Can quadratic functions tell us whether the Australian Government is trying to save taxpayers' money?

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Aim is to present some ideas of possible relevance for getting students interested in mathematics and how it can be used

- an application of high school mathematics to an issue of government policy
- the idea came from reading a newspaper article
- tries to illustrate mathematics as a way of **thinking** about a problem, as opposed to calculation.
- gives an idea of how mathematics arises when conflicting tendencies are balanced against each other.
- give an idea of vocational relevance of mathematicis, and its applications to other areas
- may provide projects for students interested in pursuing maths beyond the formal syllabus.

Ross Gittins argued in the *Sydney Morning Herald* of 23^{rd} March 2005 that the funding arrangements for private schools positively encourage parents to move their children from the public system.

The Federal Minister for Education, Dr Brendan Nelson, in a letter to the *Herald* of $25^{th} - 27^{th}$ March, responded by saying that students in private schools save the tax payer 4 billion per year.

Dr Nelson supplied the following figures:

- 68% of all school students go to state schools
- state school students receive 76% of Government funds allocated to the totality of all students attending schools
- The policy of subsidizing students who went to a private school resulted in taxpayer savings of $4 billion

Let m students go to state schools.

Let n students go to private schools.

Let the government pay an amount a for each state school student.

Let the Government pay an amount s, called a subsidy, for each private school student.

The proportion of students going to state schools is θ, where

$$\theta = \frac{m}{m+n}.$$

The proportion of government funds spent on state schools is ϕ, where

$$\phi = \frac{ma}{ma+ns}.$$

A calculation shows that

$$\frac{s}{a} = \frac{1/\phi - 1}{1/\theta - 1}.$$

According to the Minister, under current policy

$$\theta = .68 \quad \text{and} \quad \phi = .76.$$

So,

$$\frac{s}{a} = \frac{2}{3}$$

or

$$s = \frac{2a}{3}.$$

The mathematical analysis of savings

Note that

$$0 \leq s \leq a.$$

For each private student the government saves

$$a - s.$$

Definition. At subsidy s, let $g(s)$ denote the number of students who enroll in private schools.

The **total amount** saved by the government is $a - s$ for each student times $g(s)$, the corresponding number of students in private schools.

So, the total amount saved at the subsidy s is

$$\boxed{S(s) = g(s)(a - s), \quad \text{where} \quad 0 \leq s \leq a.}$$

Note that

$$S(a) = 0.$$

Savings will be maximised when the function S has a maximum over the interval $[0, a]$.

NOTATION. N_0 is the number of private school students at the subsidy 0.

N_0' is the number of private school students at the maximum subsidy a.

Of course,

$$N_0' \geq N_0.$$

DEFINITION. If $N_0 = 0$ we make the definition that $\rho = \infty$. If $N_0 > 0$, make the definition that

$$\rho = \frac{N_0'}{N_0} > 1.$$

For the time being, ρ is kept as a given constant value.

