An application of functions of the form $(ax + b)/(cx + d)$ to modelling a problem of resource allocation

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ABSTRACT

In 1998, the West report on higher education considered possible changes to the allocation of monies to universities on the basis of research and teaching. The underlying problem is one of allocation of resources in general, and is to mathematically model the resource allocation process, and analyze the effects if the weighting between various criteria in the process are changed. Functions given by an expression $(ax+b)/(cx+d)$ arise in this analysis.

It is hoped that this talk (i) may indicate how mathematics may arise in public discussion in a hidden way, and (ii) show the potential of school-level mathematics to contribute to obtaining broad qualitative insights into practical questions of public policy.

AIMS

- Model a resource allocation problem

- Investigate how recipients are affected when the relative weighting of criteria is changed

- Point out that the ideas can apply to rating performance against given criteria, instead of allocation of funds

- Increase awareness of the possibilities for applying elementary mathematics to current public policy issues

- Show how mathematical *thinking* can be applied to problems, as distinct from regarding mathematics primarily as calculation

Background to the problem

An allocating agency has a fixed amount of “money” or ”recognition” which it splits up among recipients according to given criteria.

For each criterion, a definite sum is set aside to be allocated amongst the recipients on the basis of how each recipient meets that critierion. The total amount set aside for allocation according to a given criterion reflects the importance the agency places upon that criterion in relation to the other criteria.

The recipients may have strengths regarding some criteria and have weaknesses regarding others, and these areas of strength and weakness may vary from one recipient to another.

One day, the funding agency decides to alter the relative importance it places upon the criteria—some criteria are to be increased in importance, and others are to be decreased. However, the criteria themselves do not change.

So, in relative terms, more funding is to be allocated on the basis of some of the criteria and less on the basis of some of the other criteria.

Also, the funding agency considers varying the total amount of funds it allocates to the recipients.

Statement of the problem

The following questions then arise.

(i) How does the total funding for each recipient change?

(ii) How does the proportional funding for each recipient change?

(iii) Does the perceived status of a recipient change under the new allocation?

(iv) If the total effect of the changes is considered to have had unforeseen and undesirable consequences, is there a different or fairer method of allocation which is more suited to achieving desired outcomes?

We would expect, that if a recipient is very strong in relation to a criterion to which more funds are to be allocated, then that recipient will receive more total funds if the other factors remain constant.

But what is the **precise** manner in which the total funding to the recipients depends upon the change to placing more relative importance upon a particular criterion?

If we **do** know the precise manner in which funding to the recipients depends upon the relative importance of the criteria and the total funding available, can we decide how much variation in outcomes can be achieved by varying the parameters of the process?

A specific case

The above situation arose in 1998 when, in the West Report on tertiary education, options were proposed for changing the proportion of funds given to universities on the basis of **two** criteria: research and teaching.

An article by David Phillips, a former Head of the Higher Education Division of the Department of Employment, Education, Training and Youth Affairs, on the consequences of implementing these options, appeared in *The Australian* of July 1st 1998.

Phillips considered the implications of increasing the total amount allocated to universites for research (called the "research quantum").

Assuming that the funding for both teaching and research purposes to the totality of universities remained constant, this would have meant the total allocation to the universities for teaching would have had to decrease.

Phillips calculated the effects upon the total allocations to 36 universities for every 1% increase in the research quantum, pointing out that the effects could be significant.

For example, for every 1% increase in the research quantum, he calculated that Melbourne University would gain $3.2 million and the University of Western Sydney would lose $1.6 million.

Analysis of the problem

In the analysis, we consider the allocation of funds to a number n of recipients subject to **two** criteria, denoted by X and Y.

The changes in comparative allocations are described by what are known mathematically as "linear transformations" and ratios of such transformations. These give an exact form to what happens under any change in the parameters of this procedure of allocating funds.

