MATH201: Multivariate and Vector Calculus

Assignment Session 1 2008

The assignment is due by 5pm, Monday in week 12, May19th.
The assignment should be placed in the MATH201 assignment box in the northern wing of building 15, near room 15.107.

Family Name ....
First Name ....
Student Number ....
Signature ....

Question 1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function given by

$$f(x, y) = x^2y^2 + (2x + y)^3.$$

(i) Calculate $D_1f(x, y)$ and $D_2f(x, y)$.

(ii) If $u = (1/\sqrt{2}, -1/\sqrt{2})$, prove that u is a unit vector and calculate the directional derivative $D_uf(1, 1)$.

(iii) Describe all unit vectors $v \in \mathbb{R}^2$ such that $D_vf(1, 1) = 0$.

Question 2. For $(x, y) \in \mathbb{R}^2$, put

$$P(x, y) = 3x^2 - 8xy + y^3, Q(x, y) = -4x^2 + 3xy^2.$$

Let F be the vector field on $\mathbb{R}^2$ given by $F = (P, Q)$. Prove that F is exact and find a function $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\text{grad}\phi = F.$$

Then, calculate $\text{div}F$.

Question 3. Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a given continuously differentiable function. Let $u, v \in \mathbb{R}^n$, let $[a, b]$ be an interval and let

$$r = (r_1, r_2, \ldots, r_n); [a, b] \longrightarrow \mathbb{R}^n$$

be a curve C such that $r(a) = u$ and $r(b) = v$. Note that $\phi \circ r : [a, b] \longrightarrow \mathbb{R}$.

(i) Explain why, for all $t \in [a, b]$,

$$(\phi \circ r)'(t) = \sum_{j=1}^{n} D_j\phi(r(t))r_j'(t).$$

That is, carefully stating any results you are using, explain why,

$$(\phi \circ r)'(t) = \sum_{j=1}^{n} \frac{\partial \phi}{\partial x_j}(r(t))r_j'(t).$$

(ii) Prove that

$$\int_C \text{grad}\phi \cdot dr = \phi(v) - \phi(u).$$

Question 4

Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

$$g(x, y) = (x + y^2, -x + y^2).$$

(a) Calculate the Jacobian $J(g)(x, y)$ of g at (x, y).

(b) Sketch the region A in $\mathbb{R}^2$ corresponding to the integral

$$\int_0^1 \left(\int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \right) dx.$$

(c) Sketch the region B in $\mathbb{R}^2$ corresponding to the integral

$$\int_0^1 \left(\int_{-u}^{u} e^{v/u} \, dv \right) du.$$

(d) Prove that g maps A onto B.

(e) By using the substitutions

$$u = x + y^2 \text{ and } v = -x + y^2,$$

and by using the result in (a) above, prove that

$$\int_0^1 \int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \, dx = \frac{1}{4} \int_0^1 \int_{-u}^{u} e^{v/u} \, dv \, du.$$

(f) By using (e), or otherwise, deduce the value of

$$\int_0^1 \int_0^{\sqrt{1-x}} y e^{\frac{-x+y^2}{x+y^2}} \, dy \, dx.$$

Question 5. Let A be the inside and boundary of the triangle in $\mathbb{R}^2$ whose vertices are $(0, 0)$, $(1, -1)$ and $(1, 1)$. Let C be the curve obtained by proceeding around the boundary of A in an anti-clockwise direction. Prove that

$$\int_C P dx + Q dy = \int \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

That is, prove Green's Theorem for the triangle A. [**Hint:** the lecture notes have a proof for when A is a rectangle. So, the idea is is to give a similar proof where we have this triangle A in place of a rectangle.]