A Tilt at TILFs

Rod Nillsen
University of Wollongong

This talk is dedicated to
Gary H. Meisters

Abstract

In this talk I will endeavour to give an overview of some aspects of the theory of Translation Invariant Linear Forms (TILFs) and associated Hilbert spaces of functions. In particular, I will discuss some of the early ideas and results of Gary Meisters in this area, and try to explain how these ideas have led to applications in various areas of harmonic analysis.

1 Definitions

Let $\mathbb{T}$ denote the circle group, $\mathbb{Z}$ the group of integers and $\mathbb{R}^n$ the n-dimensional Euclidean group. Let G denote any one of these groups(or any locally compact abelian group). Let $M(G)$ denote the measures on G of finite variation, and let$X$ denote a vector space of functions or distributions on G such that $\mu \in M(G)$ and $f \in L^2(G) \Longrightarrow$ the convolution $\mu * f$ is defined and $\mu * f \in X$. Let X' denote the algebraic dual of X and let $S \subseteq M(G)$.

1.1 Main Definitions

1. A linear form $L \in X'$ is S-*invariant* if $L(\mu * f) = L(f)$ for all $\mu \in S$ and all $f \in X$.
2. If $x \in G$ and $f \in L^2(G)$, $\delta_x * f$ is the function $t \longmapsto f(t-x)$ and is called the *translation of f by x*. Then, when $S = \{\delta_x : x \in G\}$, an S-invariant form is called *translation* invariant or a TILF.

3. $\mathcal{D}(X,S)$ denotes the vector subspace of X spanned by all vectors of the form $f - \mu * f$, for some $\mu \in S$ and $f \in X$. Thus, for $f \in X$, $f \in \mathcal{D}(X,S) \iff$ there are $n \in \mathbb{N}$, $g_1, g_2, \ldots, g_n \in X$ and $\mu_1, \mu_2, \ldots, \mu_n \in S$ such that $f = \sum_{j=1}^{n}(g_j - \mu_j * g_j)$. When $S = \{\delta_x : x \in G\}$, $\mathcal{D}(X,S)$ is denoted by $\mathcal{D}(X)$. A space $\mathcal{D}(X,S)$ is called a *difference space*.

1.2 Comments

1. If $\mu \in X'$, μ is S-invariant $\iff \mu\left(\mathcal{D}(X,S)\right) = \{0\}$.
2. Note that a function $f - \delta_x * f$ is given by $t \longmapsto f(t) - f(t-x)$, a *first order difference*, sometimes used to approximate the derivative of a function.

2 The basic problems

1. Identify all S-invariant forms.
   - (i) 0 is always one, but are there others?
   - (ii) Is an S-invariant form necessarily continuous on X?
2. Characterise the space $\mathcal{D}(X,S)$ as a subspace of X.

2.1 Comment

Problem 2 may be regarded as a refinement of problem 1. For, there is a non-zero S-invariant form on $X \iff \mathcal{D}(X,S) \neq X$ and, $\mathcal{D}(X,S)$ is dense in $X \iff$ the only continuous S-invariant form on X is 0. [Meisters, J. Func.Anal. (12), 1973].

3 The circle group case

THEOREM [Meisters & Schmidt, J. Func. Anal. (11) 1972]. *If G is compact and connected, in particular if $G = \mathbb{T}$, then*

$$\mathcal{D}(L^2(G)) = \left\{f : f \in L^2(G) \text{ and } \int_G f = 0\right\}.$$

Thus in this case, $\mathcal{D}(L^2(G))$ has codimension 1 in $L^2(G)$, and every TILF on $L^2(G)$ is continuous and is a multiple of the Haar measure on G.

IDEA OF PROOF. Let $\widehat{\mu}$ denote the Fourier transform of μ. Let $\widehat{G}$ be the dual group of G ($=\mathbb{Z}$, if $G=\mathbb{T}$). Then, if $f \in L^2(G)$, $f \in \mathcal{D}(L^2(G)) \iff$ there are $x_1, x_2, \ldots, x_n \in G$ such that

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^2}{\sum_{j=1}^n |1-\gamma(x_j)|^2} d\gamma < \infty. \tag{1}$$

