G)$, the vector space $\mathcal{D}(X, S)$ is defined to be the subspace of X consisting of all functions or distributions f in X for which there exist $m \in \mathbb{N}, \mu_1, \mu_2, \ldots, \mu_m \in S$ and $f_1, f_2, \ldots, f_m \in X$ such that $f = \sum_{j=1}^{m}(f_j - \mu_j * f_j)$. The space $\mathcal{D}(X, S)$ is called a *difference space* of X. In the case when $S = \{\delta_x : x \in G\}$, the difference space $\mathcal{D}(X, S)$ is denoted by $\mathcal{D}(X)$.

A linear form L in the algebraic dual X' of X is said to be S*-invariant* if $L(\mu * f) = L(f)$ for all $f \in X$ and $\mu \in S$. Equivalently, L is S-invariant if and only if L vanishes identically on the subspace $\mathcal{D}(X, S)$ of X. If $x \in G$ and $f \in X$, $\delta_x * f$ is called the (left) *translation* of f by x. Note that if $f \in L^p(G)$, $(\delta_x * f)(t) = f(x^{-1}t)$, for almost all $t \in G$. A linear form L on X is said to be *translation invariant* if $L(\delta_x * f) = L(f)$ for all $x \in G$ and $f \in X$. Thus, L is translation invariant if and only if L vanishes identically on $\mathcal{D}(X)$. In fact, it is easy to prove, using a Hamel basis argument, that

$$\mathcal{D}(X, S) = \bigcap \Big\{ \text{kernel of } L : L \in X' \text{ and } L \text{ is } S\text{-invariant} \Big\}. \tag{1.1}$$

The preceding remarks make it clear that there is a close relationship between invariant linear forms on X and the corresponding difference space of X, and this is a main reason for interest in the difference spaces. Historically, interest seems to have centred more on the invariant forms associated with difference spaces, rather than with the difference spaces themselves. This is perhaps because of the importance of invariant integration in many different areas – for example, the Haar measure on G is the measure arising from a positive linear form which vanishes identically on a difference space of continuous functions which have compact support.

In particular, there is a large body of work on invariant linear forms on $L^\infty(G)$. If μ is a positive, continuous element of $L^\infty(G)'$ and $||\mu|| = 1$, then μ is called a *mean* on $L^\infty(G)$, and G is said to be *amenable* if $L^\infty(G)$ has a translation invariant mean. The theory of amenability on groups and other structures is now very extensive, and is the subject of recent books by Paterson [49] and Pier [50].

The emphasis in this work is more upon invariant linear forms on $L^p(G)$ for $1 \leq p < \infty$, and upon invariant forms on spaces $\mathcal{F}_p(G)$ of abstract distributions on G which are characterized by the property that their Fourier transforms are in $L^p(\widehat{G})$, where $\widehat{G}$ is the dual of G when G is abelian. Even more, the emphasis is upon the corresponding difference spaces and upon closely related spaces known as generalized difference spaces. In certain cases where G is abelian, the main results characterize, for example, a difference space $\mathcal{D}(L^2(G), S)$ as being isomorphic under the Fourier transform to a space $L^2(\widehat{G}, \mu)$, where μ is a non-negative measure on $\widehat{G}$. Such characterizations of difference spaces suggest that these spaces are of interest quite apart from their connection with invariant linear forms. A further reason for interest in these spaces, and the generalized difference spaces, is that they also serve to characterize the ranges of certain types of differential operators, including such familiar ones as the Laplace and wave operators. Also, they shed light on the behaviour of some of the singular integral operators of classical analysis such as the Riesz potential operators.

2. Differentiation, differences and the behaviour of the Fourier transform near the origin

In numerical analysis, finite differences have long been used to give local approximations to derivatives. For example, if $f : \mathbb{R} \longmapsto \mathbb{R}$ is differentiable and if its derivative $D(f) \in L^2(\mathbb{R})$, consider $x, y \in \mathbb{R}$ with y near the origin, and let $g \in L^2(\mathbb{R})$ be the function given by $g(u) = -f(u)/y$. Then $D(f)(x)$ may be approximated by

$$\frac{f(x+y) - f(x)}{y} = \frac{(\delta_{-y} * f - f)(x)}{y} = (g - \delta_{-y} * g)(x).$$

This suggests that, even globally, there may be a relationship between $D(f)$ and differences of the form $h - \delta_{-y} * h$, for $h \in L^2(\mathbb{R})$ and $y \in \mathbb{R}$.

