EXAMPLE

Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given by

\[
f(x, y) = ((x - 1)^2 - y + 1, (x - 1)^2).
\]

Let \( S \) be the subset of the domain of \( f \) as indicated in the picture.

The problem is: calculate and describe the subset \( f(S) \) of the codomain of \( f \).

Possible approach.
(1) Calculate \( A' = f(A), B' = f(B) \) and \( C' = f(C) \).

(2) On the line segment \( BC \), \( y = 0 \) and \( 0 \leq x \leq 1 \). Calculate \( f(x, 0) \) and describe the curve \( B'C' \).

(3) On the line segment \( AB \), \( x = 0 \) and \( 0 \leq y \leq 1 \). Calculate \( f(0, y) \) and describe the curve \( A'C' \).

(4) On the curve \( AC \), \( 0 \leq x \leq 1 \) and \( y = (x - 1)^2 \). Calculate \( f(x, y) \) for points \((x, y)\) on the curve and describe the curve \( A'C' \).

(5) Describe the set \( f(S) \).
We have

\[ A' = f(A) = f(0, 1) = (1, 1), \]
\[ B' = f(B) = f(0, 0) = (2, 1), \]
\[ C' = f(C) = f(1, 0) = (1, 0). \]

Now consider a point \((x, 0)\) on \(BC\). Then,

\[ f(x, 0) = ((x - 1)^2 + 1, (x - 1)^2) = (a + 1, a), \text{ where } a = (x - 1)^2. \]

Thus, if \((u, v)\) denotes a general point in the codomain \(\mathbb{R}^2\) of \(f\), we see that \(f(x, 0)\) lies on the line \(u = v + 1\) – that is the line \(v = u - 1\).

Now consider a point \((0, y)\) on \(AB\). Then,

\[ f(0, y) = (2 - y, 1). \]

Thus, we see that \(f(0, y)\) lies on the line \(v = 1\).

Finally, consider a point \((x, y)\) on \(AC\). We have \(y = (x - 1)^2\) on \(AC\). Then,

\[ f(x, y) = f(x, (x - 1)^2) = (1, (x - 1)^2). \]

Thus we see that \(f(x, y)\) in this case lies on the line \(u = 1\).

Thus, \(f\) maps \(S\) to \(S'\), where \(S'\) is the subset of \(\mathbb{R}^2\) bounded by the lines given by \(v = 1\), \(v = u - 1\) and \(u = 1\), as indicated in the picture.

A further question concerning \(f\) is whether it has an inverse function. We have

\[ f(x, y) = (u, v) \iff ((x - 1)^2 - y + 1, (x - 1)^2) = (u, v) \iff u = (x - 1)^2 - y + 1 \text{ and } v = (x - 1)^2. \]
\[ u = v - y + 1 \text{ and } x - 1 = \sqrt{v} \text{ or } x - 1 = -\sqrt{v} \]
\[ y = v - u + 1 \text{ and } x = 1 + \sqrt{v} \text{ or } x = 1 - \sqrt{v} \]
\[ (x, y) = (1 + \sqrt{v}, v - u + 1) \text{ or } (x, y) = (1 - \sqrt{v}, v - u + 1). \]

We see that for a given \((u, v)\), the equation \(f(x, y) = (u, v)\) has in general two different solutions for \((x, y)\). So, \(f\) is not one-to-one and \(f\) does not have an inverse function. Note that the above argument shows that the range of \(f\) is

\[ \{(u, v) : v \geq 0\} = \mathbb{R} \times \mathbb{R}_+. \]

Rod Nillsen
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