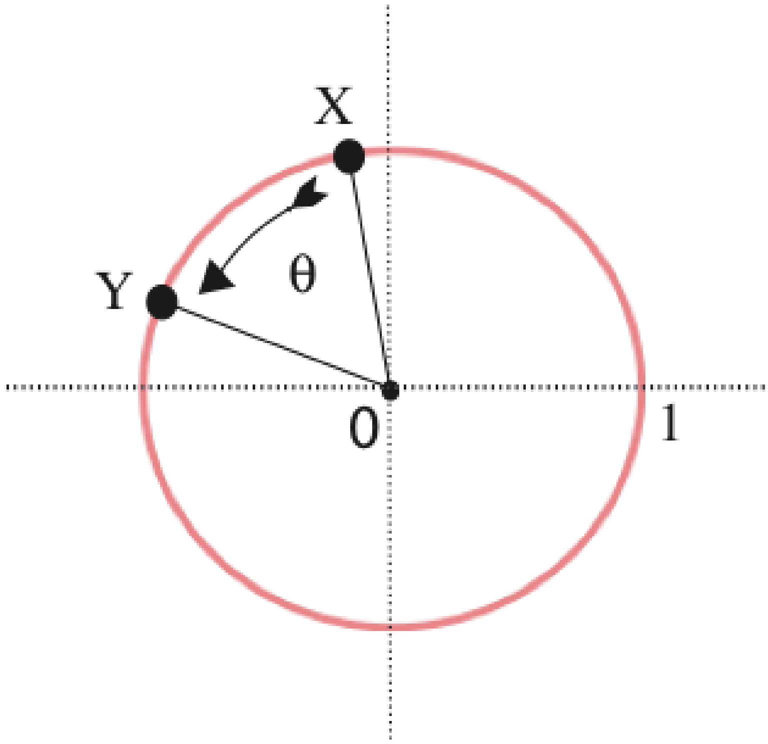
MATH201

EXAMPLE ON LINEAR FUNCTIONS

ROTATIONS ARE LINEAR FUNCTIONS



The Figure illustrates an anticlockwise rotation R_θ about the origin through the angle θ in $\mathbb{R}^2$. The rotation takes an arbitrary vector X in $\mathbb{R}^2$ and rotates it anti-clockwise about the origin through the angle θ to give the vector Y in $\mathbb{R}^2$. The rotation does this for **every** point in $\mathbb{R}^2$. Thus R_θ is a function with domain $\mathbb{R}^2$ and codomain $\mathbb{R}^2$, so we write

$$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

Now suppose that X is the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Put $Y = R_\theta(X)$ and suppose that Y is the vector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. We can calculate the coordinates y_1, y_2 of Y in terms of θ and the coordinates x_1, x_2 of X. We find that

$$\begin{aligned} y_1 &= \cos\theta x_1 - \sin\theta\, x_2, \text{ and} \\ y_2 &= \sin\theta\, x_1 + \cos\theta\, x_2. \end{aligned}$$

These equations can be written in matrix form: we have

$$R_\theta(X) = Y$$

$$
\begin{aligned}
&= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\, x_1 - \sin\theta\, x_2 \\ \sin\theta\, x_1 + \cos\theta\, x_2 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= A(\theta)X, \qquad (1)
\end{aligned}
$$

where $A(\theta)$ is the 2×2 matrix

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(1) shows that

$$R_\theta(X) = A(\theta)X, \text{ for all } X \in \mathbb{R}^2.$$

Thus, R_θ is linear, by a result in lectures.