Sharp results concerning the expression of functions as sums of finite differences

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Abstract

Let $s \in \mathbb{N}$, let $*$ denote convolution and let δ_x denote the Dirac measure at a point x. A function in $L^2(\mathbb{R})$ is called a *difference of order s* if it is of the form $\sum_{k=0}^{s}(-1)^k\binom{s}{k}\delta_{kx}*g$ for some $x \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$. In fact, the concept of a "difference of order s" may be defined in a similar manner for each $s > 0$. Then, it is known that a function f in $L^2(\mathbb{R})$ is a finite sum of differences of order s if and only if $\int_{\infty}^{\infty} |\widehat{f}(x)|^2|x|^{-2s}dx < \infty$, and the vector space of all such functions is denoted by $\mathcal{D}_s(L^2(\mathbb{R}))$. Every function in $\mathcal{D}_s(L^2(\mathbb{R}))$ is a sum of $\mathrm{int}(2s)+1$ differences of order s, where $\mathrm{int}(t)$ denotes the integer part of a real number t. Thus, every function in $\mathcal{D}_1(L^2(\mathbb{R}))$ is a sum of three first order differences, but it was proved in 1994 that there is a function in $\mathcal{D}_1(L^2(\mathbb{R}))$ which is never the sum of two first order differences. This complemented, for the group $\mathbb{R}$, the corresponding result for first order differences obtained by Meisters and Schmidt in 1972 for the circle group $\mathbb{T}$. In this paper, it is shown that there is a vector subspace of $L^2(\mathbb{R})$ which has dimension equal to that of the continuum such that, for each $s \geq 1/2$, every non-zero function in this subspace is a sum of $\mathrm{int}(2s)+1$ differences of order s but is never the sum of $\mathrm{int}(2s)$ differences of order s. The proof depends upon extending to higher dimensions the following result set in two dimensions and obtained by Schmidt in 1972 in connection with Heilbronn's problem: if $x_1, \ldots, x_n$ are points in the unit square, $\sum_{1\leq i<j\leq n} |x_i - x_j|^{-2} \geq 200^{-1}n^2 \ln n$. Following on from the work of Meisters and Schmidt, the results presented here further develop a connection between certain estimates in combinatorial geometry and some questions of sharpness in harmonic analysis.

1 Introduction

Let G be a locally compact abelian group, and let $L^2(G)$ denote the Hilbert space of all complex-valued, square integrable functions on G with respect to the Haar measure on G.

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Let the Dirac measure at a point $x \in G$ be denoted by δ_x, and let $*$ denote the operation of convolution where such an operation is defined. Thus, if $x \in G$ and $f \in L^2(G)$, then $\delta_x * f$ is given in $L^2(G)$ by

$$(\delta_x * f)(t) = f(t - x),$$

for almost all $t \in G$. A function on G which is of the form $f - \delta_x * f$, for some $x \in G$ and $f \in L^2(G)$, is called a *first order difference* of $L^2(G)$. Then $\mathcal{D}_1(L^2(G))$ denotes the vector subspace of $L^2(G)$ consisting of all finite sums of first order differences. In other words, a function g in $L^2(G)$ belongs to $\mathcal{D}_1(L^2(G))$ if and only if there exist an $n \in \mathbb{N}$, points $x_1, x_2, \ldots, x_n \in G$ and functions $f_1, f_2, \ldots, f_n \in L^2(G)$ such that

$$g \;=\; \sum_{j=1}^{n} (f_j - \delta_{x_j} * f_j). \tag{1.1}$$

Now, let the dual group of G be denoted by $\widehat{G}$. Letting $\mathbb{R}$ denote the additive group of real numbers, we have $\widehat{\mathbb{R}} = \mathbb{R}$. Also, letting $\mathbb{T}$ denote the circle group consisting of all complex numbers with modulus one, we have $\widehat{\mathbb{T}} = \mathbb{Z}$, the group of all integers. If $f \in L^2(G)$, then $\widehat{f}$ denotes the Fourier transform of f, which is a function in $L^2(\widehat{G})$ [11, Theorem 1.6.1].

