

**The Fourier transform
and the
expression of functions
as sums of
finite differences**

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Abstract

If f is a function in $L^2(\mathbb{R})$, whether f is a finite sum of difference functions of order s is determined entirely by the behaviour of the Fourier transform \widehat{f} of f near the origin. Namely, f is a finite sum of difference functions of order s if and only if

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{|x|^{2s}} dx < \infty,$$

in which case f is a sum of $2s + 1$ differences of order s . The problem discussed here is whether $2s + 1$ is the smallest possible number. For $s = 1$, this is already known for the circle group and for \mathbb{R} (Meisters and Schmidt 1972, Nilsen 1994).

Let $L^2(\mathbb{R})$ denote the space of all (measurable) functions $f : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{1/2} < \infty.$$

$L^2(\mathbb{R})$ is a Hilbert space, and its inner product is

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx.$$

Sobolev spaces

Let $\mathcal{S}(\mathbb{R})$ denote the space of all infinitely differentiable functions ϕ on \mathbb{R} such that

$$\lim_{|x| \rightarrow \infty} \phi^{(k)}(x)x^n = 0,$$

for all $k, n = 1, 2, \dots$. Then, if $f \in L^2(\mathbb{R})$, f is in the first order Sobolev space $W^1(\mathbb{R})$ if there is $h \in L^2(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} f(x)\phi'(x)dx = - \int_{-\infty}^{\infty} h(x)\phi(x)dx,$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Then, h is the derivative of f ,

$$D(f) = h.$$

Note that $\mathcal{S}(\mathbb{R}) \subseteq W^1(\mathbb{R})$.

If $s \in \mathbb{N}$, the set of natural numbers, the **Sobolev space of order s** is the subspace $W^s(\mathbb{R})$ of $L^2(\mathbb{R})$ consisting of all functions in $L^2(\mathbb{R})$ whose (distributional) derivatives up to and including the order s all exist and are in $L^2(\mathbb{R})$. In particular, if $s = 1$ and D denotes differentiation, we have

$$D : W^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R}).$$

The Sobolev space $W_s(\mathbb{R})$ is a Hilbert space, with the inner product given by and integral involving Fourier transforms (see later).

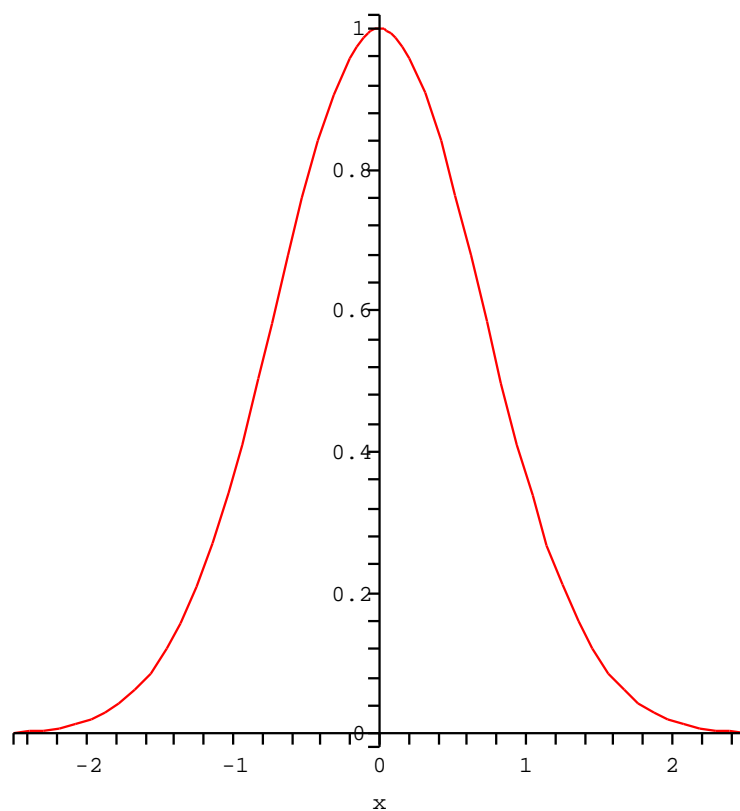
If f is a function on \mathbb{R} , and if $x \in \mathbb{R}$, the **convolution** of f by the Dirac measure δ_x is the function $\delta_x * f$ given by