THE LINEAR MODEL

The simplest choice for g is a linear function. That is g is given by

$$g(s) = N_0 + \left(\frac{N_0' - N_0}{a}\right) s.$$

The graph of g is a straight line,

$$g(0) = N_0 \quad \text{and} \quad g(a) = N_0'.$$

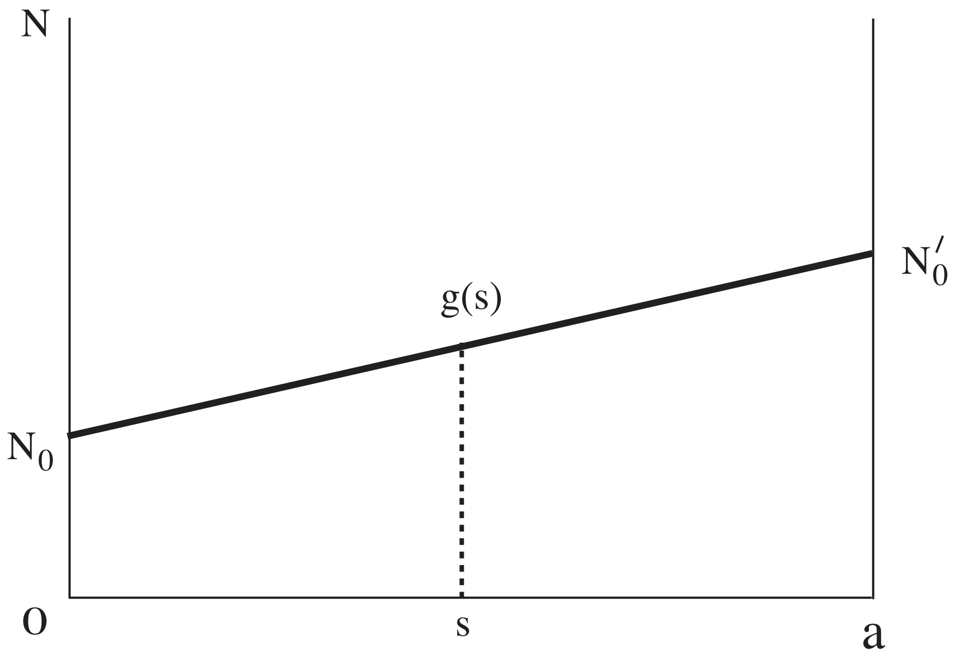

Figure 1. The figure shows the graph of g, where

$$g(s) = N_0 + \left(\frac{N_0' - N_0}{a}\right) s, \text{ and}$$

$$\begin{aligned} S(s) &= \left(N_0 + \left(\frac{N_0' - N_0}{a}\right) s\right)(a - s) \\ &= \frac{N_0}{a}\left(a + \left(\frac{N_0'}{N_0} - 1\right) s\right)(a - s), \text{ so} \\ S(s) &= \frac{N_0}{a}\,(a + (\rho - 1)\, s)\,(a - s) \end{aligned}$$

.

Thus, if $\rho < \infty$ the savings are given by

$$S(s) = \frac{N_0}{a}\,(a + (\rho - 1)s)\,(a - s).$$

S is a **quadratic function** . The zeros of S are at

$$a \quad \text{and} \quad \frac{-a}{\rho - 1}.$$

S has a maximum value at the mid-point of these two zeros,

$$\frac{a}{2}\left(\frac{\rho - 2}{\rho - 1}\right).$$

This point is negative if $\rho < 2$, is 0 if $\rho = 2$, and is in $(0, a/2)$ if $\rho > 2$. We only consider $0 \leq s \leq a$.

If $\rho = \infty$,

$$S(s) = \frac{N_0'}{a} s(a - s).$$

S is again a **quadratic function**. When $\rho = \infty$ we see that S has zeros at 0 and a, and that S has a maximum at $a/2$ whose value is