When (1) holds, there are $f_1, f_2, \ldots, f_n$ such that $f = \sum_{j=1}^n (f_j - \delta_{x_j} * f_j)$. However, the trouble is that for a given f, it is hard to tell whether there are $x_1, x_2, \ldots, x_n \in G$ such that (1) holds. Let $\widehat{f}(0) = 0$ & consider

$$\begin{aligned}
&\int_{G^n}\left(\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^2}{\sum_{j=1}^n |1-\gamma(x_j)|^2} d\gamma\right) dx_1 \ldots dx_n \\
=& \int_{\widehat{G}} |\widehat{f}(\gamma)|^2 \left(\int_{G^n} \frac{dx_1 dx_2 \ldots dx_n}{\sum_{j=1}^n |1-\gamma(x_j)|^2}\right) d\gamma \\
=& \left(\int_{\widehat{G}} |\widehat{f}(\gamma)|^2 d\gamma\right)\left(\int_{[-\pi,\pi]^n} \frac{d\theta_1 d\theta_2 \ldots d\theta_n}{\sum_{j=1}^n |1-e^{i\theta_j}|^2}\right) \\
=& \left(\int_{\widehat{G}} |\widehat{f}(\gamma)|^2 d\gamma\right)\left(\int_{[-\pi,\pi]^n} \frac{d\theta_1 d\theta_2 \ldots d\theta_n}{4\sum_{j=1}^n \sin^2 \theta_j/2}\right) < \infty,
\end{aligned}$$

for $n \geq 3$.

Thus, for almost all $(x_1, x_2, x_3) \in G^3$, (1) holds and if follows that any $f \in \mathcal{D}(L^2(G)$ is the sum of 3 first order differences (here, 3 is best possible).

Meisters and Schmidt in fact showed that on any compact group with a finite number of components, any TILF on $L^2(G)$ is a multiple of Haar measure and so continuous. Meisters (1973) had shown that the L^2-space of the Cantor group had discontinuous TILFs. Further results of Meisters & Bagget (1983) and Johnson (1983), produced a characterisation of the compact abelian groups G such that every TILF on $L^2(G)$ is continuous. Bourgain(1986) showed that for $1 < p < \infty$, every TILF on $L^p(\mathbb{T})$ is continuous (extended to other groups by Lo in 1996).

The result of Meisters and Schmidt suggests a reformulation of the basic problem for $L^2(G)$. For, in the circle group case let μ be the measure on $\widehat{\mathbb{T}} = \mathbb{Z}$ given by

$$\mu(A) = \begin{array}{ll} \text{number of elements in } A, & \text{if } 0 \notin A; \\ \infty, & \text{if } 0 \in A. \end{array}$$

Then observe that for $f \in L^2(\mathbb{T})$, $\int_{\mathbb{Z}} |\widehat{f}|^2 d\mu < \infty \iff \widehat{f}(0) = 0$. So, their result can be stated as: *the Fourier transform maps* $\mathcal{D}(L^2(\mathbb{T}))$ *bijectively onto* $L^2(\widehat{\mathbb{T}}, \mu)$. Then, on a LCA group G the basic problem for $L^2(G)$ becomes: *describe a measure* μ *on* $\widehat{G}$ *such that the Fourier transform maps a difference space* $\mathcal{D}(L^2(G), S)$ *bijectively onto* $L^2(\widehat{G}, \mu)$.

4 The case of the real line

In 1973, Meisters had proved the following:

$$f \in \mathcal{D}(L^2(\mathbb{R})) \Longrightarrow \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2} dx < \infty. \tag{2}$$

It follows from this that $\mathcal{D}(L^2(\mathbb{R}))$ is a dense, proper subspace of $L^2(\mathbb{R})$. He deduced: *there are non-zero TILFs on* $L^2(\mathbb{R})$ *and every such TILF is discontinuous.* Further results for non-compact groups were obtained by Woodward(1974), Saeki(1984) and RN(1994), all concerning the existence and profusion of discontinuous TILFs in non-compact cases such as $\mathbb{R}$. Now (2) shows that Fourier transforms of functions in $\mathcal{D}(L^2(\mathbb{R}))$ have a certain precise behaviour *near the origin.* In fact, the functions in $L^2(\mathbb{R})$ which are in $\mathcal{D}(L^2(\mathbb{R}))$ are *characterized* by the behaviour expressed in (2) [RN, J. Func. Anal. 1993]. We therefore have

THEOREM. *Let* $f \in L^2(\mathbb{R})$. *Then*

$$f \in \mathcal{D}(L^2(\mathbb{R})) \iff \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2} dx < \infty.$$

The space $\mathcal{D}(L^2(\mathbb{R}))$ *is Hilbert, with the inner product given by, for*

$$f, g \in \mathcal{D}(L^2(\mathbb{R})), \langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^{-2}) dx$$