$$t \longmapsto f(t - x).$$

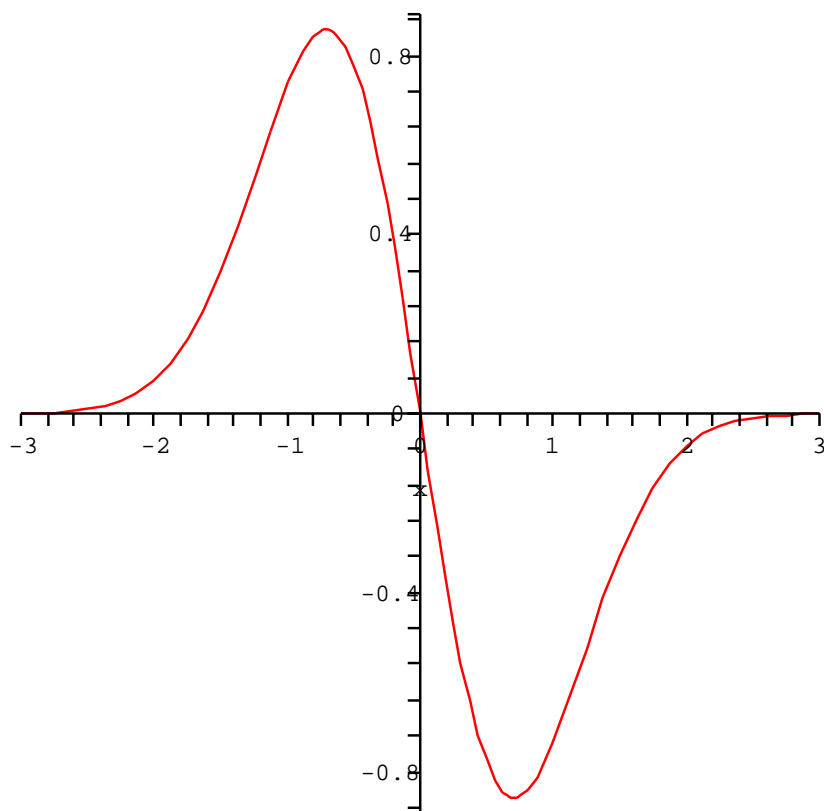
A **first order difference** in $L^2(\mathbb{R})$ is a function of the form

$$g - \delta_x * g,$$

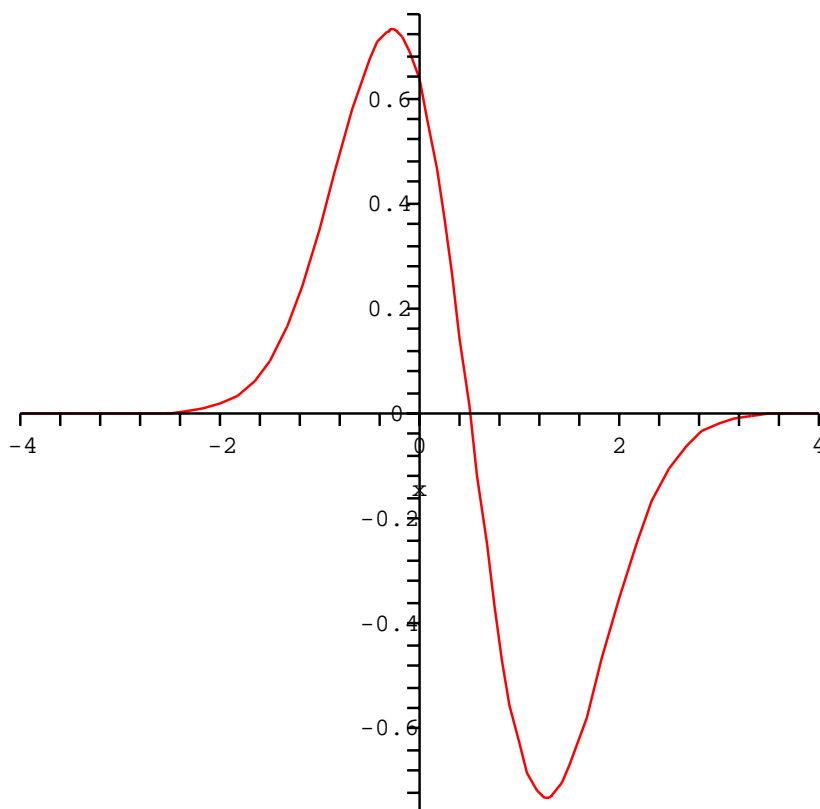
for some $x \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$.



The function $x \mapsto e^{-x^2}$. It belongs to the Sobolev space $W^s(\mathbb{R})$ for all $s \in \mathbb{N}$. The function vanishes at ∞ and $-\infty$ and, having a maximum or a minimum only at a single point, it is a wave with a single “crest”.