$$\frac{aN_0'}{4}.$$

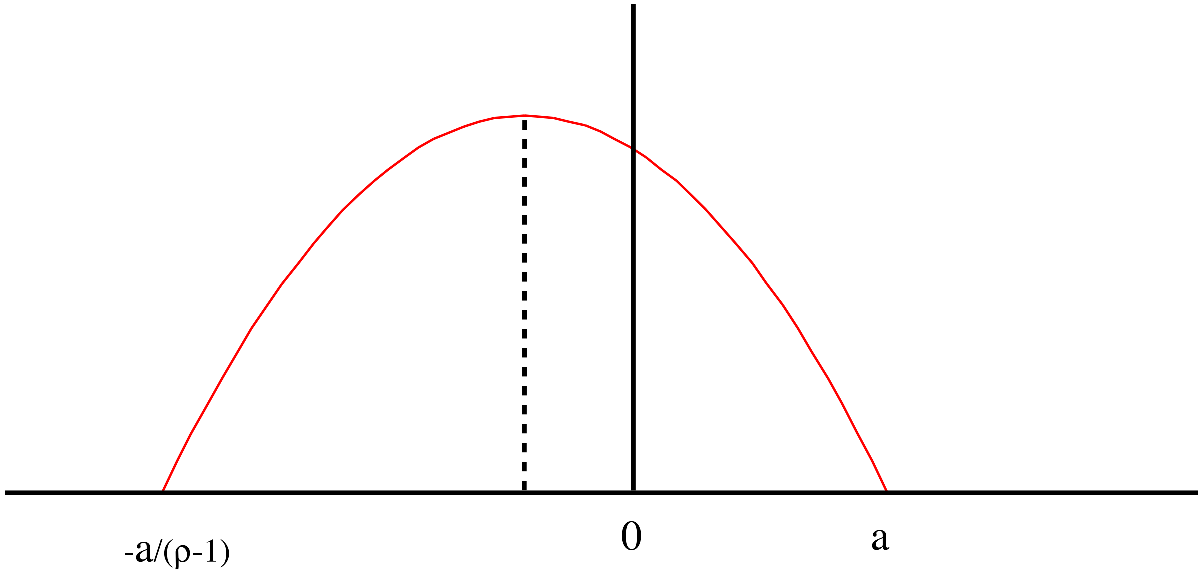

Figure 2. The picture shows the savings S against the subsidy s, when $1 < \rho < 2$. That is, it illustrates the graph of the quadratic

$$S(s) = \frac{N_0}{a}\,(a + (\rho - 1)s)\,(a - s),$$

for $1 < \rho < 2$.

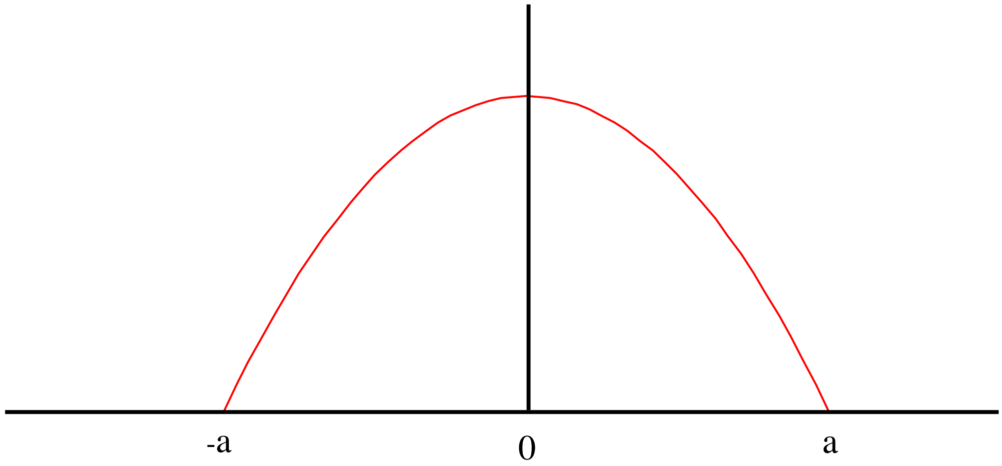

Figure 3. The picture shows the savings S against the subsidy s, in a case where $\rho = 2$. That is the graph of the quadratic

$$S(s) = \frac{N_0}{a}(a+s)(a-s) = \frac{N_0}{a}\left(a^2 - s^2\right).$$

The maximum savings are when there is no subsidy.

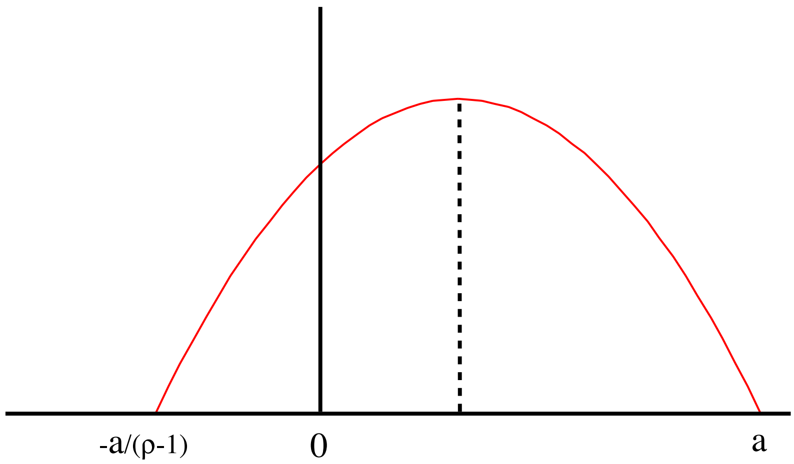

Figure 4. The figure shows the graph of the savings against the subsidy when $2 < \rho < \infty$. The maximum occurs somewhere between 0 and half the maximum possible subsidy.

