MATH 201 Autumn 2008

Example illustrating Stokes' Theorem

Let S be the set

$$S = \Big\{(x, y, z) : z \geq 0 \text{ and } x^2 + y^2 + z^2 = 1\Big\}.$$

That is, S is a hemisphere, the upper part of the sphere of centre 0 and radius 1 in $\mathbb{R}^3$. The boundary of S is the circle of centre 0 and radius 1 in the $X - Y$ plane; that is the set

$$\Big\{(x, y, 0) : x^2 + y^2 = 1\Big\}.$$

Then the boundary is given by the curve C where

$$C(t) = (\cos t, \sin t, 0), \text{ for } 0 \leq t \leq 2\pi.$$

We have

$$C'(t) = (-\sin t, \cos t, 0), \text{ for } 0 \leq t \leq 2\pi.$$

Let F be the vector field on $\mathbb{R}^3$ given by

$$F(x, y, z) = (-y, x, 1), \text{ for } (x, y, z) \in \mathbb{R}^3.$$

We have

$$\text{curl}F(x, y, z) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 1 \end{pmatrix} = 2e_3 = (0, 0, 2).$$

Stokes Theorem says that

$$\int_C \langle F, C' \rangle = \int\int_S \langle \text{curl}F, n \rangle \, dS.$$

We will calculate form the definitions the integrals on each side of this equation, verifying Stokes' Theorem in this case.

NOW

$$\begin{aligned} \int_C \langle F, C' \rangle &= \int_0^{2\pi} \langle F(C(t), C'(t) \rangle \, dt \\ &= \int_0^{2\pi} \langle (-\sin t, \cos t, 1), (-\sin t, \cos t, 0) \rangle \, dt \\ &= \int_0^{2\pi} ((-\sin t)^2 + \cos^2 t) \, dt \\ &= \int_0^{2\pi} 1 \, dt \\ &= 2\pi. \end{aligned} \qquad (*)$$

WE USE using the formula from lectures for calculating surface integrals – that is,

$$\int_S \phi \, dS = \int \int_D \phi(x, y, h(x, y)) \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \, dxdy,$$

where

$$D = \{(x, y, 0) : x^2 + y^2 \leq 1\} \text{ and } h(x, y) = \sqrt{1 - x^2 - y^2}.$$

WE HAVE

$$\int \int_S \langle \text{curl } F, n\rangle \, dS$$

$$= \int \int_{x^2+y^2 \leq 1} \langle (0, 0, 2), (x, y, \sqrt{1 - x^2 - y^2})\rangle \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \, dxdy$$

$$= 2 \int \int_{x^2+y^2 \leq 1} \sqrt{1 - x^2 - y^2} \sqrt{1 + \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\right)^2 + \left(\frac{y}{\sqrt{1 - x^2 - y^2}}\right)^2} dxdy$$

$$= 2 \int \int_{x^2+y^2 \leq 1} \sqrt{1 - x^2 - y^2} \, \frac{1}{\left(\sqrt{1 - x^2 - y^2}\right)^2} dxdy$$

$$= 2 \int \int_{x^2_+ y^2 \leq 1} 1 \, dxdy$$

$$= 2\pi. \qquad (**)$$

From $(*)$ and $(**)$ we have that in this example

$$\int_C \langle F, C'\rangle = \int \int_S \langle \text{curl } F, n\rangle \, dS,$$

so we have verified Stokes' Theorem in this case, by calculating directly from the definitions involved the values of the integrals.