Partial derivatives of inverse functions – the classical approach

This discussion helps you to complete Problem 33 in Exercise 4.10

Given the function \((x, y) \mapsto z = f(x, y)\) and the transformation from \(\mathbb{R}^2\) to \(\mathbb{R}^2\) given by \((u, v) \mapsto (x, y)\), where

\[ x = x(u, v) \quad \text{and} \quad y = y(u, v), \quad (1) \]

so that \(z = f((x(u, v), y(u, v))\), the Chain Rule may be used either as

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.
\]

or

\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.
\]

For the first pair, partial derivatives \(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial x}{\partial v}, \text{ and } \frac{\partial y}{\partial v}\) may be determined from \(x = x(u, v)\) and \(y = y(u, v)\). However, for the second pair, partial derivatives \(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \text{ and } \frac{\partial v}{\partial y}\) are required. These belong to the inverse transformation of \(u,v\) into \(x,y\), and can be obtained by differentiating the transforming equations implicitly.

First, differentiate the transforming equations (1) partially with respect to \(x\). We obtain

\[
1 = \frac{\partial x}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial x},
\]

\[0 = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial x}.
\]

Solving these as simultaneous equations in \(\frac{\partial u}{\partial x}\) and \(\frac{\partial v}{\partial x}\) gives

\[u_x = \frac{y_v}{x_u y_v - x_v y_u} \]

and \(v_x = \frac{-y_u}{x_u y_v - x_v y_u}\). (2)
Secondly, by differentiating the transforming equations (1) partially with respect to \( y \), the following equations are obtained:

\[
\begin{align*}
  u_y &= \frac{-x_v}{x_u y_v - x_v y_u} \quad \text{and} \quad v_y = \frac{x_u}{x_u y_v - x_v y_u}.
\end{align*}
\]  

(3)

The common denominator obtained in these results is the \textbf{Jacobian} of the transformation from \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \) given by

\[
(u, v) \mapsto (x, y),
\]

and it is the determinant

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix}
  x_u & x_v \\
  y_u & y_v
  \end{vmatrix},
\]

Now using (2) and (3) observe that

\[
\frac{\partial (u, v)}{\partial (x, y)} = \left| \begin{array}{cc}
  u_x & u_y \\
  v_x & v_y
  \end{array} \right| = \left| \begin{array}{cc}
  y_v & -x_v \\
  -y_u & x_u
  \end{array} \right| = \frac{1}{\frac{x_u y_v - x_v y_u}{\frac{\partial (x, y)}{\partial (u, v)}}}
\]

confirming the identity obtained from the Inverse Function Theorem in the notes.

***************

\textbf{NOW}, in the above argument, the roles of \( x, y \) and \( u, v \) could be interchanged. Then, we would get the equivalent formulas

\[
\begin{align*}
  x_u &= \frac{v_y}{u_x v_y - u_y v_x}, \\
  y_u &= \frac{-v_x}{u_x v_y - u_y v_x}, \\
  x_v &= \frac{-u_y}{u_x v_y - u_y v_x} \\
  \text{and} \quad y_v &= \frac{u_x}{u_x v_y - u_y v_x}.
\end{align*}
\]  

(4)

Formulas (2), (3) and (4) can also be derived more formally from the Inverse Function Theorem by using the identity

\[
\begin{pmatrix}
  \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
  \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix} = \left( \begin{pmatrix}
  \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
  \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix} \right)^{-1}.
\]