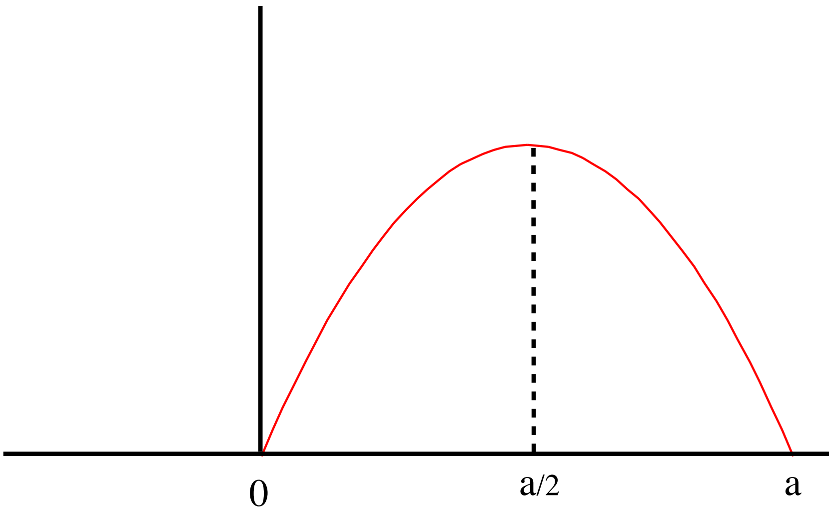

Figure 5. The figure shows the graph of the savings S against the subsidy s, in a case where $\rho = \infty$. The maximum savings occur at half the maximum possible subsidy.

Put

$$S_{max} = \max\{S(s) : 0 \le s \le a\}.$$

If $1 \le \rho \le 2$,

$$S_{max} = S(0) = aN_0.$$

If $2 \le \rho < \infty$,

$$\begin{aligned}
S_{max} &= S\left(\frac{a}{2}\left(\frac{\rho-2}{\rho-1}\right)\right) \\
&= \frac{N_0}{a}\left(a + (\rho-1)\frac{a}{2}\left(\frac{\rho-2}{\rho-1}\right)\right)\left(a - \frac{a}{2}\left(\frac{\rho-2}{\rho-1}\right)\right) \\
&= \frac{aN_0}{4(\rho-1)}\left(2+\rho-2\right)\left(2\rho-2-\rho+2\right) \\
&= \frac{aN_0\rho^2}{4(\rho-1)}.
\end{aligned}$$

Let S_{gov} denote the savings under government policy. That is,

$$\begin{aligned} S_{gov} &= S\left(\frac{2a}{3}\right) \\ &= \frac{N_0}{a}\left(a + (\rho - 1)\frac{2a}{3}\right)\left(a - \left(\frac{2a}{3}\right)\right) \\ &= \frac{N_0 a}{9}(2\rho + 1). \end{aligned}$$

Thus, when $2 < \rho < \infty$,

$$\boxed{S_{max} = \frac{aN_0\rho^2}{4(\rho-1)} \text{ and } S_{gov} = \frac{N_0 a}{9}(2\rho+1).}$$

So,

$$\begin{aligned}\frac{S_{max}}{S_{gov}} &= \frac{aN_0\rho^2}{4(\rho-1)} \cdot \frac{9}{N_0 a(2\rho+1)} \\ &= \frac{9}{4}\left(\frac{\rho^2}{2\rho^2-\rho-1}\right).\end{aligned}$$

We find that

$$S_{max} - S_{gov} = \left(\frac{S_{max}}{S_{gov}} - 1 \right) S_{gov}$$
$$= \frac{1}{4} \cdot \left(\frac{\rho^2 + 4\rho + 4}{2\rho^2 - \rho - 1} \right) S_{gov}.$$

There is a similar calculation for $1 < \rho < 2$.

So far, ρ has been kept fixed, we do not know the exact value of ρ.

