

Partial derivatives of inverse functions – the classical approach
This discussion helps you to complete Problem 33 in Exercise 4.10

Given the function $(x, y) \mapsto z = f(x, y)$ and the transformation from \mathbb{R}^2 to \mathbb{R}^2 given by $(u, v) \mapsto (x, y)$, where

$$x = x(u, v) \text{ and } y = y(u, v), \quad (1)$$

so that $z = f(x(u, v), y(u, v))$, the Chain Rule may be used either as

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \quad \text{and} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \end{aligned}$$

or

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \quad \text{and} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}. \end{aligned}$$

For the first pair, partial derivatives $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial x}{\partial v}$ and $\frac{\partial y}{\partial v}$ may be determined from $x = x(u, v)$ and $y = y(u, v)$. However, for the second pair, partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ are required. These belong to the inverse transformation of u, v into x, y , and can be obtained by differentiating the transforming equations implicitly.

First, differentiate the transforming equations (1) partially with respect to x . We obtain

$$\begin{aligned} 1 &= \frac{\partial x}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial x} \\ 0 &= \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x}. \end{aligned}$$

Solving these as simultaneous equations in $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ gives

$$\begin{aligned} u_x &= \frac{y_v}{x_u y_v - x_v y_u} \\ \text{and } v_x &= \frac{-y_u}{x_u y_v - x_v y_u}. \end{aligned} \quad (2)$$

Secondly, by differentiating the transforming equations (1) partially with respect to y , the following equations are obtained:

$$u_y = \frac{-x_v}{x_u y_v - x_v y_u} \quad \text{and} \quad v_y = \frac{x_u}{x_u y_v - x_v y_u}. \quad (3)$$

The common denominator obtained in these results is the **Jacobian** of the transformation from \mathbb{R}^2 into \mathbb{R}^2 given by

$$(u, v) \mapsto (x, y),$$

and it is the determinant

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix},$$

Now using (2) and (3) observe that

$$\begin{aligned} \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= \frac{1}{(x_u y_v - x_v y_u)^2} \begin{vmatrix} y_v & -x_v \\ -y_u & x_u \end{vmatrix} \\ &= \frac{1}{x_u y_v - x_v y_u} \\ &= \frac{1}{\frac{\partial(x, y)}{\partial(u, v)}}, \end{aligned}$$

confirming the identity obtained from the Inverse Function Theorem in the notes.

NOW, in the above argument, the roles of x, y and u, v could be interchanged. Then, we would get the equivalent formulas

$$\begin{aligned} x_u &= \frac{v_y}{u_x v_y - u_y v_x} \\ y_u &= \frac{-v_x}{u_x v_y - u_y v_x}, \\ x_v &= \frac{-u_y}{u_x v_y - u_y v_x} \\ \text{and } y_v &= \frac{u_x}{u_x v_y - u_y v_x}. \end{aligned} \quad (4)$$

Formulas (2), (3) and (4) can also be derived more formally from the Inverse Function Theorem by using the identity

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}^{-1}.$$