

**Partial derivatives of inverse functions – the classical approach**  
**This discussion helps you to complete Problem 33 in Exercise 4.10**

Given the function  $(x, y) \mapsto z = f(x, y)$  and the transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  given by  $(u, v) \mapsto (x, y)$ , where

$$x = x(u, v) \text{ and } y = y(u, v), \quad (1)$$

so that  $z = f((x(u, v), y(u, v)))$ , the Chain Rule may be used either as

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \quad \text{and} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \end{aligned}$$

or

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \quad \text{and} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}. \end{aligned}$$

For the first pair, partial derivatives  $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial x}{\partial v}$  and  $\frac{\partial y}{\partial v}$  may be determined from  $x = x(u, v)$  and  $y = y(u, v)$ . However, for the second pair, partial derivatives  $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$  and  $\frac{\partial v}{\partial y}$  are required. These belong to the inverse transformation of  $u, v$  into  $x, y$ , and can be obtained by differentiating the transforming equations implicitly.

First, differentiate the transforming equations (1) partially with respect to  $x$ . We obtain

$$\begin{aligned} 1 &= \frac{\partial x}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial x} \\ 0 &= \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x}. \end{aligned}$$

Solving these as simultaneous equations in  $\frac{\partial u}{\partial x}$  and  $\frac{\partial v}{\partial x}$  gives

$$\begin{aligned} u_x &= \frac{y_v}{x_u y_v - x_v y_u} \\ \text{and } v_x &= \frac{-y_u}{x_u y_v - x_v y_u}. \end{aligned} \quad (2)$$

Secondly, by differentiating the transforming equations (1) partially with respect to  $y$ , the following equations are obtained:

$$u_y = \frac{-x_v}{x_u y_v - x_v y_u} \quad \text{and} \quad v_y = \frac{x_u}{x_u y_v - x_v y_u}. \quad (3)$$

The common denominator obtained in these results is the **Jacobian** of the transformation from  $\mathbb{R}^2$  into  $\mathbb{R}^2$  given by

$$(u, v) \mapsto (x, y),$$

and it is the determinant

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix},$$

Now using (2) and (3) observe that

$$\begin{aligned} \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= \frac{1}{(x_u y_v - x_v y_u)^2} \begin{vmatrix} y_v & -x_v \\ -y_u & x_u \end{vmatrix} \\ &= \frac{1}{x_u y_v - x_v y_u} \\ &= \frac{1}{\frac{\partial(x, y)}{\partial(u, v)}}, \end{aligned}$$

confirming the identity obtained from the Inverse Function Theorem in the notes.

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**NOW**, in the above argument, the roles of  $x, y$  and  $u, v$  could be interchanged. Then, we would get the equivalent formulas

$$\begin{aligned} x_u &= \frac{v_y}{u_x v_y - u_y v_x} \\ y_u &= \frac{-v_x}{u_x v_y - u_y v_x}, \\ x_v &= \frac{-u_y}{u_x v_y - u_y v_x} \\ \text{and } y_v &= \frac{u_x}{u_x v_y - u_y v_x}. \end{aligned} \quad (4)$$

Formulas (2), (3) and (4) can also be derived more formally from the Inverse Function Theorem by using the identity

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}^{-1}.$$