Now S_{gov} denotes the savings under government policy. That is,

$$S_{gov} = S\left(\frac{2a}{3}\right) = \$4 \text{ billion.}$$

We estimate

$$S_{max} - S_{gov}.$$

Define function u by

$$\boxed{u(\rho) = \begin{cases} \dfrac{8-2\rho}{2\rho+1} & \text{, for } 1 \le \rho \le 2; \\ \dfrac{1}{4}\left(\dfrac{\rho^2+4\rho+4}{2\rho^2-\rho-1}\right), & \text{for } 2 < \rho < \infty; \\ \dfrac{1}{8}, & \text{for ``}\rho = \infty\text{''}. \end{cases}}$$

Then, a calculation shows that

$$S_{max} - S_{gov} = u(\rho) S_{gov}.$$

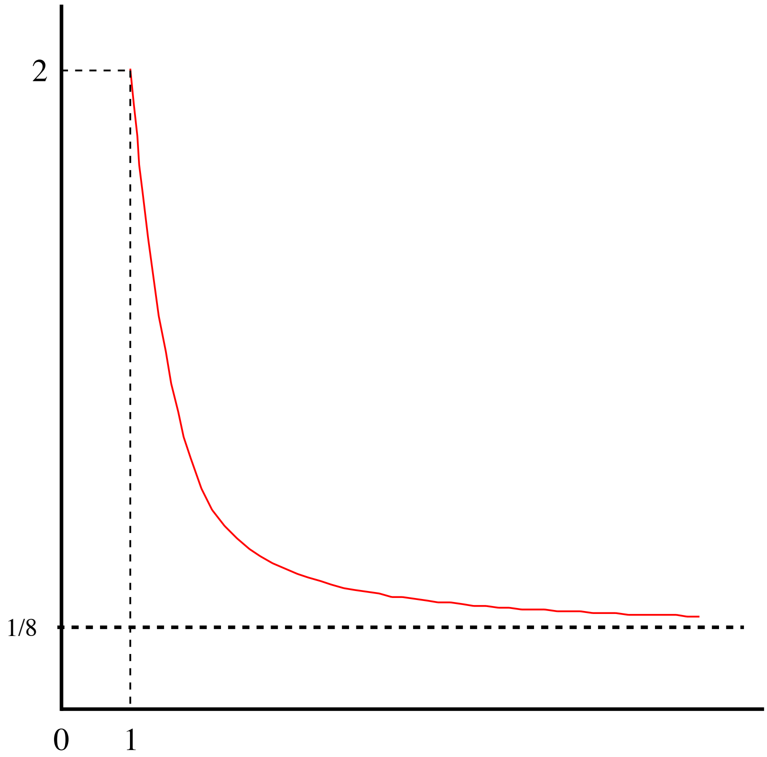

Figure 6. The graph of the function u given by

$$u(\rho) = \begin{cases} \dfrac{8-2\rho}{2\rho+1} & \text{, for } 1 \le \rho \le 2; \\ \dfrac{1}{4}\left(\dfrac{\rho^2+4\rho+4}{2\rho^2-\rho-1}\right), & \text{for } 2 < \rho < \infty; \end{cases}$$

Note that ρ decreases to $1/8$ as $\rho \longrightarrow \infty$.

THEOREM. *For all* $1 \leq \rho \leq \infty$,

$$S_{max} - S_{gov} = u(\rho)S_{gov} \geq u(\infty)S_{gov} = \frac{1}{8}S_{gov}.$$

The government appears to be trying to save more money when $\rho = \infty$. We then have

$$S_{max} - S_{gov} = \frac{1}{8}S_{gov} = \frac{1}{8}(\text{ \$4 billion dollars}) = \text{\$500 million} .$$

It follows from the Theorem that whatever the value of ρ,

$$S_{max} - S_{gov} \geq \text{\$500 million} .$$

Thus, under the linear model, $500 million is the smallest amount **more** that can be saved compared with what the Government is currently saving.

ESTIMATING THE VALUE OF ρ

Using the Minister's data, and the linear formula for g, we get

$$\rho = \frac{12}{25}\left(\frac{N}{N_0}\right) - \frac{1}{2}.$$

NOTE: N is the *total* number of students.

TABLE I. Estimates of ρ in the linear model

N_0/N	N/N_0 (upper bound for ρ)	Estimate of ρ	$u(\rho)$	Estimate of N_0'/N	$S_{max} - S_{gov}$ \$ million
0	∞	∞	0.125	0.48	500
0.05	20	9.1	0.198	0.46	792
0.1	10.0	4.3	0.313	0.43	1,252
0.12	8.33	3.5	0.378	0.42	1,512
0.15	6.66	2.7	0.501	0.41	2,004
0.18	5.56	2.2	0.681	0.39	2,724
0.2	5	1.9	0.875	0.38	3,500