The Fourier transform maps $\mathcal{D}(L^2(\mathbb{R}))$ *isometrically onto* $L^2(\mathbb{R}, (1+|x|^{-2})dx$.
Now the first order Sobolev space is $H^1(\mathbb{R})$ and consists of the functions in $L^2(\mathbb{R})$ whose derivatives are in $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$, $f \in H^1(\mathbb{R}) \iff \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^2 dx < \infty$, so functions in $H^1(\mathbb{R})$ are characterised by the behaviour of their Fourier transforms *at infinity.* The space $H^1(\mathbb{R})$ is Hilbert with inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^2) dx.$$

Now let D denote differentiation. Then, for $f \in H^1(\mathbb{R})$, $D(f)^\wedge(x) = ix\widehat{f}(x)$. It follows from this that *differentiation is a Hilbert space isometry from $H^1(\mathbb{R})$ onto $\mathcal{D}(L^2(\mathbb{R}))$*, so the latter space is the range of D in a natural sense. Hence

$$D(H^1(\mathbb{R})) = \bigcap \Big\{ \text{ kernel } (L) : L \text{ is a TILF} \Big\}.$$

4.1 FRACTIONAL DIFFERENCE SPACES

Now let $s > 0$ let α be a 2π-periodic function which has an absolutely convergent Fourier series and let:

(i) for some $\delta > 0$,

$$\delta |x|^s \leq |\alpha(x)| \leq \delta^{-1} |x|^s, \text{ for } x \text{ in } [-\delta, \delta];$$

(ii)

$$\int_{[-\pi,\pi]^m} \frac{dx_1 \dots dx_m}{\sum_{j=1}^m |\alpha(x_j)|^2} < \infty, \text{ some } m \in \mathbb{N}.$$

Let $S_\alpha = \{\delta_0 - \sum_{j=-\infty}^{\infty} \widehat{\alpha}(j)\delta_{-jy} : y \in \mathbb{R}\}$. Then we have: *$\mathcal{D}(L^2(\mathbb{R}), S_\alpha)$ consists of all functions $f \in L^2(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2s} dx < \infty$.* Thus, as $\mathcal{D}(L^2(\mathbb{R}), S_\alpha)$ depends upon s rather than α, denote it by $\mathcal{D}_s(L^2(\mathbb{R}))$.
THEOREM.*Let $f \in L^2(\mathbb{R})$. Then*

$$f \in \mathcal{D}_s(L^2(\mathbb{R})) \Longleftrightarrow \int_{-\infty}^{\infty} |\widehat{f}(x)|^2 |x|^{-2s} dx < \infty.$$

$\mathcal{D}_s(L^2(\mathbb{R}))$ is Hilbert, with the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{g}(x)}(1 + |x|^{-2s})dx, \text{ for } f, g \in \mathcal{D}_s(L^2(\mathbb{R})).$$

For $s \in \mathbb{N}$, D^s is an isometry from the Sobolev space $H^s(L^2(\mathbb{R}))$ onto $\mathcal{D}_s(L^2(\mathbb{R}))$.

DEFINITION. An S_α invariant linear form, with S_α as above, may be called an s-ILF. Thus, for $s \in \mathbb{N}$,

$$D^s(H^s(\mathbb{R})) = \bigcap \Big\{ \text{ kernel } (L) : L \text{ is an } s - \text{ILF} \Big\}.$$

EXAMPLE. A function f in $L^2(\mathbb{R})$ is the second derivative of some function in $L^2(\mathbb{R})$ if and only if there are $x_1,\ldots,x_5 \in \mathbb{R}$, $f_1,\ldots,f_5 \in L^2(\mathbb{R})$ with $f = \sum_{j=1}^5\Big(f_j - 2^{-1}(\delta_{x_j}+\delta_{-x_j}) * f_j\Big)$. Also, a function f in $L^2(\mathbb{R})$ is the second derivative of some function in $L^2(\mathbb{R})$ if and only if $L(f) = 0$ for every $\{2^{-1}(\delta_x+\delta_{-x}) : x \in \mathbb{R}\}$-invariant form on $L^2(\mathbb{R})$.