The derivative of $x \mapsto e^{-x^2}$. Like the original function, the derivative vanishes at ∞ and $-\infty$, but now the function is a wave with a “crest” and a “trough”.



A first order difference of $x \mapsto e^{-x^2}$.
In fact, it is the function

$$x \mapsto e^{-x^2} - e^{-(x-1)^2}.$$

Just as in the case of taking the derivative, taking a first order difference has produced a “wave” with a “crest” and a “trough”.

THEOREM (JFA, 1992). *The following conditions are equivalent for a function f in $L^2(\mathbb{R})$.*

(1) *f is in the range of D on the Sobolev space $W^1(\mathbb{R})$. That is, there is a function $g \in W^1(\mathbb{R})$ such that*

$$D(g) = f.$$

(2) *There are $x_1, x_2, \dots, x_n \in \mathbb{R}$ and $f_1, f_2, \dots, f_n \in L^2(\mathbb{R})$ such that*

$$f = \sum_{j=1}^n (f_j - \delta_{x_j} * f_j).$$

That is, f may be written as a finite sum of first order differences.

Also, if f satisfies the above conditions—that is, if f is the sum of a finite number of first order differences, there are $x_1, x_2, x_3 \in \mathbb{R}$ and $f_1, f_2, f_3 \in L^2(\mathbb{R})$ such that

$$f = \sum_{j=1}^3 (f_j - \delta_{x_j} * f_j). \quad (*)$$

That is, if f can be expressed as a finite sum of first order differences, it can be expressed as the sum of **three** first order differences. **NOTE:** the number 3 in (*) is **sharp**, in the sense that there is a function f which can be expressed as in (*), but

$$f \neq \sum_{j=1}^2 (h_j - \delta_{y_j} * h_j),$$

for all $h_1, h_2 \in L^2(\mathbb{R}), y_1, y_2 \in \mathbb{R}$. (Springer, 1994)

Also, if $f \in L^2(\mathbb{R})$ and f is a sum of first order differences, then, for *almost all* $(x_1, x_2, x_3) \in \mathbb{R}^3$, there are $f_1, f_2, f_3 \in L^2(\mathbb{R})$ such that

$$f = \sum_{j=1}^3 (f_j - \delta_{x_j} * f_j) .$$

If f is a continuous function on \mathbb{R} which is zero outside of some bounded interval, the **Fourier transform** of f is the function \hat{f} given by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt} dt, \quad x \in \mathbb{R}.$$

The Fourier transform may be extended to the whole of $L^2(\mathbb{R})$, and we find that for all $f \in L^2(\mathbb{R})$,

$$\begin{aligned} \|\hat{f}\| &= \left(\int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx \right)^{1/2} \\ &= \sqrt{2\pi} \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{1/2} \\ &= \sqrt{2\pi} \|f\|. \end{aligned}$$

The Sobolev space $W_s(\mathbb{R})$ is a Hilbert space, with the inner product $\langle \cdot, \cdot \rangle$ given by the following integral involving Fourier transforms:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \hat{f}(x) \overline{\hat{g}(x)} (1 + |x|^{2s}) dx,$$

for $f, g \in W_s(\mathbb{R})$.

A **difference of order 2**, or a **second order difference** in the space of functions $L^2(\mathbb{R})$, is defined to be a function of the form

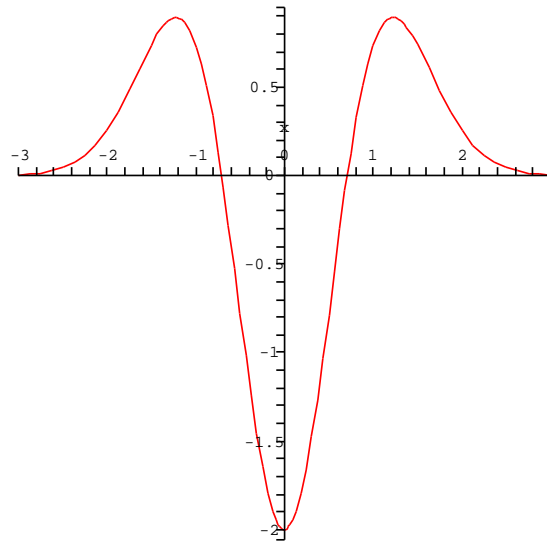
$$g - \frac{1}{2}(\delta_x + \delta_{-x}) * g,$$

for some $x \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$.