LIMITATIONS OF THE ANALYSIS

- is the linear model accurate?
- other models may be better
- takes no account of the different circumstances for the various schools in the private and state systems.
- assumes that the fees charged by private schools are not affected by subsidies

OTHER MODELS

I. Where demand is quadratic, not linear. A possible model for g is that it is a quadratic, perhaps given by

$$g(s) = N_0 + \left(\frac{N_0' - N_0}{a^2} \right) s^2, \quad \text{for} \quad 0 \leq s \leq a.$$

II. Demand changes in proportion to the number of potential customers. Another way of trying to get a formula for g would be to say that the rate at which g is changing as a function of s should be proportional to the number of potential students who have not yet enrolled at a private school. Mathematically, this would mean that there is a constant c such that

$$g'(s) = c\Big(N_0' - g(s)\Big), \text{ for } 0 \le s \le a.$$

III. A model of "Zipf's Law" type. Another model is to take

$$g(s) = \frac{N_0' a}{\rho - 1} \cdot \frac{1}{a - s + \frac{a}{\rho - 1}} = \frac{N_0' a}{\rho a - (\rho - 1)s}.$$

Still other models are possible.

The idea is in each model to carry out the analysis that has been described for the linear model, to reach conclusions about the validity of government statements about its own policy, and maybe come up with a better policy than the Government's current policy.

Other areas where the analysis is applicable

The analysis presented here has been a particular approach to a special case of a supply and demand problem. Such problems occur widely and they can be tackled by an adaptation of the techniques described in this paper. Three possible areas are described.

I. Maximizing profit in retailing. A retailer sells a particular item, which costs \$a to purchase. So, if the item sells for price p, the profit is $p - a$. The total profit if the item is sold at this price p will be

$$P(p) = g(p)(p - a).$$

The problem here is to maximize the profit P, considered as a function of the price.

II. Supply-side economics and the Laffer curve. Professor Art Laffer was an adviser on economics to President Ronald Reagan in the USA in the early 1980s. He pointed out that the amount of income tax revenue a government would receive was affected by the tax rate. Thus, if the tax rate was 0%, the government would collect nothing from income tax. But if the tax rate was 100%, the government would again get nothing, for no-one would bother to earn an income if they had to give all of it to the government. So, somewhere in between there should be a point where government revenue would be maximized.

III. International students and university revenue. Australian universities receive a significant amount of funding from international students who are charged full fees. But if these students did not have to pay full fees then more fee paying students would be encouraged to enroll. Overall revenue could increase even though the individual charge per student decreases.

Two quotes on the usefulness of quadratics

In 2003 in the *London Times* Simon Jenkins wrote

> I studied maths to 16. I could sword-fight with a slide rule and consort with logarithms....I could stalk the square on the hypotenuse and drop a surd at 50 paces. I ate quadratic equations for breakfast and lunched on differential calculus. It was completely pointless.

The following is an excerpt from the speech made by Mr McWalter in the House of Commons on 26th June 2003.

> To tell students that quadratic equations are beyond them, that they are nothing and that educated people need have no inkling of what they are is to say that it is all right if they are so ill equipped to understand modern science that they cannot even comprehend its starting point....When educators tell us we should do that, I rejoin that they have a strange view of education .....it is powerful educational medicine that something that can be expressed very simply [the quadratic equation] can be extraordinarily difficult to solve. Much of modern culture tends the other way. People are presented with enormously difficult problems in politics or economics... and they assume that such problems have a simple .... solution. The quadratic equation can teach us to be humble.... such activities, demanding as they do that the students make a real effort to change their perspective, are at the core of education; education as mountain climbing and not molehill jumping.

SUMMARY

In this talk, using high school mathematics and quadratic functions, I investigated the extent to which Government policy is designed to maximise savings in expenditure on school education with respect to state and private schools.

• origins of the problem

• the analysis assumed a linear function for demand

• this gave a quadratic function for savings

• you and your students can carry out an analysis for yourselves based on the linear model and upon other models

• an aim has been to show how mathematics is applicable to many other problems in commerce and public policy such as:

∗ price setting of a good to achieve maximum profit

∗ setting the income tax rate so as to maximize revenue for a government

∗ subsidizing students so as to maximize university revenue.

• May help to make students aware of the job relevance of mathematics

• An aim was to demonstrate mathematical **thinking**, as distinct from mathematical **calculation**.