

THE DEFINITION OF THE INTEGRAL OF A FUNCTION OF TWO VARIABLES

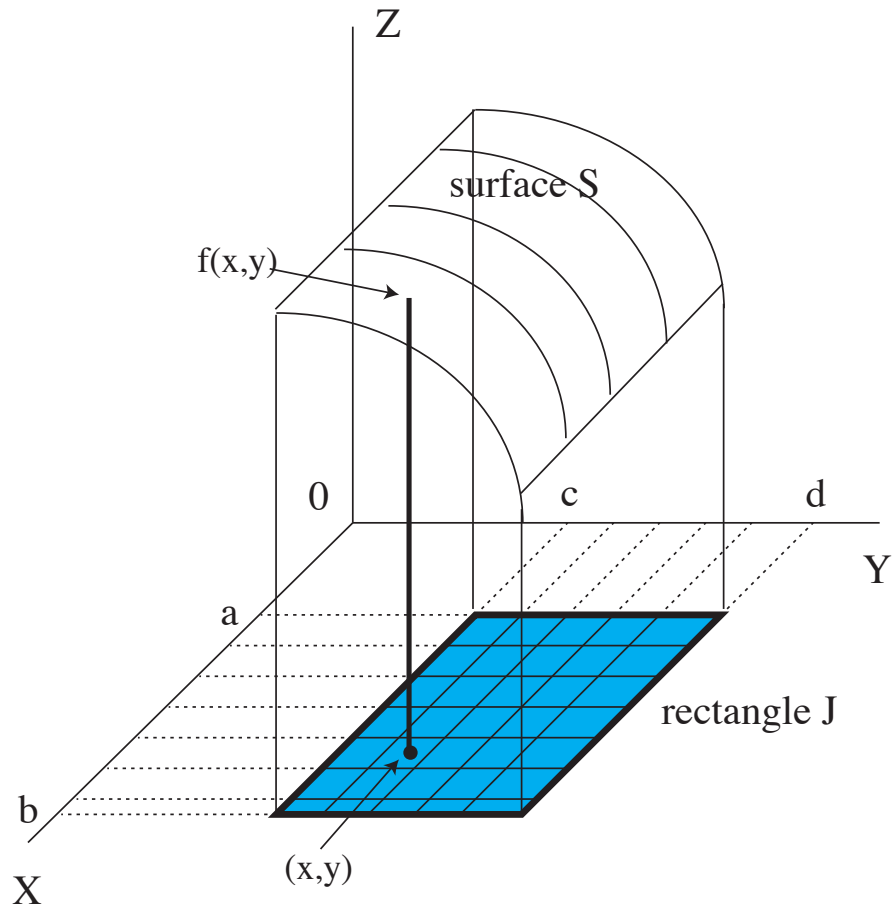


Figure IV.1. The figure depicts in the XY -plane the rectangle $J = [a, b] \times [c, d]$. The surface S is the graph of the function $f : J \rightarrow \mathbb{R}$, so

$$S = \{(x, y, f(x, y)) : (x, y, 0) \in J\}.$$

The rectangle $J = [a, b] \times [c, d]$ is partitioned into subrectangles, as indicated, using partitions on $[a, b]$ on the X -axis and on $[c, d]$ on the Y -axis. The integral of f over J represents, geometrically, the *volume* between the surface S (the graph of f) and the rectangle J in the XY -plane.

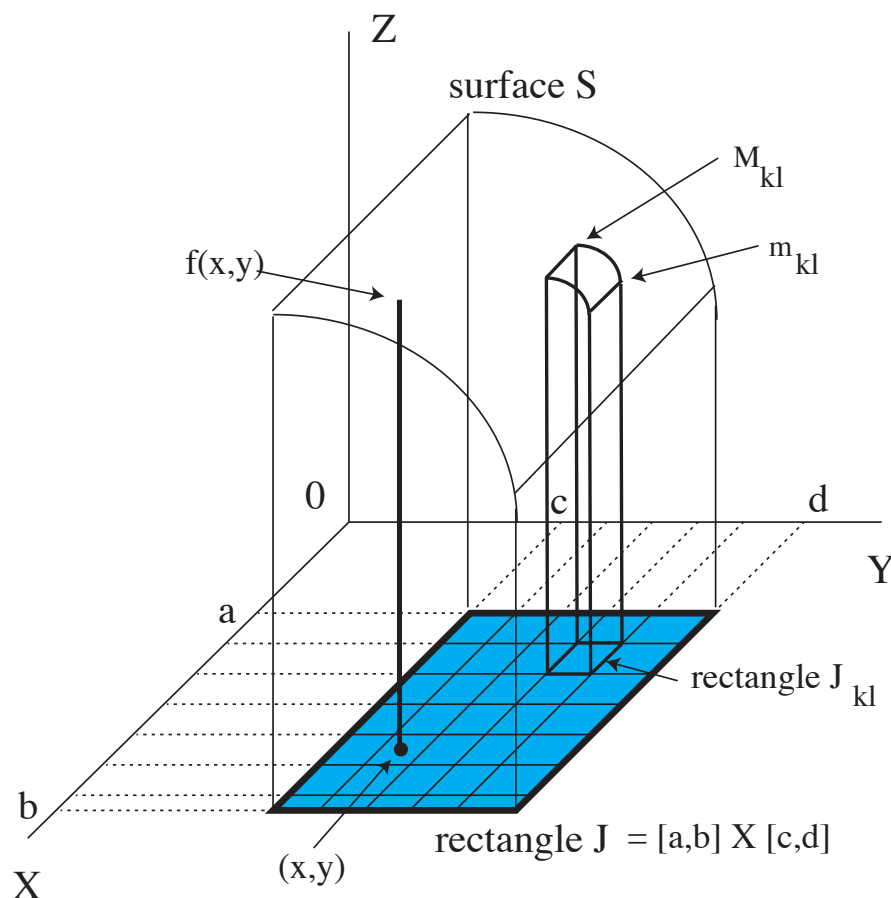


Figure IV.2. For each rectangle J_{kl} in the partition of J we consider

$$M_{kl} = \max\{f(x, y, 0) : (x, y, 0) \in J_{kl}\} \text{ and } m_{kl} = \min\{f(x, y, 0) : (x, y, 0) \in J_{kl}\}.$$

Then, $m_{kl}(\text{area of } J_{kl}) \leq \text{volume between } J_{kl} \text{ and surface } S \leq M_{kl}(\text{area of } J_{kl})$.

Thus, the total volume between J and the surface S lying above J is at least

$$\sum_{kl} m_{kl}(\text{area of } J_{kl})$$

and is at most

$$\sum_{kl} M_{kl}(\text{area of } J_{kl}).$$

The difference between these two sums is

$$\sum_{kl} (M_{kl} - m_{kl})(\text{area of } J_{kl}),$$

and this tends to the limit 0 as the partitions of J get finer and finer. As the true volume between S and J is between the two sums, this means that the volume is the limit of both

$$\sum_{kl} m_{kl}(\text{area of } J_{kl}) \text{ and } \sum_{kl} M_{kl}(\text{area of } J_{kl}),$$

as the partitions of J get finer and finer.