5 Partial Differential Operators

Let V be a subspace of $\mathbb{R}^n$, and let $e_1,\ldots,e_r$ be an orthonormal basis for V. The *V-Laplacian* Δ_V is given by $\Delta_V = \sum_{j=1}^r D_{e_j}^2$, where D_{e_j} is differentiation in direction e_j. If P_V is the orthogonal projection onto V,

$$\Delta_V(f)^\wedge(x) = -|P_V(x)|^2\widehat{f}(x).$$

Let $V_1,V_2,\ldots,V_q$ be non-zero vector subspaces of $\mathbb{R}^n$, and let $s_1,s_2,\ldots,s_q$ be q strictly positive real numbers. Let

$$\begin{aligned}
\Upsilon &= \prod_{j=1}^q |P_{V_j}|^{s_j},\\
\Psi &= \sum_{A\subseteq\{1,\ldots,q\}} \prod_{j\in A} |P_{V_j}|^{-s_j},\\
\Theta &= \sum_{A\subseteq\{1,\ldots,q\}} \prod_{j\in A} |P_{V_j}|^{s_j}.
\end{aligned}$$

Let

$$W(L^2(\mathbb{R}^n),\Psi) = \Big\{f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\widehat{f}(x)|^2\Psi^2(x)dx < \infty\Big\},$$

and similarly define $W(L^2(\mathbb{R}^n),\Theta)$. Then, using the fact that $\Upsilon\Psi = \Theta$, the operator $\prod_{j=1}^q |\Delta_{V_j}|^{s_j}$ may be defined and is an isometry from $W(L^2(\mathbb{R}^n),\Theta)$ onto $W(L^2(\mathbb{R}^n),\Psi)$. **The point is** that the space $W(L^2(\mathbb{R}^n),\Psi)$ may be described alternatively as a "generalised" difference space. A similar description of the range is valid for operators $D_{u_1}D_{u_2}\ldots D_{u_r}$, for independent vectors $u_1,\ldots,u_r \in \mathbb{R}^n$.
EXAMPLE. The Wave Operator is

$$\mathcal{W} = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) = D_{u_1}D_{u_2},$$

where $u_1 = (1, -1), u_2 = (1, 1)$. The domain of $\mathcal{W}$ is the Sobolev-type space consisiting of all $f \in L^2(\mathbb{R}^n)$ such that

$$\left\{ \int_{\mathbb{R}^2} |\widehat{f}(x)|^2 \left\{ 1 + |x-y| + |x+y| + |x-y| \cdot |x+y| \right\} \right\}^2 dxdy < \infty.$$

The range of $\mathcal{W}$ consists of those functions in $L^2(\mathbb{R}^2)$ which are the sum of 9 functions, each of which is of the form $(x, y) \longmapsto g(x, y) - g(x+a, y+a) - g(x+b, y_b) + g(x+a+b, y+a-b)$, for some $a, b \in \mathbb{R}$ and some $g \in L^2(\mathbb{R})$. The range of $\mathcal{W}$ is the intersection of the kernels of all the linear forms which are $\{\delta_{(a,a)} + \delta_{(b,-b)} - \delta_{(a+b,a-b)} : a, b \in \mathbb{R}\}$-invariant.

5.1 GENERALIZED DIFFERENCE SPACES

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ be q continuous, complex valued 2π-periodic functions on $\mathbb{R}^n$ be such that for some $\delta_1, \delta_2 > 0$,

$$\delta_1 |x|^{s_j} \leq |\alpha_j(x)| \leq \delta_2 |x|^{s_j},$$

for all $x \in [-\pi, \pi]$ and all $j = 1, 2, \ldots, q$. For each $j = 1, 2, \ldots, q$, let $m_j \in \mathbb{N}$ with $m_j > 2s_j$, and let $J_1, J_2, \ldots, J_q$ be q disjoint subintervals of $\mathbb{N}$ such that each J_j has m_j elements. Now consider the set of all functions f in $L^2(\mathbb{R}^n)$ such that f is equal to a sum of the form

$$\sum_{(k_1,\ldots,k_q) \in \prod_{j=1}^q J_j} \left(\prod_{j=1}^q \sum_{\ell=-\infty}^{\infty} \widehat{\alpha}_j(\ell) \delta_{-\ell y_{k_j}} \right) * \ h_{k_1 k_2 \ldots k_q},$$