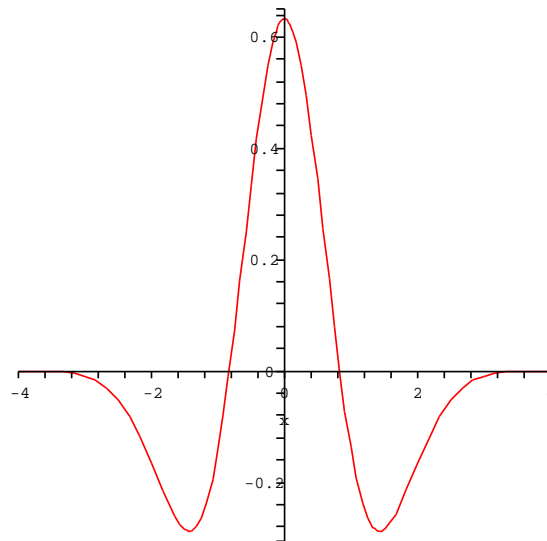
In general, for $s \in \mathbb{N}$ a function in $L^2(\mathbb{R})$ is called a **difference of order s** if it is of the form

$$\sum_{k=0}^s (-1)^k \binom{s}{k} \delta_{kx} * g$$

for some $x \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$. In fact, the concept of a “difference of order s ” may be defined in a similar manner for each $s > 0$.



The second derivative $(-2 + 4x^2)e^{-x^2}$ of e^{-x^2} .



A second order difference of $x \mapsto e^{-x^2}$. It is the function

$$x \mapsto e^{-x^2} - \frac{1}{2} \left(e^{-(x-1)^2} + e^{-(x+1)^2} \right).$$

Now, we saw that taking one derivative had the same effect on a function as taking a first order difference. What happens about higher order derivatives and differences? The figure shows the similar effect of taking a second order derivative and a second order difference on the function $x \mapsto e^{-x^2}$.

The question arises as to the **precise** relationship between derivatives of functions and finite differences of functions in $L^2(\mathbb{R})$.

Note that

$$D(\widehat{f})(x) = ix\widehat{f}(x),$$

so that differentiation increases the “high” frequencies in a function (or signal) and decreases the “low” frequencies.

Also, if h is a difference of order s given by

$$h = \sum_{k=0}^s (-1)^k \binom{s}{k} \delta_{kx} * g, \text{ then}$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{|\widehat{h}(x)|^2}{|x|^{2s}} dx \\ &= \int_{-\infty}^{\infty} |1 - e^{ixs}|^{2s} \frac{|\widehat{g}(x)|^2}{|x|^{2s}} dx \\ &= C \int_{-\infty}^{\infty} \frac{|\sin(x/2)|^{2s}}{|x|^{2s}} |\widehat{g}(x)|^2 dx \\ &\leq D \int_{-\infty}^{\infty} |\widehat{g}(x)|^2 dx < \infty. \end{aligned}$$

THEOREM (JFA, 1992). *Let f be a function in $L^2(\mathbb{R})$. Then, f is a finite sum of first order differences if and only if*

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{|x|^2} dx < \infty.$$

Moreover, in general, if $s \in \mathbb{N}$, then f is a finite sum of differences of order s if and only if

$$\int_{-\infty}^{\infty} \frac{|\widehat{f}(x)|^2}{|x|^{2s}} dx < \infty.$$

In this case there are $x_1, x_2, \dots, x_{2s+1} \in \mathbb{R}$ and $f_1, f_2, \dots, f_{2s+1} \in L^2(\mathbb{R})$ such that

$$f = \sum_{j=1}^{2s+1} \left(\sum_{k=0}^s (-1)^k \binom{s}{k} \delta_{kx_j} * f_j \right). \quad (**)$$

The preceding result has a number of consequences. Let $\mathcal{D}_s(\mathbb{R})$ denote the set of all functions in $L^2(\mathbb{R})$ which are a finite sum of differences of order s . Then, $D^s : W^s(\mathbb{R}) \longrightarrow \mathcal{D}_s(\mathbb{R})$ and this mapping is onto. Also, $\mathcal{D}_s(\mathbb{R})$ is a Hilbert space with the inner product given by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x) \overline{\widehat{g}(x)} \left(1 + \frac{1}{|x|^{2s}} \right) dx,$$

The question here is : we know (**) is sharp for $s = 1$. Is it sharp for general s ?

The question was answered in the affirmative in joint work with Susumu Okada (Proc. Lond. Math. Soc. 2004).