Mathematical formulation of the problem

Currently, assume under criterion X that the total current allocation is A, and that under criterion Y the total current allocation is B. Then, if T is the total current allocation under both criteria,

$$T = A + B. \tag{1}$$

Put

$$\rho = \frac{A}{B}. \tag{2}$$

ρ measures the relative importance of criteria X and Y in the mind of the allocator.

In the current allocation, assume that recipient j receives

a_j under criterion X and b_j under criterion Y.

Then,

$$A = \sum_{j=1}^{n} a_j \quad \text{and} \quad B = \sum_{j=1}^{n} b_j. \tag{3}$$

Put

$$\rho_j = \frac{a_j}{b_j}, \tag{4}$$

for $j = 1, 2, \ldots, n$.

Then, ρ_j measures the extent to which recipient j meets criterion X compared with criterion Y, under the current allocation.

Lemma 1

EITHER $\rho_1 = \rho_2 = \cdots = \rho_n = \rho$,

OR there are j, k **such that**

$$\rho_j < \rho < \rho_k.$$

$$\rho = \frac{A}{B}$$

$$\rho_j = \frac{a_j}{b_j}$$

	current allocation	
recipient	X	Y
1	a_1	b_1
2	a_2	b_2
⋮	⋮	⋮
n	a_n	b_n

sum	sum
A	B

$$A+B=T$$

Now, the allocator is proposing to allocate a total amount T' instead of T, A' in place of A under criterion X, and B' in place of B under criterion Y. Then, there is $\eta > 0$ such that

$$T' = \eta T. \tag{5}$$

If $\eta > 1$, the total funding is increased, if $\eta < 1$ the total funding is decreased, if $\eta = 1$, $T' = T$ and there is no change in the total funding. Also

$$T' = A' + B'. \tag{6}$$

Let

$a'_j =$ new amount under X for recipient j,

$b'_j =$ new amount under Y for recipient j.

Then,

$$A' = \sum_{j=1}^{n} a'_j, \quad B' = \sum_{j=1}^{n} b'_j.$$

$$\rho = \frac{A}{B}$$

$$\rho_j = \frac{a_j}{b_j}$$

	current allocation		proposed allocation	
recipient	X	Y	X	Y
1	a_1	b_1	a'_1	b'_1
2	a_2	b_2	a'_2	b'_2
⋮	⋮	⋮	⋮	⋮
n	a_n	b_n	a'_n	b'_n
	sum	sum	sum	sum
	A	B	A'	B'

$$A + B = T \qquad A' + B' = T'$$

$$T' = \eta T$$

Now in the new allocations, only the **relative balance** between X and Y is changed. Thus, there are $\theta, \phi > 0$ such that

$$A' = \theta A, \quad a'_j = \theta a_j, \quad B' = \phi B, \quad b'_j = \phi b_j, \tag{7}$$

for all $j = 1, 2, \ldots, n$. We have

$$\theta A + \phi B = A' + B' = \eta T = \eta A + \eta B. \tag{8}$$

We have from (8),

$$\begin{aligned} \phi &= \frac{\eta A + \eta B - \theta A}{B} \\ &= \eta \frac{A}{B} + \eta - \theta \frac{A}{B} \\ &= \eta\rho + \eta - \theta\rho. \end{aligned} \tag{9}$$

We have from (7), (8) and (9),

$$B' = \eta T - \theta A, \quad b'_j = (\eta\rho + \eta - \theta\rho) b_j, \quad (10)$$

for $j = 1, 2, \ldots, n$.

We now consider how funding to recipients changes under the new allocation.

