EXAMPLE

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be given by

$$f(x,y,z) = (x^2 + y, y^2 + z, z^2 + x).$$

We calculate the derivative matrix of f. We get

$$f'(x,y,z) = \begin{pmatrix} 2x & 1 & 0 \\ 0 & 2y & 1 \\ 1 & 0 & 2z \end{pmatrix}$$

So, for example, we have

$$f'(1,-1,2) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 4 \end{pmatrix} \text{ and } f'(3,-2,8) = \begin{pmatrix} 6 & 1 & 0 \\ 0 & -4 & 1 \\ 1 & 0 & 16 \end{pmatrix}$$

The Jacobian of f at (x,y,z) is

$$Jf(x,y,z) = \det(f'(x,y,z)) = 2x(4yz) - (-1) = 8xyz + 1.$$

We have

$$Jf(1,-1,2) = -16 + 1 = -15 \neq 0,$$
$$Jf(3,-2,8) = -384 + 1 = -383 \neq 0.$$

The Jacobian of f at each of the points $(1,-1,2)$ and $(3,-2,8)$ is non-zero. So, by the Inverse Function Theorem, about each of these points, there is an open sphere on which the function f has a differentiable inverse. Then, noting that

$$f(1,-1,2) = (0,3,5),$$

from this Theorem we know that

$$(f^{-1})'(0,3,5) = (f'(1,-1,2))^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 4 \end{pmatrix}^{-1} = \frac{1}{15}\begin{pmatrix} 8 & 4 & -1 \\ -1 & -8 & 2 \\ -2 & -1 & 4 \end{pmatrix}.$$

Rod Nillsen
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