In 1972, G. Meisters and W. Schmidt [6] characterised the space $\mathcal{D}_1(L^2(\mathbb{T}))$ as the subspace of $L^2(\mathbb{T})$ whose functions have Fourier transforms which have a certain behaviour at the identity 0 in the dual group $\mathbb{Z}$ of $\mathbb{T}$. More generally and specifically, they proved that if G is a connected, compact abelian group and $\widehat{e}$ denotes the identity of $\widehat{G}$, then

$$\mathcal{D}_1(L^2(G)) \;=\; \Big\{ f : f \in L^2(G) \quad \text{and} \quad \widehat{f}(\widehat{e}) = 0 \Big\}. \tag{1.2}$$

The proof of (1.2) given in [6] revealed some fine detail concerning the representation of functions in the form (1.1). In particular, it was shown that for a compact connected abelian group G the number n in (1.1) may always be taken to be three. That is,

$$\mathcal{D}_1(L^2(G)) \;=\; \left\{ \sum_{j=1}^{3} (f_j - \delta_{x_j} * f_j) : x_1, x_2, x_3 \in G \;\text{ and }\; f_1, f_2, f_3 \in L^2(G) \right\}. \tag{1.3}$$

However, for the circle group $\mathbb{T}$ the number three in (1.3) is sharp in the sense that it is shown in [6] that there is a function $f \in \mathcal{D}_1(L^2(\mathbb{T}))$ such that for all $x_1, x_2 \in \mathbb{T}$ and all $f_1, f_2 \in L^2(\mathbb{T})$

$$f \;\neq\; \sum_{j=1}^{2} (f_j - \delta_{x_j} * f_j). \tag{1.4}$$

A linear form J on $L^2(G)$ is called *translation invariant* if $J(\delta_x * f) = J(f)$ for all $x \in G$ and $f \in L^2(G)$. Equation (1.2) shows that under the stated conditions, $\mathcal{D}_1(L^2(G))$ has codimension one in $L^2(G)$. It then follows, as proved by Meisters and Schmidt, that if G is connected, compact and abelian then every translation invariant linear form J on $L^2(G)$ is a constant multiple of the Haar measure G, and so must be continuous. Therefore, (1.2) and (1.3) may be regarded as refinements of the statement that each translation invariant linear form on $L^2(G)$ is continuous. Further results in the compact group case also were obtained by Meisters [5]. Subsequently, B. Johnson [3] characterised those compact abelian groups G for which every translation invariant linear form on $L^2(G)$ is continuous. In [2], J. Bourgain showed that every translation invariant linear form on $L^p(\mathbb{T})$, $1 < p < \infty$, is continuous, and the generalisation of this to certain compact abelian groups was studied by Lo [4].

Development of the preceding ideas for non-compact abelian groups has been carried out in [7, 8, 9, 10]. However, the following remarks are restricted to the group $\mathbb{R}$. For each $s \in (0, \infty)$, a vector space $D_s(L^2(\mathbb{R}))$ may be defined to consist of all finite sums of functions which are "*differences of order* s". For $s = 1$, a *difference of order* 1 is the same as a first order difference defined earlier. In the case $s = 2$, a *difference of order* 2, or a *second order difference*, is defined to be a function of the form $f - 2^{-1}(\delta_x + \delta_{-x}) * f$ for some $x \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$. Thus, a function f in $L^2(\mathbb{R})$ belongs to $\mathcal{D}_2(L^2(\mathbb{R}))$ if and only if there are $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $f_1, f_2, \ldots, f_n \in L^2(\mathbb{R})$ such that

$$f \;=\; \sum_{j=1}^{n}(f_j - 2^{-1}(\delta_{x_j} + \delta_{x_j^{-1}}) * f_j). \tag{1.5}$$

For the spaces $\mathcal{D}_s(L^2(\mathbb{R}))$, $s > 0$, the result corresponding to (1.2) is the following:

$$\mathcal{D}_s(L^2(\mathbb{R})) \;=\; \left\{ f : f \in (\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{|x|^{2s}}\, dx < \infty \right\}. \tag{1.6}$$

It has been shown in [7, Theorem 2] and in [8, pp.80-81] (but see [9, Theorem 3] for a more accessible proof) that for each $s \in (0, \infty)$, every function f in $\mathcal{D}_s(L^2(\mathbb{R}))$ is the sum of $\operatorname{int}(2s) + 1$ "differences of order s" where $\operatorname{int}(2s)$ stands for the integer part of $2s$. In particular, each function in $\mathcal{D}_1(L^2(\mathbb{R}))$ may be written as a sum of three first-order differences, and each function in $\mathcal{D}_2(L^2(\mathbb{R}))$ may be written as a sum of five second-order differences. These statements for the spaces $\mathcal{D}_1(L^2(\mathbb{R}))$ and $\mathcal{D}_2(L^2(\mathbb{R}))$ in the non-compact case of the group $\mathbb{R}$ correspond to the result in (1.3) which applies for the space $\mathcal{D}_1(L^2(\mathbb{T}))$ in the case of the compact group $\mathbb{T}$.