The **absolute change for recipient j** is

$$\begin{aligned}
& a'_j + b'_j - a_j - b_j \\
= \; & \theta a_j + \phi b_j - a_j - b_j \\
= \; & \theta a_j + (\eta\rho + \eta - \theta\rho) b_j - a_j - b_j \\
= \; & (\theta - 1) a_j + (\eta\rho + \eta - \theta\rho - 1) b_j \\
= \; & b_j \left((\theta - 1) \frac{a_j}{b_j} + \eta\rho + \eta - \theta\rho - 1 \right) \\
= \; & b_j \Big((\theta - 1)\rho_j - (\theta - 1)\rho + \eta\rho + \eta - \rho - 1 \Big) \\
= \; & b_j \Big((\theta - 1)\rho_j - (\theta - 1)\rho + (\eta - 1)\rho + \eta - 1 \Big) \\
= \; & b_j \Big((\theta - 1)(\rho_j - \rho) + (\eta - 1)(\rho + 1) \Big).
\end{aligned} \tag{11}$$

THE CASE $\eta = 1$ and $\theta > 1$

That is $T' = T$ and there is no change in the total funding. If the total funding is the same, and more is allocated under X, then less must be allocated under Y. This is expressed by the form taken by the equation (8), which becomes

$$(\theta - 1)A = -(\phi - 1)B.$$

The **absolute change** for recipient j is from (11)

$$a'_j + b'_j - a_j - b_j \;=\; b_j(\theta - 1)\left(\rho_j - \rho\right). \quad (12)$$

As $\theta > 1$, we see that recipient j will get an increase if $\rho_j > \rho$, otherwise there is a decrease.

Also, given b_j and given θ, the absolute change for recipient j is proportional to $\rho_j - \rho$.

Also, given θ and given ρ_j, the absolute change for recipient j is proportional to b_j.

We have from (12)

Theorem 1 **Assume that $\eta = 1$ and $\theta > 1$. Then the following hold.**

(1) Recipient j receives an increase when $\rho_j > \rho$, and a decrease when $\rho_j < \rho$.

(2) Given ρ_j, this increase or decrease for recipient j is proportional to b_j, the amount allocated to recipient j under Y.

(3)At least one recipient will receive an increase and at least one recipient will receive a decrease.

The Theorem shows that recipients who are stronger in criterion X, which is to be given more importance, are rewarded, while the others are "punished".

This is to be expected, because when the total amount of funds stays the same, the existence of "winners" means that there must also be "losers". This is not necessarily the case if the total funds are increased.

Also, the **proportional change P_j for recipient j** is

$$
\begin{aligned}
P_j &= \frac{a_j' + b_j' - a_j - b_j}{a_j + b_j} \\
&= \frac{(\theta - 1)a_j + (\rho - \theta\rho)b_j}{a_j + b_j} \\
&= \frac{(\theta - 1)\rho_j + \rho(1 - \theta)}{\rho_j + 1} \\
&= (\theta - 1)\left(\frac{\rho_j - \rho}{\rho_j + 1}\right). \qquad (13)
\end{aligned}
$$

How does P_j vary with j?

