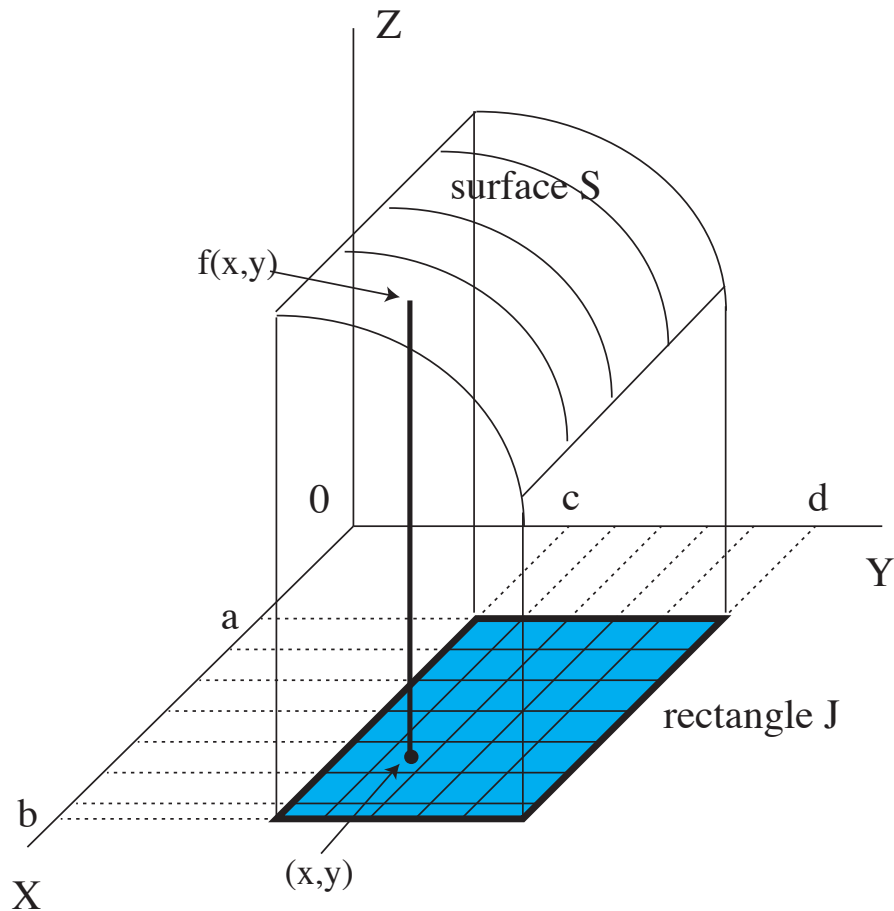


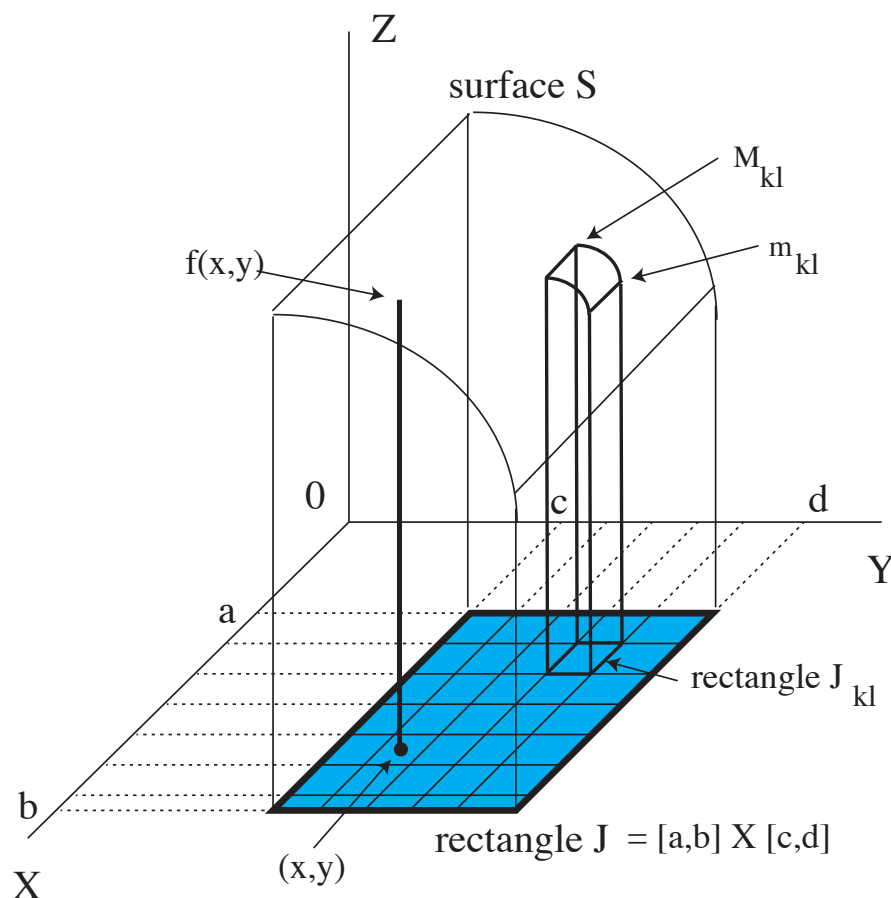
THE DEFINITION OF THE INTEGRAL OF A FUNCTION OF TWO VARIABLES



**Figure IV.1.** The figure depicts in the  $XY$ -plane the rectangle  $J = [a, b] \times [c, d]$ . The surface  $S$  is the graph of the function  $f : J \rightarrow \mathbb{R}$ , so

$$S = \{(x, y, f(x, y)) : (x, y, 0) \in J\}.$$

The rectangle  $J = [a, b] \times [c, d]$  is partitioned into subrectangles, as indicated, using partitions on  $[a, b]$  on the  $X$ -axis and on  $[c, d]$  on the  $Y$ -axis. The integral of  $f$  over  $J$  represents, geometrically, the *volume* between the surface  $S$  (the graph of  $f$ ) and the rectangle  $J$  in the  $XY$ -plane.



**Figure IV.2.** For each rectangle  $J_{kl}$  in the partition of  $J$  we consider

$$M_{kl} = \max\{f(x, y, 0) : (x, y, 0) \in J_{kl}\} \text{ and } m_{kl} = \min\{f(x, y, 0) : (x, y, 0) \in J_{kl}\}.$$

Then,  $m_{kl}(\text{area of } J_{kl}) \leq \text{volume between } J_{kl} \text{ and surface } S \leq M_{kl}(\text{area of } J_{kl})$ .

Thus, the total volume between  $J$  and the surface  $S$  lying above  $J$  is at least

$$\sum_{kl} m_{kl}(\text{area of } J_{kl})$$

and is at most

$$\sum_{kl} M_{kl}(\text{area of } J_{kl}).$$

The difference between these two sums is

$$\sum_{kl} (M_{kl} - m_{kl})(\text{area of } J_{kl}),$$

and this tends to the limit 0 as the partitions of  $J$  get finer and finer. As the true volume between  $S$  and  $J$  is between the two sums, this means that the volume is the limit of both

$$\sum_{kl} m_{kl}(\text{area of } J_{kl}) \text{ and } \sum_{kl} M_{kl}(\text{area of } J_{kl}),$$

as the partitions of  $J$  get finer and finer.