Example (one-to-one functions). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(x) = e^{-x^2}.$$

Then

$$f(-x) = e^{-x^2} = f(x).$$

As x and $-x$ are distinct unless $x = 0$, we see that x and $-x$ are generally distinct points, so f is not one to one. BUT, if the domain of f is changed to the set where $x \geq 0$, then the restriction of f to $\{x : x \geq 0\}$ is a function g that *is* one-to-one. Also, this restriction $g : [0, \infty) \longrightarrow (0, 1]$ is onto. Show that

$$g^{-1} : (0, 1] \longrightarrow [0, \infty),$$

and check that

$$g^{-1}(x) = \sqrt{-\ln x}.$$

Example (one-to-one functions). Let : $r : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$r(x) = x^2 + 2x + a.$$

(Here, a is a given number.) Prove that

$$f(-h-1) = f(h-1),$$

for all $h \in \mathbb{R}$. This shows that r is not one-to-one. But the restriction of r to $[-1, \infty)$ is one-to-one. For, if $x, y \in [-1, \infty)$ and $r(x) = r(y)$,

$$\begin{aligned} x^2 + 2x + a = y^2 + 2y + a &\implies x^2 - y^2 = 2(y - x) \\ &\implies (x - y)(x + y) = -2(x - y) \\ &\implies x + y = -2 \text{ or } x = y \\ &\implies x = y = -1 \text{ or } x = y, \text{ as } x, y \in [-1, \infty) \\ &\implies x = y. \end{aligned}$$

Thus the restriction of r to $[-1, \infty)$ is one-to-one (and all this can be illustrated graphically). Thus, the restriction of r to $[-1, \infty)$ has an inverse. Maybe calculate this inverse.

Example (composing functions). On the real line let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$f(x) = 2x - 5 \text{ and } g(x) = 3x + 1.$$

Then, $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ and

$$(g \circ f)(x) = g(f(x)) = 3(2x - 5) + 1 = 6x - 14.$$

Example (inverse functions). In effect, the calculation on pages 12-13 of the notes shows that the function $f : (-\infty, 2) \cup (2, \infty) \longrightarrow \mathbb{R}$ given by

$$f(x) = \frac{x-3}{x-2}$$

has an inverse function

$$f^{-1} : (-\infty, 1) \cup (1, \infty) \longrightarrow (-\infty, 2) \cup (2, \infty)$$

given by

$$f^{-1}(y) = \frac{-2y+3}{1-y}.$$

Example (composing functions). Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$f(x,y) = (2x-5, y^2) \text{ and } g(x,y) = (x^2, 3y+1).$$

Then, $g \circ f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and

$$(g \circ f)(x,y) = g(f(x,y)) = g(2x-5, y^2) = ((2x-5)^2, 3y^2+1) = (4x^2-20x+25, 3y^2+1).$$

Example (addition of vectors etc).
If $x = (1,-2,-6) \in \mathbb{R}^3$ and $y = (-3,2,7) \in \mathbb{R}^3$, then $x+y \in \mathbb{R}^3$ and

$$x+y = (-2,0,1).$$

Also, $5x \in \mathbb{R}^3$ and

$$5x = (5,-10,-30).$$

Example (a non-linear function). Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$f(x,y) = (x^2, 3x+2y).$$

Then f is non-linear. For, if $X = (x,y)$ and $Y = (u,v)$ we have

$$\begin{aligned} f(X+Y) &= f(x+u, y+v) \\ &= ((x+u)^2, 3(x+u)+2(y+v)) \\ &= (x^2+u^2+2xu, 3x+2y+3u+2v) \\ &= (x^2, 3x+2y) + (u^2, 3u+2v) + (2xu, 0) \\ &= f(X) + f(Y) + (2xu, 0) \\ &\neq f(X) + f(Y), \end{aligned}$$

if $xu \neq 0$. So, as $f(X+Y) \neq f(X) + f(Y)$ for many cases of X, Y, we see that f is not linear.