Theorem 2 Put

$$P(x) = (\theta - 1)\left(\frac{x-\rho}{x+1}\right).$$

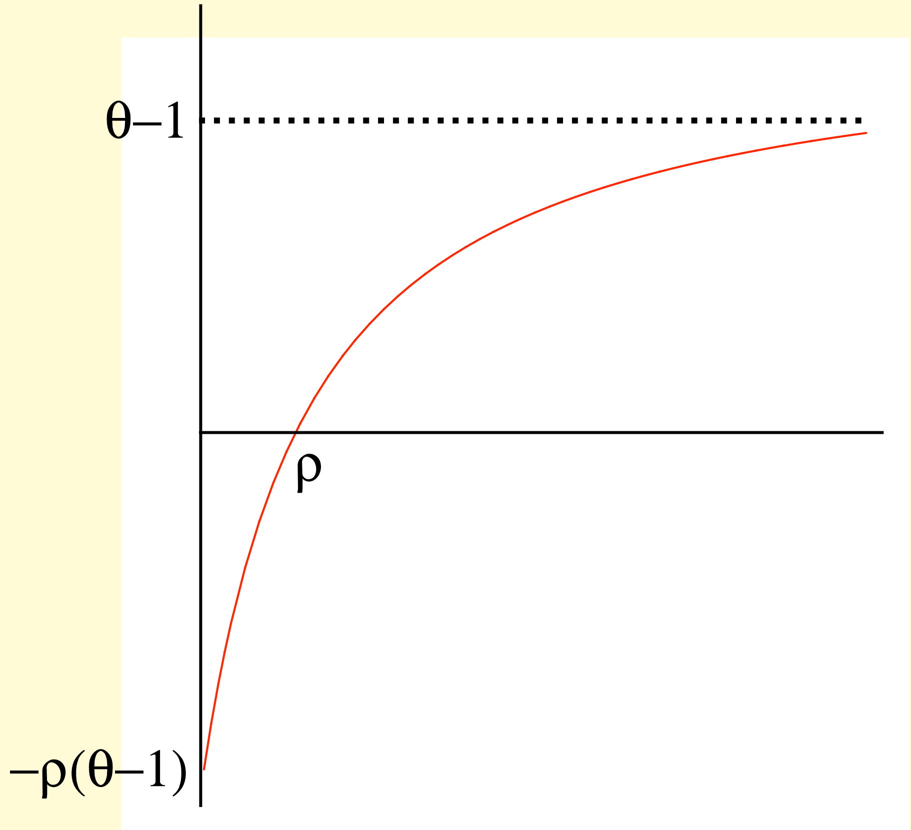
Then, for $j = 1, 2, \ldots, n$,

proportional change for recipient $j = P_j = P(\rho_j)$.

PROOF. Immediate from (13). $\square$

Thus, how P_j varies with j becomes a question of how $P(x)$ varies with x.

Note that $P(x)$ is of the form $(ax+b)/(cx+d)$.



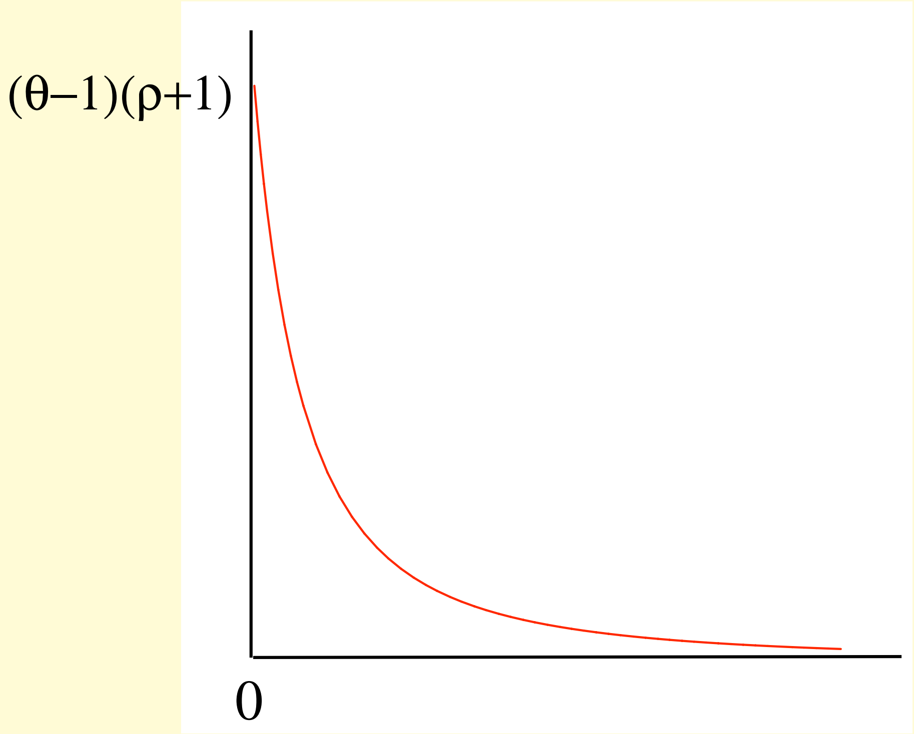
The graph is of the function P where

$$P(x) = (\theta - 1)\left(\frac{x-\rho}{x+1}\right).$$

P increases more rapidly for small values of x.

