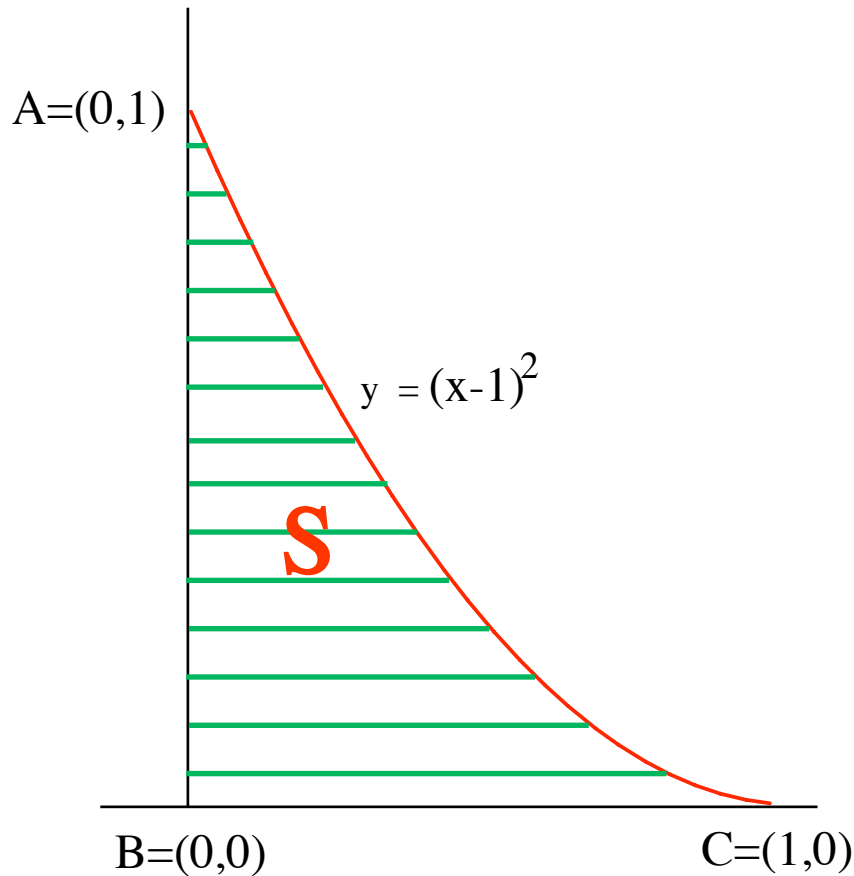


EXAMPLE

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by

$$f(x, y) = ((x - 1)^2 - y + 1, (x - 1)^2).$$

Let S be the subset of the domain of f as indicated in the picture.



The problem is: calculate and describe the subset $f(S)$ of the codomain of f .

Possible approach.

(1) Calculate

$$A' = f(A), B' = f(B) \text{ and } C' = f(C).$$

(2) On the line segment BC , $y = 0$ and $0 \leq x \leq 1$. Calculate $f(x, 0)$ and describe the curve $B'C'$.

(3) On the line segment AB , $x = 0$ and $0 \leq y \leq 1$. Calculate $f(0, y)$ and describe the curve $A'C'$.

(4) On the curve AC , $0 \leq x \leq 1$ and $y = (x - 1)^2$. Calculate $f(x, y)$ for points (x, y) on the curve and describe the curve $A'C'$.

(5) Describe the set $f(S)$.

We have

$$A' = f(A) = f(0, 1) = (1, 1),$$

$$B' = f(B) = f(0, 0) = (2, 1),$$

$$C' = f(C) = f(1, 0) = (1, 0).$$

Now consider a point $(x, 0)$ on BC . Then,

$$f(x, 0) = ((x - 1)^2 + 1, (x - 1)^2) = (a + 1, a), \text{ where } a = (x - 1)^2.$$

Thus, if (u, v) denotes a general point in the codomain \mathbb{R}^2 of f , we see that $f(x, 0)$ lies on the line $u = v + 1$ – that is the line $v = u - 1$.

