Example (a linear function arising in Einstein's special theory of relativity). In system A events are observed to occur at a position x and at a time t. That is, in system A an event is identified with an element $(x,t) \in \mathbb{R}^2$.

In system B events are observed to occur at a position x' and at a time t'. That is, in system B an event is identified with an element $(x',t') \in \mathbb{R}^2$.

Now let c denote the speed of light, and assume that system B is moving with a constant velocity v with respect to system A. According to Einstein's special theory of relativity, $|v| < c$, and if the same event is observed in system A and system B, respectively interpreted as the elements (x,t) and (x',t') of $\mathbb{R}^2$, we have

$$\begin{aligned} x' &= \frac{1}{\sqrt{1-(v/c)^2}}\Big(x - vt\Big), \text{ and} \\ t' &= \frac{1}{\sqrt{1-(v/c)^2}}\Big(-\frac{v}{c^2}x + t\Big). \end{aligned}$$

The transformation given by these equations, which changes (x,t) into (x',t'), is known as the *Lorentz tranformation* (corresponding to v). The Lorentz transformation maps $\mathbb{R}^2$ into $\mathbb{R}^2$. The Lorentz transformation can be written as

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{pmatrix} 1 & -v \\ \frac{-v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = A(v) \begin{pmatrix} x \\ t \end{pmatrix},$$

where $A(v)$ is the 2×2 matrix

$$A(v) = \frac{1}{\sqrt{1-(v/c)^2}} \begin{pmatrix} 1 & -v \\ & \\ \dfrac{-v}{c^2} & 1 \end{pmatrix}.$$

We deduce that *the Lorentz transformation* is a linear function from $\mathbb{R}^2$ into $\mathbb{R}^2$ whose matrix is $A(v)$. Note that

$$\det(A(v)) = \sqrt{1 - \frac{v^2}{c^2}} \neq 0,$$

so that $A(v)$ is an invertible matrix and the Lorentz transformation is an invertible transformation.

Exercise (optional and not examinable). Prove that if $|v_1| < c$ and $|v_2| < c$, then

$$\left| \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \right| < c$$

and

$$A(v_1)A(v_2) = A\left(\frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}\right).$$

Deduce Einstein's formula $v_1 \oplus v_2$ for the addition of velocities: namely

$$v_1 \oplus v_2 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}},$$

and explain what it means. Also, deduce that $A(-v) = A(v)^{-1}$ and explain what this means.