MATH201: Multivariate and Vector Calculus

Assignment Session 1 2009

The assignment is due by 5pm, Wednesday 27th May in week 12.

The assignment should be placed in the MATH201 assignment box in the northern wing of building 15, near room 15.107.

Family Name ....
First Name ....
Student Number ....
Signature ....

Question 1. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$g(x,y) = (x+y^2, -x+y^2).$$

(a) Calculate the Jacobian $J(g)(x,y)$ of g at (x,y).

(b) Sketch the region A in $\mathbb{R}^2$ corresponding to the integral

$$\int_0^1 \left(\int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \right) dx.$$

(c) Sketch the region B in $\mathbb{R}^2$ corresponding to the integral

$$\int_0^1 \left(\int_{-u}^{u} e^{v/u} \, dv \right) du.$$

(d) Prove that g maps A onto B.

(e) By using the substitutions

$$u = x + y^2 \text{ and } v = -x + y^2,$$

and by using the result in (a) above, prove that

$$\int_0^1 \int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \, dx = \frac{1}{4} \int_0^1 \int_{-u}^{u} e^{v/u} \, dv \, du.$$

(f) By using (e), or otherwise, deduce the value of

$$\int_0^1 \int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \, dx.$$

Question 2. Let C be the curve whose range is the line segment proceeding from $(1,1)$ to $(2,3)$. Let F be the vector field on $\mathbb{R}^2$ given by

$$F(x,y) = (x^2 - y, 2x + 3y).$$

(a) Find functions x and y and an interval $[a, b]$ such that the curve C is given by

$$C(t) = (x(t), y(t)), \text{ for } a \leq t \leq b.$$

(b) Calculate the value of

$$\int_C F \cdot dr.$$

Question 3. Let C_1, C_2 be the curves in $\mathbb{R}^2$ given by

$$C_1(t) = (t, t), \text{ for } 0 \leq t \leq 1,$$

and

$$C_2(t) = (t, t^2), \text{for } 0 \leq t \leq 1.$$

Let $F = (M, N)$ be a vector field on $\mathbb{R}^2$.

(a) Sketch a picture of C_1 and C_2.

(b) Calculate an expression for

$$\int_{C_1} Mdx + Ndy \text{ in terms of an integral over } [0, 1].$$

Note that because M, N are general functions and we are not told what they are, the expression will involve M, N.

(c) Calculate an expression for

$$\int_{C_2} Mdx + Ndy \text{ in terms of an integral over } [0, 1].$$

(d) Let C be the curve obtained by proceeding along C_2 from $(0, 0)$ to $(1, 1)$ and then proceeding along C_1 (but in the opposite direction) from $(1, 1)$ to $(0, 0)$. NOTE that C is a closed curve and goes in an anti-clockwise direction. Using (b), (c) above, calculate an expression for

$$\oint_C Mdx + Ndy.$$

(e) Let R be the region inside the curve C. Calculate an expression for

$$\int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

(f) Deduce from (d) and (e) that

$$\oint_C Mdx + Ndy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

That is, deduce Green's Theorem for the curve C and the region R. [NOTE: to prove this you will need to get *every detail* correct. Compare with the proof of Green's Theorem for a rectangle given in the notes. You may also need to "brush up" on your integration by substitution from first year.]

Question 4. For $(x, y) \in \mathbb{R}^2$, put

$$P(x, y) = 3x^2 - 8xy + y^3, Q(x, y) = -4x^2 + 3xy^2.$$

Let F be the vector field on $\mathbb{R}^2$ given by $F = (P, Q)$. Prove that F is exact and find a function $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\text{grad}\phi = F.$$

Then, calculate div$F$.