Now consider a point $(0, y)$ on AB . Then,

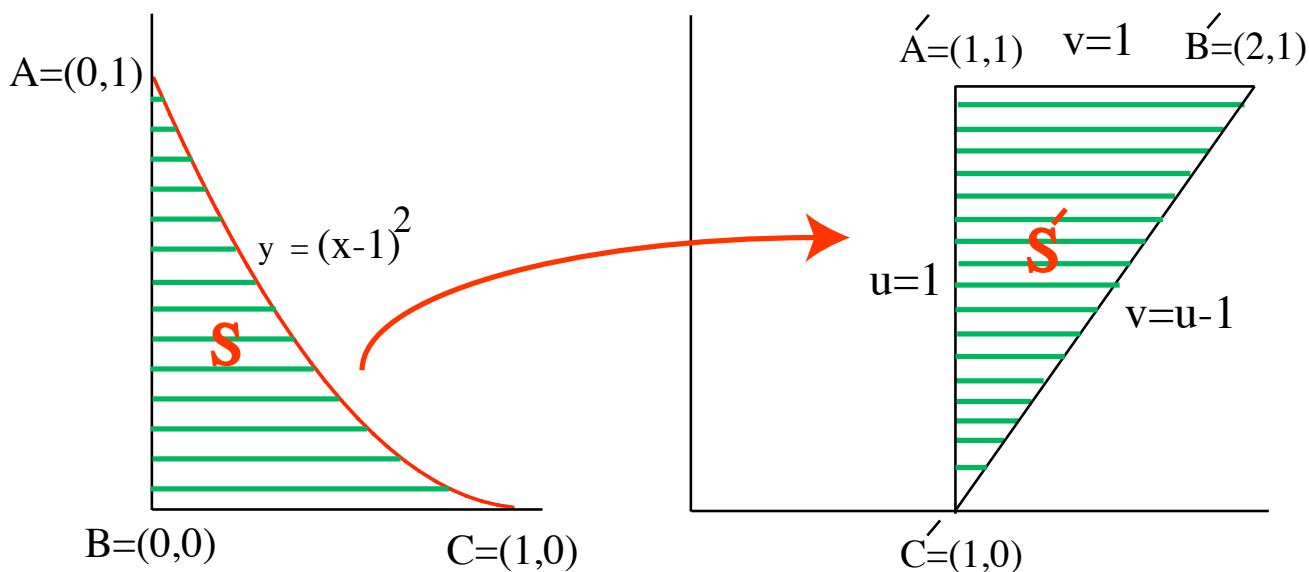
$$f(0, y) = (2 - y, 1).$$

Thus, we see that $f(0, y)$ lies on the line $v = 1$.

Finally, consider a point (x, y) on AC . We have $y = (x - 1)^2$ on AC . Then,

$$f(x, y) = f(x, (x - 1)^2) = (1, (x - 1)^2).$$

Thus we see that $f(x, y)$ in this case lies on the line $u = 1$.



Thus, f maps S to S' , where S' is the subset of \mathbb{R}^2 bounded by the lines given by $v = 1$, $v = u - 1$ and $u = 1$, as indicated in the picture.

A further question concerning f is whether it has an inverse function. We have

$$\begin{aligned} f(x, y) = (u, v) &\iff ((x - 1)^2 - y + 1, (x - 1)^2) = (u, v) \\ &\iff u = (x - 1)^2 - y + 1 \text{ and } v = (x - 1)^2 \end{aligned}$$

$$\begin{aligned}\Leftrightarrow & u = v - y + 1 \text{ and } x - 1 = \sqrt{v} \text{ or } x - 1 = -\sqrt{v} \\ \Leftrightarrow & y = v - u + 1 \text{ and } x = 1 + \sqrt{v} \text{ or } x = 1 - \sqrt{v} \\ \Leftrightarrow & (x, y) = (1 + \sqrt{v}, v - u + 1) \text{ or } (x, y) = (1 - \sqrt{v}, v - u + 1).\end{aligned}$$

We see that for a given (u, v) , the equation $f(x, y) = (u, v)$ has in general *two different* solutions for (x, y) . So, f is not one-to-one and f does not have an inverse function. Note that the above argument shows that the range of f is

$$\{(u, v) : v \geq 0\} = \mathbb{R} \times \mathbb{R}_+.$$

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