The question now arises as to whether or not the number $\operatorname{int}(2s)+1$ is sharp in estimating the number of "differences of order s"needed to represent a general function in $\mathcal{D}_s(L^2(\mathbb{R}))$. In the case $s = 1$, it was proved in [9, Proposition II.6.11] that 3 is a sharp estimate for the space $\mathcal{D}_1(L^2(\mathbb{R}))$ in the sense that there is a function f in $\mathcal{D}_1(L^2(\mathbb{R}))$ such that for all $x_1, x_2 \in \mathbb{R}$ and all $f_1, f_2 \in L^2(\mathbb{R})$

$$f \;\neq\; \sum_{j=1}^{2}(f_j - \delta_{x_j} + f_j). \tag{1.7}$$

So,(1.7) expresses the fact that the number 3 is sharp for $\mathcal{D}_1(L^2(\mathbb{R}))$, whereas (1.4) expresses the fact that the number 3 is sharp for $\mathcal{D}_1(L^2(\mathbb{T}))$. The main aim of this paper is to show that the number $\operatorname{int}(2s) + 1$ is a sharp estimate in each space $\mathcal{D}_s(L^2(\mathbb{R}))$, $s \geq \frac{1}{2}$, in the following sense: every function in $\mathcal{D}_s(L^2(\mathbb{R}))$ is a sum of $\operatorname{int}(2s) + 1$ "differences of order s", but there is a function f in $\mathcal{D}_s(L^2(\mathbb{R}))$ such that, for each $s \geq \frac{1}{2}$, f is never the sum of $\operatorname{int}(2s)$ "differences of order s". In particular, there is $f \in \mathcal{D}_2(L^2(\mathbb{R}))$ such that for all

$x_1, x_2, x_3, x_4 \in \mathbb{R}$ and $f_1, f_2, f_3, f_4 \in L^2(\mathbb{R})$

$$f \neq \sum_{j=1}^{4} \Big(f_j - 2^{-1}(\delta_{x_j} + \delta_{-x_j})\Big) * f_j. \tag{1.8}$$

The proof of (1.4), to which (1.7) is a non-compact counterpart, was based by Meisters and Schmidt upon an earlier result of Schmidt [12] concerning the distribution of points in the unit square. The techniques used here for proving sharpness results such as (1.8) in the spaces $\mathcal{D}_s(L^2(\mathbb{R}))$, $s \geq \frac{1}{2}$, depend upon extending this earlier result of Schmidt to r dimensions. Letting $|\cdot|$ denote the usual Euclidean norm in $\mathbb{R}^r$, we prove the following result. When $r \in \mathbb{N}$, there is a constant $C_r > 0$ such that for all $n \in \mathbb{N}$ and all distinct points $x_1, x_2, \ldots, x_n \in [0,1]^r$

$$\sum_{1 \leq i < j \leq n} \frac{1}{|x_i - x_j|^r} \quad \geq \quad C_r n^2 \log_2 n.$$

A connection is then made between this estimate and the divergence of certain integrals which determine whether a function can be expressed as the sum of a finite number of "differences of order s".

In fact, the preceding ideas are carried out in a more general way which constructs a vector space for which each non-zero function in the space is simultaneously in all of the spaces $\mathcal{D}_s(L^2(\mathbb{R})), s \geq 1/2$, but no such function can be expressed as a finite sum of $\text{int}(2s)$ "differences of order s" for any $s \geq 1/2$. The ideas are then extended to spaces of distributions characterized by the requirement that their Fourier transforms are in some given space $L^p(\mathbb{R}), 1 \leq p < \infty$. In [13], T. Tao has described some applications of combinatorics in analysis, especially to the boundedness of Fourier integral operators. The methods of this paper further emphasise the rôle of combinatorial arguments in harmonic analysis.

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