A major motivation for the n-dimensional results presented here was to develop a theory of difference spaces, or a theory of spaces related to such spaces, which would suffice to characterize the ranges of a reasonably large class of partial differential operators. Each of these operators was envisaged as being defined on a suitable Sobolev-type subspace of $L^2(\mathbb{R}^n)$ (say), and the range of each one was envisaged as being a difference space or some closely related space. The connection between these spaces and differential operators comes via the Fourier transform and, because of the centrality of this idea to this work, it is now described in more detail. The Fourier transform of f is denoted by $\widehat{f}$ or $f^\wedge$.

If $f \in L^2(\mathbb{R})$ let $D(f)$ denote the derivative of f in the sense of Schwartz distributions. Consider $f \in L^2(\mathbb{R})$ such that $D(f) \in L^2(\mathbb{R})$. Then $\widehat{f} \in L^2(\mathbb{R})$ and $D(f)^\wedge(x) = ix\widehat{f}(x)$, for almost all $x \in \mathbb{R}$. Plancherel's theorem then shows that $D(f)^\wedge$ is small near the origin in the sense that

$$\int_{-\infty}^{\infty} \frac{|D(f)^\wedge(x)|^2}{|x|^2} dx < \infty.$$

In fact, it is easy to see that

$$D(L^2(\mathbb{R})) \cap L^2(\mathbb{R}) = \left\{ g : g \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{|\widehat{g}(x)|^2}{|x|^2} dx < \infty \right\}. \tag{2.1}$$

On the other hand, if $f \in L^2(I\!\!R)$ and $y \in I\!\!R$, $(f - \delta_y * f)^\wedge(x) = (1 - e^{-ixy})\widehat{f}(x)$, for almost all $x \in I\!\!R$, so that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{|(f-\delta_y * f)^\wedge(x)|^2}{|x|^2} dx &= \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 \left|\frac{1-e^{-ixy}}{x}\right|^2 dx, \\ &= y^2 \int_{-\infty}^{\infty} \left(\frac{\sin \frac{xy}{2}}{\frac{xy}{2}}\right)^2 |\widehat{f}(x)|^2 dx, \\ &\leq y^2 \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 dx, \text{ as } |\sin x| \leq |x|, \\ &< \infty.\end{aligned}$$

It follows that for all $h \in \mathcal{D}(L^2(I\!\!R))$,

$$\int_{-\infty}^{\infty} \frac{|\widehat{h}(x)|^2}{|x|^2} dx < \infty,$$

an observation originally made in 1973 by Meisters, who deduced from it that $\mathcal{D}(L^2(I\!\!R))$ is a proper, dense subspace of $L^2(I\!\!R)$ and that there are discontinuous linear forms on $L^2(I\!\!R)$ which are translation invariant. More recently, the author has proved that, in fact,

$$\mathcal{D}(L^2(I\!\!R)) = \left\{h : h \in L^2(I\!\!R) \text{ and } \int_{-\infty}^{\infty} \frac{|\widehat{h}(x)|^2}{|x|^2} dx < \infty\right\}. \tag{2.2}$$

This shows that the functions in $L^2(I\!\!R)$ which are in $\mathcal{D}(L^2(I\!\!R))$ are completely characterized by the behaviour of their Fourier transforms near the origin. Equation (2.2) also shows that $\mathcal{D}(L^2(I\!\!R))$ is a Hilbert space in the norm $|||\cdot|||$ given by $|||f||| = \left(\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 (1 + |x|^{-2}) dx\right)^{1/2}$.

A comparison of (2.1) and (2.2) shows that $\mathcal{D}(L^2(I\!\!R)) = D(L^2(I\!\!R)) \cap L^2(I\!\!R)$, a result which describes $\mathcal{D}(L^2(I\!\!R))$ as the range of the differentiation operator D on the Sobolev space consisting of those functions in $L^2(I\!\!R)$ whose derivatives are in $L^2(I\!\!R)$. This Sobolev space is a Hilbert space in the norm $||\cdot||$ which is given by $||f|| = \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 (1 + |x|^2) dx)^{1/2}$, and the differentiation operator is an isometry from this space onto the space $\mathcal{D}(L^2(I\!\!R))$ in the norm $|||\cdot|||$. The theory of generalized multiplication spaces and generalized difference spaces presented in Chapter II was motivated by an attempt to extend this description of the range of the differentiation operator, and its description as an isometry, to more general differential operators. Such results may be regarded as putting the idea of approximating derivatives by finite differences within the framework of functional analysis and operator theory.