Using the quotient rule we find that

$$P'(x) = \frac{(\theta - 1)(\rho + 1)}{(x + 1)^2}.$$



The graph is of the function P' where

$$P'(x) = \frac{(\theta - 1)(\rho + 1)}{(1 + x)^2}.$$

P' decreases in x means that $P(x)$ changes more rapidly for small x.

The effect of the method of allocation when $\eta = 1$, $\theta > 1$

The proposed change in allocation method gives absolute priority to X over Y. That is, those recipients who are "above average" on X are rewarded, those who are "below average" on X are punished. In terms of incentives, the new method of allocation encourages everyone to become better than the others at X. The method would produce more **similarity** of objectives amongst recipients, rather than promote **diversity** of objectives.

THE CASE $\eta \neq 1$ and $\theta \neq 1$

When $\eta = 1$, the total funds remain the same, and some recipients will get an increase and some a decrease. This may be politically unacceptable if, for example, those getting a decrease are in marginal seats.

A government in these circumstances might consider increasing the funds, so that all recipients get an increase under the new allocation method. This means, in our terms, that $\eta > 1$.

The question is: what is the minimum amount of total funds available that would ensure that all recipients get an increase?

In the case when η is not necessarily 1, we get from (11) that

$$a'_j+b'_j-a_j-b_j = b_j(\theta-1)\left(\rho_j - \rho + (\rho+1)\cdot\left(\frac{\eta-1}{\theta-1}\right)\right).$$

Thus when $\eta > 1$ and $\theta > 1$, the condition for the funds to recipient j to increase is

$$\rho_j > \rho - (\rho+1)\left(\frac{\eta-1}{\theta-1}\right). \qquad (14)$$

Put

$$\sigma = \min\{\rho_1, \rho_2, \ldots, \rho_n\}.$$

Then, the condition for **every** recipient to get an increase is, from (14),

$$\sigma > \rho - (\rho+1)\left(\frac{\eta-1}{\theta-1}\right).$$

Equivalently, the condition for **every** recipient to get an increase is,

$$\sigma > \rho - (\rho + 1)\left(\frac{\eta - 1}{\theta - 1}\right).$$

That is, every recipient will get an increase precisely when

$$\eta > \frac{1}{1+\rho}\Big\{\sigma + 1 + \theta(\rho - \sigma)\Big\},$$

or equivalently when

$$\eta > \frac{\theta\rho + [1 + \sigma(\theta - 1)]}{\rho + 1}.$$

Theorem 3 **Assume that $\theta > 1$. Then all recipients will receive an increase under the changed allocation precisely when**

$$\eta > \frac{\theta\rho + [1 + \sigma(\theta - 1)]}{\rho + 1}. \qquad (15)$$

Note that (15) tells us by how much the total funds must be increased to ensure that every recipient gets an increase.

Note also that the right hand side of (15) is of the form

$$\frac{a\rho + b}{c\rho + d}.$$

Theorem 4 **Put, for $j = 1, 2, \ldots, n$,**

$$\rho'_j = \frac{a'_j}{b'_j}.$$

Then, $\frac{\rho'_j}{\rho_j}$ has the same value for all j. In fact

$$\frac{\rho'_j}{\rho_j} = 1 + \frac{(\theta - \eta)(\rho + 1)}{\eta\rho + \eta - \theta\rho}.$$

Note that again, $\frac{\rho'_j}{\rho_j}$ has the form

$$\frac{a\rho + b}{c\rho + d}.$$

COMMENTS

- The analysis enables us to understand the process as whole, by understanding the changes resulting from changes in the parameters

- The analysis reveals effects of the allocation procedure that were not originally envisaged

- The problem is posed as to devising a procedure that would encourage diversity, if that is indeed an aim of policy

- The analysis can be adapted to other applications, maybe in more complex situations — for example where there are more than 2 criteria

Rod Nillsen, June 2006