Family Name ....
First Name ....
Student Number ....
Signature ....

UNIVERSITY OF WOLLONGONG

SCHOOL OF MATHEMATICS AND APPLIED STATISTICS

MATH201 : Multivariate and Vector Calculus

Autumn Session Examination 2007

Time Allowed: 3 hours 15 minutes
Number of Questions: 6

DIRECTIONS TO CANDIDATES

1. Each question is to be attempted.
2. The questions are of equal value.
3. Examination paper is printed on both sides.
4. All notation is as given in lectures.
5. All necessary working is to be shown.

EXAMINATION MATERIALS/AIDS ALLOWED

Non-alphanumeric calculators are permitted

EXAMINATION MATERIALS/AIDS TO BE SUPPLIED

None

THIS EXAMINATION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Question 1

(a) Write down definitions of the following.

(i) The inner product $\langle x, y\rangle$ of the vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$.

(ii) The length $|x|$ of a vector x in $\mathbb{R}^n$.

(iii) The cross product $x \times y$ of two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3$.

(iv) A linear function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

(b) Calculate the equation of the plane that goes through the points $(-1, 2, 3)$, $(1, 1, 2)$ and $(2, 1, 2)$, and show that for any number a, the vector $(a, 2, 3)$ lies in this plane.

(c) Consider the equation

$$4x^2 + 9y^2 - z = 0.$$

The equation describes a surface S in $\mathbb{R}^3$.

(i) Identify the type of surface that S is.

(ii) Assuming $h > 0$, show that the plane $z = h$ intersects the surface S in an ellipse, and calculate the semi-axes of the ellipse.

(iii) Sketch a picture of the surface S.

Question 2

(a) Let x be a given point in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a given function. Write down the definition of when f is differentiable at the point x.

(b) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a given function. Write down the definition of the partial derivative $D_1 f(x, y)$, also denoted by f_x and $\frac{\partial f}{\partial x}$.

(c) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a given function, whose value at (x, y) is denoted by $f(x, y)$. Consider the change of variables in $\mathbb{R}^2$ given by

$$u = x^2 - y^2 \quad \text{and} \quad v = x^2 + y^2.$$

(i) Use the Chain Rule to prove that

$$f_x = 2xf_u + 2xf_v \quad \text{and} \quad f_y = -2yf_u + 2yf_v.$$

(ii) Using (i) and the Chain Rule, show that

$$f_{xx} = 2f_u + 2f_v + 8x^2 f_{uv} + 4x^2(f_{uu} + f_{vv}).$$

Question 3

Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$g(x, y) = (x + y^2, -x + y^2).$$

(a) Write down the coordinate functions of g.
(b) Write down the derivative matrix $g'(x, y)$ of g at the point (x, y).
(c) Calculate the Jacobian $J(g)(x, y)$ of g at (x, y).
(d) Sketch the region in $\mathbb{R}^2$ corresponding to the integral

$$\int_0^1 \left(\int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \right) dx.$$

(e) Sketch the region in $\mathbb{R}^2$ corresponding to the integral

$$\int_0^1 \left(\int_{-u}^{u} e^{v/u} \, dv \right) du.$$

(f) By using the substitutions

$$u = x + y^2 \text{ and } v = -x + y^2,$$

and by calculating the appropriate Jacobian, prove that

$$\int_0^1 \int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \, dx = \frac{1}{4} \int_0^1 \int_{-u}^{u} e^{v/u} \, dv \, du.$$

(g) By using (f), or otherwise, deduce the value of

$$\int_0^1 \int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \, dx.$$

Question 4

(a) State Green's Theorem.
(b) Let C be the closed curve which is the circle of centre 0 and radius 1 that is traversed anticlockwise. Using Green's Theorem, or otherwise, calculate

$$\oint_C (x^3 + y)dx + (x^2 - y^3)dy.$$

(c) Describe an underlying reason which ensures that for a vector field F in $\mathbb{R}^2$,

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

whenever C_1 and C_2 are two curves with the same starting points and the same end points.

(d) Let F be the vector field in $\mathbb{R}^2$ given by $F(x, y) = (M(x, y), N(x, y))$, where

$$M(x, y) = 2x - 6y \quad \text{and} \quad N(x, y) = -6x + 6y.$$

(i) Prove that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

(ii) Find a function $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\frac{\partial \phi}{\partial x} = M \text{ and } \frac{\partial \phi}{\partial y} = N.$$

Question 5

(a) Write down the definition of a vector field on $\mathbb{R}^n$.

(b) State the Divergence Theorem, also known as Gauss' Theorem.

(c) Let F be the vector field on $\mathbb{R}^3$ given by

$$F(x, y, z) = (2x + 3y, z, 3y + z).$$

(i) Calculate the divergence of F and the curl of F.

(ii) Let V be the region in $\mathbb{R}^3$ enclosed by the plane $x + 2y + z = 3$ and the XY, YZ and XZ planes. Let S denote the closed surface that is the boundary of this region V. Sketch a picture of V and S. Then, using the Divergence Theorem, or otherwise, calculate

$$\oiint_S \langle F, n \rangle \, dS.$$

Question 6

(a) Write down a statement of Stokes' Theorem.

(b) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$f(x, y) = xe^{-x^2 - y^2}.$$

Calculate all critical points of f and find all relative and absolute maxima and minima of f, giving reasons.

(c) Using the method of Lagrange multipliers, calculate all points (x, y, z) such that $x^4 y^6 z^2$ has a maximum or a minimum subject to the constraint that $x^2 + y^2 + z^2 = 1$.

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