where for each $k \in \{1, 2, \ldots, q\}$, $y_k \in V_k$; and for each $(k_1, \ldots, k_q) \in \prod_{j=1}^q J_j$, $h_{k_1 k_2 \ldots k_q} \in L^2(\mathbb{R}^n)$. This set of functions is a subset of $L^2(\mathbb{R}^n)$ which depends upon $V_1, \ldots, V_q$, $s_1, \ldots, s_q$ but is *independent* of $\alpha_1, \ldots, \alpha_q$ and $m_1, \ldots, m_q$. Accordingly, this set of functions is denoted by $\mathcal{D}_{s_1, s_2, \ldots, s_q}(L^2(\mathbb{R}^n), V_1, \ldots, V_q)$.
THEOREM. $\mathcal{D}_{s_1, s_2, \ldots, s_q}(L^2(\mathbb{R}^n), V_1, \ldots, V_q)$
is a vector subspace of $L^2(\mathbb{R}^n)$*, and it is a Hilbert space in the inner product* $[\cdot, \cdot]$ *given by*

$$[f, g] = \int_{\mathbb{R}^n} \left(\sum_{A \subseteq \{1,2,\ldots,q\}} \left[\prod_{j \in A} |P_{V_j}|^{-s_j} \right] \right) \widehat{f}(x) \overline{\widehat{g}(x)} \ dx,$$

in $\mathbb{R}^2$*, if* V_1 *is the subspace spanned by* $(-1, 1)$ *and* V_2 *is the one spanned by* $(1, 1)$*, the range of* $\mathcal{W}$*, as before, is the space* $\mathcal{D}_{1,1}(L^2(\mathbb{R}^n), V_1, V_2)$.

More generally, if Θ, Ψ are the functions as before, the Theorem shows that $W(L^2(\mathbb{R}^n), \Psi) = \mathcal{D}_{s_1,s_2,\ldots,s_q}(L^2(\mathbb{R}^n), V_1, \ldots, V_q)$. Thus,
THEOREM. $\prod_{j=1}^{q} |\Delta_{V_j}|^{s_j}$ *is an isometry from* $W(L^2(\mathbb{R}^n), \Theta)$ *onto*

$$\mathcal{D}_{s_1,s_2,\ldots,s_q}(L^2(\mathbb{R}^n), V_1, \ldots, V_q).$$

6 Multiplier Operators on LCA Groups

Partial differential operators are special cases of (unbounded) multiplier operators with mutipliers of the form $\prod_{j=1}^{r} |P_{V_j}|^{s_j}$ or $\prod_{j=1}^{r} \langle \cdot, e_j \rangle$, in the present context. A multiplier operator T on a space X is one for which there is a function ϕ such that $T(f)^\wedge = \phi \widehat{f}$, all $f \in X$. Recent work with Susumu Okada has shown that the ranges of a large class of multiplier operators on LCA groups may be desribed by means of "generalised difference spaces" on these groups. For example,
THEOREM (S. Okada & RN, 1997).*If T is a bounded multiplier operator on $L^2(G)$ with multiplier ϕ, there is a family of pseudomeasures $S = \{\mu_a : a \in \mathbb{R}\}$ on G such that:*

1. *$\mu_a * \mu_b = \mu_{a+b}$ for all $a, b \in \mathbb{R}$;*

2. *the range of T is the difference space $\mathcal{D}(L^2(G), S)$ and this space is a Hilbert space in the inner product $\langle \cdot, \cdot \rangle$ given by*

$$\langle f, g \rangle = \int_{\widehat{G}} \widehat{f} \ \overline{\widehat{g}} \ (1 + |\phi|^{-2}) \ d\mu_{\widehat{G}},$$

   for all f, g in the range of T; and

3. *for each f in the range of T, there are $a_1, a_2, a_3 \in \mathbb{R}$ and $f_1, f_2, f_3 \in L^2(G)$ such that $f = \sum_{j=1}^{3} (f_j - \mu_{a_j} * f_j)$.*

Results with S. Okada have also been obtained which extend this sort of result to unbounded multiplier operators.

7 Singular Integral Operators

7.1 The Riesz potential operators.

Let $n, s \in \mathbb{N}$ with $0 < s < n/2$. The Riesz potential operator I_s of order s is given by $I_s(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} dy$, for $x \in \mathbb{R}^n$, and $I_s(f)^\wedge(x) = \widehat{f}(x)/|x|^{n-s}$ (Stein, Singular Intergals &...). The Sobolev space $W^s(L^2(\mathbb{R}^n))$ of order s on $\mathbb{R}^n$ consists of all functions $f \in L^2(\mathbb{R}^n)$ such that

$$||f|| = \left(\int_{\mathbb{R}^n} \left(1 + |\widehat{f}(x)|^2 |x|^{2s} \right) dx \right)^{1/2} < \infty,$$

and it is Hilbert in this norm $|| \cdot ||$. The Laplace operator Δ is given by $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$.

