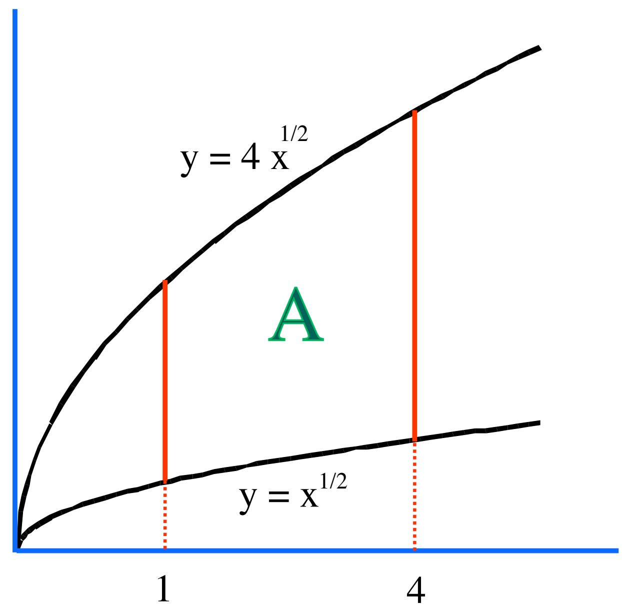
MATH 201: EXAMPLE ON INTEGRATION BY SUBSTITUTION IN TWO VARIABLES



The subset A of $\mathbb{R}^2$ is illustrated in the picture. It is the region bounded by the graphs of $y = 4\sqrt{x}$ and $y = \sqrt{x}$ and by $x = 1$ and $x = 4$. Let us make the substitutions

$$u = x \text{ and } v = \frac{y}{\sqrt{x}}.$$

From these equations we have

$$x = u \text{ and } y = v\sqrt{u}.$$

[Note that this gives us the *inverse* of the transformation $(x, y) \longmapsto (u, v) = \left(x, \frac{y}{\sqrt{x}}\right)$.] Thus, for the Jacobian we will have

$$\frac{\partial(x,y)}{\partial(u,v)} = det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = det \begin{pmatrix} 1 & 0 \\ \frac{v}{2\sqrt{u}} & \sqrt{u} \end{pmatrix} = \sqrt{u}.$$

We can check that the transformation mapping $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$(x, y) \longmapsto (u, v) = \left(x, \frac{y}{\sqrt{x}}\right),$$

maps the set A onto the rectangle $B = [1, 4] \times [1, 4]$. Then if $f : A \longrightarrow \mathbb{R}$ is a given function, we will have

$$\begin{aligned} \int\int_A f(x,y)dxdy &= \int\int_B f(x(u,v), y(u,v))\left|\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right|dudv \\ &= \int_1^4 \left(\int_1^4 f(x(u,v), y(u,v))\sqrt{u}du\right) dv \\ &= \int_1^4 \left(\int_1^4 f(u, v\sqrt{u})\sqrt{u}du\right) dv. \end{aligned}$$

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