The preceding discussion also establishes a connection between the range of the differentiation operator D and invariant linear forms. For, since the range of

D, when D is restricted to the appropriate Sobolev subspace of $L^2(\mathbb{R})$, is equal to $\mathcal{D}(L^2(\mathbb{R}))$, it follows from (1.1) that

$$D(L^2(\mathbb{R})) \cap L^2(\mathbb{R}) = \bigcap \Big\{ \text{kernel of } L : L \in L^2(\mathbb{R})' \text{ and } L \text{ is translation invariant} \Big\}.$$

A similar description is also valid for the ranges of differential operators belonging to a class of operators which includes the Laplace and wave operators. To the extent that one regards amenability as the study of invariant linear forms and associated concepts, such results may be considered to establish a connection between amenability and the theory of partial differential operators.

3. Multiplication spaces

Let G be a Hausdorff locally compact abelian group with dual $\widehat{G}$. Let F be a family of complex valued Borel measurable functions on G. Then if $1 \leq p \leq \infty$, the vector space $\mathcal{M}(L^p(G), F)$ is defined to consist of those functions f defined almost everywhere on G for which there exist $m \in \mathbb{N}$, $f_1, f_2, \ldots, f_m \in F$ and $g_1, g_2, \ldots, g_m \in L^p(G)$ such that $f = \sum_{j=1}^m f_j g_j$. The space $\mathcal{M}(L^p(G), F)$ is called a *multiplication space* of $L^p(G)$. The Fourier transform changes convolutions into multiplications, so it follows that if $S \subseteq M(G)$, if $F = \{1 - \widehat{\mu} : \mu \in S\}$ and if $1 \leq p < \infty$, then the Fourier transform is a bijection from $\mathcal{D}(\mathcal{F}_p(G), S)$ onto $\mathcal{M}(L^p(\widehat{G}), F)$.

Corresponding to the generalized difference spaces are the generalized multiplication spaces. The basic problem is to characterize a multiplication space, or a generalized multiplication space of $L^p(G)$, as a space $L^p(G, \nu)$ for some measure ν on G.

Characterizing a multiplication space of $L^p(G)$ in this way is usually easier than the corresponding problem for difference spaces of $L^p(G)$. This is because characterizing a space $\mathcal{M}(L^p(G), F)$ is usually no harder than characterizing $\mathcal{M}(L^2(G), F)$, but the Fourier transform is not generally a bijection from $L^p(G)$ onto a space $L^q(\widehat{G})$, a fact familiar from considerations arising from the Hausdorff-Young theorem (see [29, p.146], for example). This difficulty is the motivation for introducing the spaces $\mathcal{F}_p(G)$ of abstract distributions on G, and for considering difference spaces of $\mathcal{F}_p(G)$.

4. Arrangement of the work

Chapter I is largely concerned with multiplication spaces and sets, with difference spaces and sets, and with the characterization of these in terms of the finiteness of certain integrals. The relationship between difference spaces and invariant linear forms is discussed, the emphasis being on how the two types of object are related to each other. The relationship between the multiplication and difference spaces is given by the Fourier transform, and this raises the problem of constructing measures having a certain type of prescribed Fourier transform. This is discussed for subsequent application.

In Chapter II are presented the basic characterizations of generalized multiplication and difference spaces on $\mathbb{R}^n$. This leads to deeper and more useful characterizations of these sets than was possible in the more general setting in Chapter I. Some sharpness aspects of these characterizations are presented, duality aspects are discussed, and the chapter concludes with discussions of other representations of some of the difference spaces, and with establishing a connection between these spaces and the theory of wavelets.

Chapter III is concerned with applications to describing the ranges of various types of partial differential operators in terms of the corresponding difference spaces. In this formulation the operators are isometries from an appropriate Sobolev-type space onto a corresponding generalized difference space. Applications to various singular integral operators are discussed, including the Hilbert and Riesz transforms and the Riesz potential operators.

Difference spaces of $L^p(G)$, where G is a locally compact group (not necessarily abelian) and $1 \leq p \leq \infty$, are discussed in Chapter IV. Owing to the difficulty associated with the Fourier transform in this more general setting, the results concentrate on comparison of difference spaces with each other, rather than with explicit characterizations as were possible for $\mathbb{R}^n$. These results may be interpreted equally as results about various types of invariant linear forms on $L^p(G)$.

Detailed references and comments are kept for the notes section at the end of each chapter, except where they are considered essential for elucidating the main text.

The main aim of the work has been to present new results on difference spaces, multiplication spaces, invariant linear forms and associated applications. However, every effort has been made to make the whole work, and each chapter, as self contained as seemed reasonable. It is hoped that this will enhance readability of the work. A reader familiar with integration theory, Fourier theory and the theory of distributions on $\mathbb{R}^n$, should find Chapters II and III quite accessible. A reader familiar with basic harmonic analysis on locally compact groups should find Chapters I and IV readily accessible. It is hoped that the work as a whole will be of interest to graduate students and researchers in the areas of Fourier analysis, abstract harmonic analysis, partial differential equations and singular integral operators.

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