THEOREM. *The operator $|\Delta|^{s/2}$ is an isometry from $W^s(L^2(\mathbb{R}^n))$ onto the difference space $\mathcal{D}_s(L^2(\mathbb{R}^n))$, and its inverse is the Riesz potential operator of order s. Also, $\mathcal{D}_s(L^2(\mathbb{R}^n))$ consists of the functions f in $L^2(\mathbb{R}^n)$ such that $I_s(f) \in L^2(\mathbb{R}^n)$.*

7.2 The Hilbert transform & related operators.

The Hilbert transform on $L^2(\mathbb{R})$ arises from convolution by the kernel $x \longmapsto 1/\pi x$. Now let s be an even non-negative integer, and consider the function

$$K_{s,y} : x \longmapsto \frac{1}{\pi x \prod_{k=1}^{s/2} (x^2 - k^2 y^2)}.$$

Owing to the identity

$$\sum_{k=0}^{s} \binom{s}{k} \frac{(-1)^k}{x - ky} = \frac{(-1)^s s! y^s}{\prod_{k=0}^{s} (x - ky)},$$

convolution by $K_{s,y}$ defines a bounded operator $H_{s,y}$ on $L^2(\mathbb{R})$ in the same way as the Hilbert transform. In fact the Hilbert transform is the case $s = 0$.

THEOREM. *Let $y \in \mathbb{R}, y \neq 0$. The operator $H_{2,y}$ on $L^2(\mathbb{R})$ is given by convolution by the kernel $x \longmapsto 1/\pi x(x^2 - y^2)$. This operator has multiplier $x \longmapsto -2iy^{-2} sign(x) \sin^2(xy/2)$. The range of this operator consists of all functions in $L^2(\mathbb{R})$ which can be expressed in the form $g - 2^{-1}(\delta_y + \delta_{-y}) * g$ for some $g \in L^2(\mathbb{R})$. That is, the range is the intersection of the kernels of all the $\{\delta_y + \delta_{-y})/2\}$-invariant linear forms on $L^2(\mathbb{R})$.*

Whereas this result describes the range of $H_{2,y}$ in terms of certain second order differences, convolution by the kernel $x \longmapsto 1/\pi(x^2 - y^2)$ has a range which can be expressed in terms of *first* order differences.

8 Wavelets

Let $\mathbb{R}^*$ denote the non-zero real numbers. If $h \in L^2(\mathbb{R})$, we define a function U_h from $L^2(\mathbb{R})$ into the functions on $\mathbb{R} \times \mathbb{R}^*$ by

$$U_h(f)(a,b) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x)\, \overline{h\left(\frac{x-b}{a}\right)}\, dx,$$

for all $f \in L^2(\mathbb{R})$ and all $a \in \mathbb{R}, b \in \mathbb{R}^*$. The function U_h is linear and is called the *wavelet transform* with *wavelet h*. A standard identity in the theory of the wavelet transform, which is analogous to the Plancherel Theorem, is

$$\int_{-\infty}^{\infty}\int_{\infty}^{\infty} \frac{|U_h(f)(a,b)|^2}{|a|^2}\, dadb = \left(\int_{-\infty}^{\infty} \frac{|\widehat{h}(x)|^2}{|x|} dx\right) \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right).$$

This singles out the wavelets h which have the property that

$$\int_{-\infty}^{\infty} |\widehat{h}(x)|^2 |x|^{-1} dx < \infty,$$

and such wavelets are called *admissible*, in which case, the wavelet transform maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R} \times \mathbb{R}^*, |a|^{-2} dadb)$. It is clear from what has been said earlier that *h is admissible* $\Longleftrightarrow h \in \mathcal{D}_{1/2}(L^2(\mathbb{R}))$. Equivalently, h is admissible $\Longleftrightarrow$ there is $g \in W^{1/2}(L^2(\mathbb{R}))$ such that $|D|^{1/2}(g) = h$. Alternatively, if we think of a TILF as being a "1-invariant linear form", h is an admissible wavelet $\Longleftrightarrow$ for every 1/2-ILF, L say, $L(h) = 0$.

9 Conclusion

Results of Willis–& earlier Meisters. Takahashi

10 Some Notation

$\widehat{G}$ = dual group of G.
$\widehat{\mathbb{R}^n} = \mathbb{R}^n, \quad \widehat{\mathbb{T}} = \mathbb{Z}, \; \widehat{\mathbb{Z}} = \mathbb{T}$.

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